4-CYCLES AND AN EQUILIBRIUM POINT IN A PIECEWISE LINEAR MAP WITH INITIAL CONDITIONS ON THE POSITIVE Y-AXIS

Nattakan Rungjang, Wiphawee Khamhom and Wirot Tikjha*

Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanulok 65000

*E-mail: wirottik@psru.ac.th

Received: 2024-04-02
Revised: 2024-08-18
Accepted: 2024-08-21

ABSTRACT

This article examines the behavior of a piecewise linear map, particularly when initial conditions are set along the positive y-axis. Our focus is on the emergence of 4-cycles and an equilibrium point within the map. We identify specific regions on the y-axis where the solutions tend to move toward these 4-cycles and the equilibrium point. By partitioning the positive y-axis into smaller segments, we analyze the solution behavior through a combination of direct calculation and induction methods. This approach allows us to demonstrate that the solutions consistently reach a prime period of 4 and an equilibrium point. Notably, this finding holds true regardless of the application of stability theorems, indicating a robust pattern in the solution dynamics. This study looks closely at how these solutions change over time, giving a clear picture of their paths. By carefully dividing the positive y-axis and studying each part, we find out why the solutions move towards the 4-cycles and the equilibrium point. This detailed look shows that the map's behavior is predictable and follows a certain pattern no matter where we start on the positive y-axis. Our research helps us understand piecewise linear systems better, and these insights could be useful for other similar systems. Our findings prove that periodic behavior and equilibrium are common in these maps, making it easier to predict how they will act over a long time.

Keywords: Difference equation, Periodic solution, Equilibrium point

Introduction

The Lozi map (Lozi, 1978) is a widely recognized example of a piecewise linear map (PWL), representing the most basic form of a piecewise smooth map (PWS). Functioning as a simplified variant of the Hénon map, the Lozi map displays a peculiar attractor. An example of a piecewise linear map involving absolute values is part of this category. It is acknowledged that PWS maps can exhibit phenomena such as multistability (Simpson, 2010; Zhusubaliyev et al., 2008) and an abundance of coexisting attractors (Simpson, 2014a; Simpson, 2014b). A noteworthy open problem related to a piecewise linear system was outlined by Grove et al. (2012) as follows:

$$\begin{cases} x_{n+1} = |x_n| + ay_n + b \\ y_{n+1} = x_n + c|y_n| + d \end{cases}, n = 0, 1, 2, \ldots \tag{1}$$

where parameters a, b, c and d are in $\{-1, 0, 1\}$ and the initial condition $(x_0, y_0) \in \mathbf{R}^2$. Grove et al. (2012) discovered that each solution of a specific instance of the system described in System (1) eventually becomes a prime period 3 solution, excluding the unique equilibrium solution. In works by Tikjha et al. (2010; 2015; 2017) and Tikjha & Lapiere (2020), various special cases of System (1) were investigated, revealing the existence of periodic attractors. Through direct calculations and inductive statements, they demonstrated that every solution ultimately converges to either these attractors or the equilibrium point. Our objective is to extend the generalization to the parameter b in the specified case of System (1), as outlined below:

$$\begin{cases} x_{n+1} = |x_n| - y_n + b \\ y_{n+1} = x_n - |y_n| + 1 \end{cases}, n = 0, 1, 2, \ldots \tag{2}$$

where b is any real number. In their research, Krinket & Tikjha (2015) examined a unique instance of System (2) where they set b equal to -1. They observed that solutions, under specific initial conditions on the axis, eventually become prime period 4. In their study, Tikjha & Piasu (2020) explored the outcomes of an alternate scenario in System (2) where they assigned b a value of -3:

$$\begin{cases} x_{n+1} = |x_n| - y_n - 3 \\ y_{n+1} = x_n - |y_n| + 1 \end{cases}, n = 0, 1, 2, \ldots \tag{3}$$

They determined that, when the initial condition lies within a specific region in the first quadrant, excluding the positive y-axis, the solutions eventually converge to either an equilibrium point or exhibit a prime period 4. As a result, we proceed to examine System (3) with the initial condition located on the positive y-axis.

