

SOME JACOBSTHAL AND JACOBSTHAL-LUCAS IDENTITIES BY MATRIX APPROACH AND THEIR APPLICATIONS

Piyanut Puangjumpa^{1*} and Sa-at Muangchan²

¹Department of Mathematics and Statistics, Faculty of Science and Technology,
Surindra Rajabhat University, Surin, 32000.

²Department of Mathematics and Statistics, Faculty of Science and Technology,
Sakon Nakhon Rajabhat University, Sakon Nakhon, 47000.

*E-mail: piyanut@math.sut.ac.th

Received: 2018-09-28

Revised: 2018-12-04

Accepted: 2019-01-10

ABSTRACT

In this research, we studied and found some identities involving Jacobsthal and Jacobsthal-Lucas numbers by using matrix approach. In the applications, we got the solutions of some Diophantine equations by applying these identities.

Keywords: Jacobsthal numbers, Jacobsthal-Lucas numbers, Diophantine equations, Binet's formula.

Introduction

There are many research articles about the sequence of integers and its application (Bergum et al., 1985) and (Horadam, 1988). Koken & Bozkurt (2008) defined the Jacobsthal Lucas E -matrix and R -matrix alike to the Fibonacci Q -matrix. They found some equalities and Binet-like formula for the Jacobsthal and Jacobsthal-Lucas numbers. Srisawat et al. (2015) defined the k -Jacobsthal S -matrix and k -Jacobsthal W -matrix. And then, by using this matrix representation, they obtain some identities and the Binet's formula for k -Jacobsthal numbers. Srisawat & Sriprad (2016) studied on some identities involving Pell and Pell-Lucas numbers by using matrix methods and they presented the solution of some Diophantine equations by applying these

identities. Suvarnamani (2017) searched properties of the (p, q) - Fibonacci numbers by using matrix methods. These motivation research articles lead us to the new idea for Jacobsthal and Jacobsthal-Lucas numbers.

In this work, we are interested to discover some Jacobsthal and Jacobsthal-Lucas by matrix approach and also their applications. The Jacobsthal sequence $\{J_n\}$ is an additive sequence similar to the Fibonacci sequence, defined by the recurrence relation $J_n = J_{n-1} + 2J_{n-2}$ for all $n \geq 2$ with initial values $J_0 = 0$ and $J_1 = 1$. The first few terms of $\{J_n\}$ are 0, 1, 1, 3, 5,..... (sequence A001045 in the On-Line Encyclopedia of Integer Sequences or OEIS). A number in the sequence is called a Jacobsthal number and we denoted the n^{th} Jacobsthal numbers by J_n . The Jacobsthal numbers for negative subscripts are defined as $J_{-n} = -(-2)^{-n}J_n$. Similarly, The Jacobsthal-Lucas sequence is the sequence of integers $\{j_n\}$ defined by the recurrence relation $j_n = j_{n-1} + 2j_{n-2}$ for all $n \geq 2$ with initial values $j_0 = 2$ and $j_1 = 1$. The first few terms of $\{j_n\}$ are 2, 1, 5, 7, 17,..... (sequence A014551 in the OEIS). The numbers this sequence are called Jacobsthal-Lucas numbers and we denoted the n^{th} Jacobsthal-Lucas numbers by j_n . The Jacobsthal-Lucas numbers for negative subscripts are defined as $j_{-n} = (-2)^{-n}j_n$. It can be seen that $j_n = J_n + 4J_{n-1}$ and $j_n = J_{n+1} + 2J_{n-1}$ for any integer n . For $\{J_n\}$ and $\{j_n\}$, the characteristic equation is $x^2 - x - 2 = 0$ with roots $\alpha = 2$ and $\beta = -1$, the Binet's formula are $J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $j_n = \alpha^n + \beta^n$ for all $n \geq 0$ (Bergum et al.,1985) and (Horadam,1988).

Methods

In this section, we will derive some that identities for Jacobsthal and Jacobsthal-Lucas numbers by using matrix approach and we begin with the following Lemma.

Lemma 2.1 Let X be a square matrix with $X^2 = X + 2I$, then $X^n = J_n X + 2J_{n-1} I$ for any integer n .