Methods

We intend to employ direct calculations by substituting specific initial conditions into System (3). An "iteration" refers to the process of repeatedly applying a function or map to a point or state of the system. By verifying the negativity or non-negativity of the system in each iteration, we will utilize this property to compute the subsequent iteration. The subsequent definitions (Grove & Ladas, 2005) will be applied in this paper. A two-dimensional system of difference equations of the first order is a system of the form:

$$\begin{cases} x_{n+1} = f(x_n, y_n) \\ y_{n+1} = g(x_n, y_n) \end{cases}$$

where f and g are continuous functions which map $\mathbf{R}^2$ into $\mathbf{R}$ and $n \geq 0$. A **solution** of the system of difference equations is a sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ which satisfies the system for all $n \geq 0$. If we prescribean $(x_0, y_0) \in \mathbf{R}^2$ thensolutions $(x_1, y_1) = (g(x_0, y_0), f(x_0, y_0))$, $(x_2, y_2) = (g(x_1, y_1), f(x_1, y_1)), \ldots$. A solution of the system of difference equations which is constant for all $n \geq 0$ is called an equilibrium solution. If $(x_n, y_n) = (\bar{x}, \bar{y})$ for all $n \geq 0$ is an equilibrium solution of the system of difference equations, then $(\bar{x}, \bar{y})$ is called an equilibrium point, or simply an equilibrium of the system of difference equations. A solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of a system of difference equations is called an eventually equilibrium point if there exists an integer $N > 0$ such that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is the equilibrium point $(\bar{x}, \bar{y})$; that is, $(x_n, y_n) = (\bar{x}, \bar{y})$ for all $n \geq N$. A solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of a System (3) is called an eventually periodic with prime *period* p or an eventually prime period p solution (or p-cycle) if there exists an integer $N > 0$ and p is the smallest positive integer such that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is periodic with period p; that is, $(x_{n+p}, y_{n+p}) = (x_n, y_n)$ for all $n \geq N$. We denote $\{((a,b),(c,d),(e,f),(g,h))\}$ as 4-cycle which consists of 4 consecutive points: (a,b), (c,d), (e,f) and (g,h) in xy plane. It is worth noting that a solution is eventually periodic with period p (a solution is an eventually 4-cycle) when orbit (forward iterations) contains a point of the cycle (eventually 4-cycle).

Results and Discussions

We will examine the dynamics of System (3) when the initial condition is situated the positive y-axis, that is $l_1 \cup l_2$ where $l_1 = \{(0, y) \in \mathbf{R}^2 \mid y \geq 1\}$ and $l_2 = \{(0, y) \in \mathbf{R}^2 \mid 0 < y < 1\}$. We assert that System (3) possesses an equilibrium point $(-1,-1)$, determined by solving equations $\bar{x} = |\bar{x}| - \bar{y} - 3$

and $\overline{y} = \overline{x} - |\overline{y}| + 1$. We also found two prime period 4 solutions (or 4-cycles):

$P_{4.1} = \{((-5,-1),(3,-5),(5,-1),(3,5))\}$ and $P_{4.2} = \{((1,-1),(-1,1),(-3,-1),(1,-3))\}$

of System (3). We will begin our investigation by calculating the first iteration for $(x_0, y_0) \in l_1$:

$x_1 = |x_0| - y_0 - 3 = -y_0 - 3 < 0$ and $y_1 = x_0 - |y_0| + 1 = -y_0 + 1 \le 0$ (4)

$x_2 = 2y_0 - 1 > 0$ and $y_2 = -2y_0 - 1 < 0$, (5)

$x_3 = 4y_0 - 3 > 0$ and $y_3 = -1$, (6)

$x_4 = 4y_0 - 5$ and $y_4 = 4y_0 - 3 > 0$ (7)

If $(0, y_0)$ is in the region for which $y_0 \ge \frac{5}{4}$ then $x_4 \ge 0$ and so

$x_5 = -5$ and $y_5 = -1$

So, we have the following lemma.