Proof. If $n = 0$, then the proof is obvious. Next, we will use the principle of mathematical induction to show that $X^n = J_n X + 2J_{n-1} I$ for all $n \in \mathbb{N}$. When $n = 1$, we have

$$X = (1)X + (0)I = (1)X + 2(0)I = J_1 X + 2J_0 I$$

It is seen to be true. We assume that it is true for $n = k$. We will show that it is true for $n = k + 1$ as follows:

$$\begin{aligned} X^{k+1} &= X^k X \\ &= (J_k X + 2J_{k-1} I) X \\ &= J_k X^2 + 2J_{k-1} X \\ &= J_k (X + 2I) + 2J_{k-1} X \\ &= J_{k+1} X + 2J_k I. \end{aligned}$$

Hence, $X^n = J_n X + 2J_{n-1} I$ for any integer n .

Finally, we will show that $X^{-n} = J_{-n} X + 2J_{-(n-1)} I$ for any integer n . Let us consider

$$\begin{aligned} &(J_n X + 2J_{n-1} I) (J_{-n} X + 2J_{-(n+1)} I) \\ &= J_n J_{-n} X^2 + 2J_{n-1} J_{-n} X + 2J_n J_{-(n+1)} X + 4J_{n-1} J_{-(n+1)} I \\ &= -(-2)^n J_n^2 (X + 2I) - 2J_{n-1} (-2)^n J_n X - 2J_n (-2)^{-(n+1)} J_{n+1} X - 4J_{n-1} (-2)^{-(n+1)} J_{n+1} I \\ &= -(-2)^n J_n^2 X + (-2)^{n+1} J_n^2 I - 2(-2)^n J_{n-1} J_n X - 2(-2)^{-(n+1)} J_n J_{n+1} X - (-2)^{-(n+1)} J_{n-1} J_{n+1} I \\ &= -(-2)^n \left[J_n (-J_n + 2J_{n-1}) + J_{n+1} \right] X - 2 \left(\left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right)^2 - \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} \right) \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \right) \right) I \\ &= -(-2)^n \left[J_n (-J_{n+1} + J_{n+1}) X - 2 \left(\frac{-2(\alpha\beta)^n + \alpha^{n-1}\beta^{n+1} + \alpha^{n+1}\beta^{n-1}}{(\alpha - \beta)^2} \right) I \right] \\ &= I. \end{aligned}$$

In similarly, we have $(J_{-n} X + 2J_{-(n+1)} I)(J_n X + 2J_{n-1} I) = I$.

Thus, $X^n = J_n X + 2J_{n-1} I$.

This completes the proof of the Lemma.

From Lemma 2.1, we can give Corollary 2.2 easily. For more information about the matrices, one can see (Koken & Bozkurt, 2008) and (Srisawat et al., 2015).

Corollary 2.2 Let $F = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$. Then $F^n = \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix}$ for any integer n .

Next, let us define the matrix Z as in the following Lemma and by using the matrix, we obtain some identities for Jacobsthal and Jacobsthal-Lucas numbers.

Lemma 2.3 Let $Z = \begin{bmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix}$. Then $Z^n = \begin{bmatrix} (1/2)j_n & (3/2)J_n \\ (3/2)J_n & (1/2)j_n \end{bmatrix}$ for any integer n .

Proof. Note that $Z^2 = \begin{bmatrix} 5/2 & 3/2 \\ 3/2 & 5/2 \end{bmatrix} = Z + 2I$. By the Lemma 2.1,
we have $Z^n = J_n Z + 2J_{n-1} I$.

It follows that

$$\begin{aligned} Z^n &= \begin{bmatrix} (1/2)J_n + 2J_{n-1} & (3/2)J_n \\ (3/2)J_n & (1/2)J_n + 2J_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} (1/2)J_n + 4J_{n-1} & (3/2)J_n \\ (3/2)J_n & (1/2)J_n + 4J_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} (1/2)j_n & (3/2)J_n \\ (3/2)J_n & (1/2)j_n \end{bmatrix} \end{aligned}$$

This completes the proof of the Lemma.

By using the matrix Z , we get Lemma 2.4 and Lemma 2.5.

Lemma 2.4 For any integer n , the following equality holds:

$$j_n^2 - 9J_n^2 = (-2)^{n+2}$$

Proof. Since $\det(Z) = -2$ and $\det(Z^n) = \frac{1}{4}j_n^2 - \frac{9}{4}J_n^2$

It follows that $j_n^2 - 9J_n^2 = 4(-2)^n = (-2)^{n+2}$

This completes the proof of the Lemma.

Lemma 2.5 Let m and n be any integers. Then the following equality holds:

$$(i) 2j_{m+n} = j_m j_n + 9J_m J_n,$$

$$(ii) 2J_{m+n} = j_n J_m + j_m J_n.$$

Proof. Since $Z^{m+n} = Z^m Z^n$, by Lemma 2.3, we have

$$\begin{bmatrix} (1/2)j_{m+n} & (3/2)J_{m+n} \\ (3/2)J_{m+n} & (1/2)j_{m+n} \end{bmatrix} = \begin{bmatrix} (1/2)j_m & (3/2)J_m \\ (3/2)J_m & (1/2)j_m \end{bmatrix} = \begin{bmatrix} (1/2)j_n & (3/2)J_n \\ (3/2)J_n & (1/2)j_n \end{bmatrix}$$

Therefore, $2j_{m+n} = j_m j_n + 9J_m J_n$ and $2J_{m+n} = j_n J_m + j_m J_n$.

This completes the proof of the Lemma.

Lemma 2.6 For any integer n , the following equalities hold:

$$(i) \alpha^n = \alpha J_n + 2J_{n-1},$$

$$(ii) \beta^n = \beta J_n + 2J_{n-1}.$$

Proof. Let $A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ then $A^2 = A + 2I$. By the lemma 2.1,

$$\text{we have } A^n = J_n A + 2J_{n-1} I.$$

$$\text{It follows that } \begin{bmatrix} \alpha_n & 0 \\ 0 & \beta_n \end{bmatrix} = \begin{bmatrix} \alpha J_n + 2J_{n-1} & 0 \\ 0 & \beta J_n + 2J_{n-1} \end{bmatrix}$$

$$\text{This implies that } \alpha_n = \alpha J_n + 2J_{n-1} \text{ and } \beta^n = \beta J_n + 2J_{n-1}$$

This completes the proof of the Lemma.

By using Lemma 2.1 and Lemma 2.6, we get the following Lemma.

$$\text{Lemma 2.7 Let } B = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}. \text{ Then } B^n = \begin{bmatrix} \alpha_n & 0 \\ J_n & \beta^n \end{bmatrix} \text{ for any integer } n.$$

Proof. Since $B^2 = B + 2I$. Hence, by Lemma 2.1 and Lemma 2.6,

$$\begin{aligned} \text{it follows that } B^n &= J_n B + 2J_{n-1} I \\ &= \begin{bmatrix} \alpha J_n + 2J_{n-1} & 0 \\ J_n & \beta J_n + 2J_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} \alpha_n & 0 \\ J_n & \beta^n \end{bmatrix}. \end{aligned}$$

This completes the proof of the Lemma.

Remark 2.8 From the Binet's formula of Jacobsthal and Jacobsthal-Lucas numbers,

$$J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{2^n - (-1)^n}{3} \text{ and } j_n = \alpha^n + \beta^n = 2^n + (-1)^n, \text{ respectively.}$$

We can rewrite as

$$3(-1)^n J_n + 1 = (-2)^n \text{ and } (-1)^n j_n - 1 = (-2)^n, \text{ for any integer } n.$$

Now, by using Lemma 2.7 and Remark 2.8, we get the following Theorem.

Theorem 2.9 Let m and n be any integers. Then the following equality holds:

$$(-1)^n j_m^2 j_n + (-1)^m j_m j_n^2 + j_{m+n}^2 - j_m^2 - j_n^2 - 4(-1)^{m+n} j_{m+n} = j_m j_n j_{m+n} - 4.$$

Proof. Let B be a matrix as in Lemma 2.7. Then we have,

$$B^{n+1} + 2B^{n-1} = \begin{bmatrix} 3\alpha^n & 0 \\ j_n & -3\beta^n \end{bmatrix}$$

Since $(B^{m+1} + 2B^{m-1})(B^{n+1} + 2B^{n-1}) = B^{m+n+2} + 4B^{m+n} + 4B^{m+n-2}$. Then

$$\begin{aligned} 3(\alpha^n j_m - \beta^n j_n) &= J_{m+n+2} + 4J_{m+n} + 4J_{m+n-2} \\ &= J_{m+n+1} + 2J_{m+n} + 4J_{m+n} + 2(J_{m+n} - J_{m+n-1}) \\ &= J_{m+n} + 2J_{m+n-1} + 8J_{m+n} - 2J_{m+n-1} = 9J_{m+n} \end{aligned}$$

We get $3J_{m+n} = \alpha^n j_m - \beta^n j_n$. Hence,

$$\begin{aligned} 9J_{m+n}^2 &= (3J_{m+n})(3J_{n+m}) \\ &= (\alpha^n j_m - \beta^n j_n)(\alpha^m j_n - \beta^m j_m) \\ &= (\alpha^{m+n} + \beta^{m+n}) j_m j_n - (\alpha\beta)^n j_m^2 - (\alpha\beta)^m j_n^2 \\ &= j_m j_n j_{m+n} - (-2)^n j_m^2 - (-2)^m j_n^2. \end{aligned}$$

By Lemma 2.4, so that

$$j_{m+n}^2 - (-2)^{m+n+2} = j_m j_n j_{m+n} - (-2)^n j_m^2 - (-2)^m j_n^2 \quad (2.1)$$

By Remark 2.8, so that

$$\begin{aligned} j_{m+n}^2 - 4((-1)^{m+n} j_{m+n} - 1) &= j_m j_n j_{m+n} - ((-1)^n j_n - 1) j_m^2 - ((-1)^m j_m - 1) j_n^2 \\ j_{m+n}^2 - 4(-1)^{m+n} j_{m+n} + 4 &= j_m j_n j_{m+n} - (-1)^n j_m^2 j_n + j_m^2 - (-1)^m j_m j_n^2 + j_n^2 \\ (-1)^n j_m^2 j_n + (-1)^m j_m j_n^2 + j_{m+n}^2 - j_m^2 - j_n^2 - 4(-1)^{m+n} j_{m+n} &= j_m j_n j_{m+n} - 4 \end{aligned}$$

This completes the proof of the Theorem.

Theorem 2.10 Let m and n be any integers. Then the following equality holds:

$$27(-1)^n J_m^2 J_n + 27(-1)^m J_m J_n^2 - j_{m+n}^2 + 9J_m^2 + 9J_n^2 + 4(-1)^{m+n} j_{m+n} = -9J_m J_n j_{m+n} + 4$$

Proof. By (2.1), Lemma 2.4 and Lemma 2.5, we obtain

$$\begin{aligned} j_{m+n}^2 - (-2)^{m+n+2} &= (2j_{m+n} - 9J_m J_n) j_{m+n} - (-2)^n (9J_m^2 + (-2)^{m+2}) - (-2)^m (9J_n^2 + (-2)^{n+2}) \\ j_{m+n}^2 - (-2)^{m+n+2} &= 2j_{m+n}^2 - 9J_m J_n j_{m+n} - 9(-2)^n J_m^2 - (-2)^{m+n+2} - 9(-2)^m J_n^2 - (-2)^{m+n+2} \\ j_{m+n}^2 - 9(-2)^n J_m^2 - 9(-2)^m J_n^2 &= 9J_m J_n j_{m+n} + 4(-2)^{m+n} \end{aligned}$$

By Remark 2.8, so that

$$\begin{aligned} j_{m+n}^2 - 9(3(-1)^n j_n + 1) J_m^2 - 9(3(-1)^m j_m + 1) &= 9J_m J_n j_{m+n} + 4((-1)^{m+n} j_{m+n} - 1) \\ j_{m+n}^2 - 27(-1)^n J_m^2 J_n - 9J_m^2 - 27(-1)^m J_m J_n^2 - 9J_n^2 &= 9J_m J_n j_{m+n} + 4(-1)^{m+n} j_{m+n} - 4 \\ 27(-1)^n J_m^2 J_n + 27(-1)^m J_m J_n^2 - j_{m+n}^2 + 9J_m^2 + 9J_n^2 + 4(-1)^{m+n} j_{m+n} &= -9J_m J_n j_{m+n} + 4 \end{aligned}$$

This completes the proof of the Theorem.

Theorem 2.11 Let m and n be any integers. Then the following equality holds:

$$3(-1)^m j_n^2 J_m - 9(-1)^n j_n J_m^2 - 9J_m^2 + j_n^2 + 9J_m^2 - 12(-1)^{m+n} J_m = -9j_n J_m J_{m+n} + 4.$$

Proof. By similar argument as in Theorem 2.9.

$$\text{Since. } (B^{n+1} + 2B^{n-1})B^m = B^{m+n+1} + 2B^{m+n-1} = B^m(B^{n+1} + 2B^{n-1})$$

We get $j_{m+n} = \alpha^m j_n - 3\beta^n J_m$ and $j_{m+n} = 3\alpha^n J_m + \beta^m j_n$. Hence,

$$\begin{aligned} j_{m+n}^2 &= (\alpha^m j_n - 3\beta^n J_m) (3\alpha^n J_m + \beta^m j_n) \\ &= (3\alpha^{m+n} - 3\beta^{m+n}) j_n J_m + (\alpha\beta)^m j_n^2 - 9(\alpha\beta)^n J_m^2 \\ &= 9 \left(\frac{\alpha^{m+n} - \beta^{m+n}}{\alpha - \beta} \right) j_n J_m + (\alpha\beta)^m j_n^2 - 9(\alpha\beta)^n J_m^2 \\ &= 9j_n J_m J_{m+n} + (-2)^m j_n^2 - 9(-2)^n J_m^2 \end{aligned}$$

By Lemma 2.4, so that

$$\begin{aligned} 9J_m^2 + (-2)^{m+n+2} &= 9j_n J_m J_{m+n} + (-2)^m j_n^2 - 9(-2)^n J_m^2 \\ 9J_m^2 + 4(-2)^{m+n} &= 9j_n J_m J_{m+n} + (-2)^m j_n^2 - 9(-2)^n J_m^2 \end{aligned}$$

By Remark 2.8, so that

$$\begin{aligned} 9J_m^2 + 4(3(-1)^{m+n} J_m + 1) &= 9j_n J_m J_{m+n} + (3(-1)^m J_m + 1) j_n^2 - 9((-1)^n j_n - 1) J_m^2 \\ 9J_m^2 + 12(-1)^{m+n} J_m + 4 &= 9j_n J_m J_{m+n} + 3(-1)^m j_n^2 J_m + j_n^2 - 9(-1)^n j_n J_m^2 + 9J_m^2 \\ 3(-1)^m j_n^2 J_m - 9(-1)^n j_n J_m^2 - 9J_m^2 + j_n^2 + 9J_m^2 + 12(-1)^{m+n} J_m &= -9j_n J_m J_{m+n} + 4. \end{aligned}$$

This completes the proof of the Theorem.

Results

In this section, we give the solutions of some Diophantine equations by applying these above identities from Theorem 2.9, Theorem 2.10 and Theorem 2.11 as follow:

Corollary 3.1 Let m and n be any integers. Then $(x, y, z) = (j_m, j_n, j_{m+n})$ is a solution of the Diophantine equation

$$(-1)^n x^2 y + (-1)^m x y^2 + z_{m+n}^2 - x_m^2 - y_n^2 - 4(-1)^{m+n} z = xyz - 4$$

Proof. The result follows from Theorem 2.9.

Corollary 3.3 Let m and n be any integers. Then $(x, y, z) = (J_m, J_n, j_{m+n})$ is a solution of the Diophantine equation

$$27(-1)^n x^2 y + 27(-1)^m x y^2 - z^2 + 9x^2 + 9y^2 + 4(-1)^{m+n} z = -9xyz + 4$$

Proof. The result follows from Theorem 2.10.

Corollary 3.3 Let m and n be any integers. Then $(x, y, z) = (j_n, J_m, J_{m+n})$ is a solution of the Diophantine equation

$$3(-1)^m x^2 y - 9(-1)^n x y^2 - 9z^2 + x^2 + 9y^2 - 12(-1)^{m+n} z = -9xyz + 4$$

Proof. The result follows from Theorem 2.11.

Conclusions

In this research, some identities for Jacobsthal and Jacobsthal-Lucas numbers are studied and discovered by using matrix approach and for its application, we applied these identities to present the solutions of some Diophantine equations.

Acknowledgements:

The authors would like to thank the editors and referees for valuable comments and suggestions of this research.

References

Bergum, G. E., Bennett, L., Horadam, A. F. & Moore S. D. (1985). Jacobsthal polynomials and a conjecture concerning Fibonacci-like matrices. **The Fibonacci Quarterly**. 23, 240 - 248.

- Horadam, A. F. (1988). Jacobsthal and Pell. **The Fibonacci Quarterly**. 26, 79 - 83.
- Horadam, A. F. (1988). Jacobsthal representation numbers. **The Fibonacci Quarterly**. 34, 40 - 54.
- Koken, F. & Bozkurt, D. (2008) On the Jacobsthal-Lucas Numbers by Matrix Method. **Int. J. Contemp. Math Sciences**. 3, 1629 - 1633.
- Sloane, N. (1964). The On-Line Encyclopedia of Integer Sequences (OEIS). Retrieve March 2, 2019, from <https://oeis.org/>. accessed 1 March 2018.
- Srisawat, S., Sriprad, W. & Sthityanak, O. (2015). On the k-Jacobsthal numbers by matrix methods. **Science and Technol. RMUTT J.** 5(1), 70 - 76.
- Srisawat, S. & Sriprad, W. (2016). Some Pell and Pell-Lucas identities by matrix methods and their applications. **Science and Technol. RMUTT J.** 6(1), 170 - 174.
- Suvarnamani, A. (2017). Some identities of (p, q) -Fibonacci numbers by using matrix methods. **PRRJ. Science and Technol.** 12(2), 49 - 53.
-