Lemma 1 *Let* $(0, y_0) \in l_1$ *be an initial condition where* $y_0 \ge \frac{5}{4}$. *Then the fifth iteration of the solution of the system* (3) *is* $(-5,-1)$.

We will investigate the solution with the initial condition (x_0, y_0) belonging to the remain region of l_1 as the following lemma.

Lemma 2 *Let* $(0, y_0) \in l_1$ *be an initial condition where* $y_0 \in \left[1, \frac{5}{4}\right)$. *Then the solution of the system* (3) *is an eventually* $(-5,-1)$ or $(-1,1)$.

Proof. If we choose the initial condition with $y_0 = 1$ then $(x_1, y_1) = (-4, 0), (x_2, y_2) = (1, -3)$, $(x_3, y_3) = (1, -1)$ which is the member of $P_{4.2}$. Suppose that $y_0 \in \left(1, \frac{5}{4}\right)$. We have a closed form of the first four iterations in the form of equations (4) – (7) and $x_4 = 4y_0 - 5 < 0$. The following sequences are used in the inductive statement of this lemma: $a_n = \frac{2^{2n} + 1}{2^{2n}}$ and $\delta_n = 2^{2n} + 1$. Let $P(n)$ be the following statement:

“for $y_0 \in (1, a_n)$,

$x_{4n+1} = -2^{2n+1} y_0 + 2\delta_n - 5 < 0$ and $y_{4n+1} = -1,$

$x_{4n+2} = 2^{2n+1} y_0 - 2\delta_n + 3 > 0$ and $y_{4n+2} = -2^{2n+1} y_0 + 2\delta_n - 5 < 0,$

$x_{4n+3} = 2^{2n+2} y_0 - 4\delta_n + 5 > 0$ and $y_{4n+3} = -1,$

$x_{4n+4} = 2^{2n+2} y_0 - 4\delta_n + 3$ and $y_{4n+4} = 2^{2n+2} y_0 - 4\delta_n + 5 > 0.$

If $y_0 \in [a_{n+1}, a_n)$ then $x_{4n+4} \ge 0$ and so

$x_{4n+5} = -5$ and $y_{4n+5} = -1.$

If $y_0 \in (1, a_{n+1})$ then $x_{4n+4} < 0$ ”.

Firstly, we shall show that $P(1)$ is true. By letting $x_0 \in (1, a_1) = \left(1, \frac{5}{4}\right)$ we have,

$$x_{4(1)+1} = x_5 = |x_4| - y_4 - 3 = -8y_0 + 5 = -2^{2(1)+1} y_0 + 2\delta_1 - 5 < 0$$
$$y_{4(1)+1} = y_5 = x_4 - |y_4| + 1 = -1,$$
$$x_{4(1)+2} = x_6 = |x_5| - y_5 - 3 = 8y_0 - 7 = 2^{2(1)+1} y_0 - 2\delta_1 + 3 > 0$$
$$y_{4(1)+2} = y_6 = x_5 - |y_5| + 1 = -8y_0 + 5 = -2^{2(1)+1} y_0 + 2\delta_1 - 5 < 0,$$
$$x_{4(1)+3} = x_7 = |x_6| - y_6 - 3 = 16y_0 - 15 = 2^{2(1)+2} y_0 - 4\delta_1 + 5 > 0$$
$$y_{4(1)+3} = y_7 = x_6 - |y_6| + 1 = -1,$$
$$x_{4(1)+4} = x_8 = |x_7| - y_7 - 3 = 16y_0 - 17 = 2^{2(1)+2} y_0 - 4\delta_1 + 3$$
$$y_{4(1)+4} = y_8 = x_7 - |y_7| + 1 = 16y_0 - 15 = 2^{2(1)+2} y_0 - 4\delta_1 + 5 > 0.$$

If $y_0 \in [a_2, a_1) = \left[\frac{17}{16}, \frac{5}{4}\right)$ then $x_8 = 16y_0 - 17 \geq 0$ and so
$x_{4(1)+5} = x_9 = |x_8| - y_8 - 3 = -5$ and $y_{4(1)+5} = y_9 = x_8 - |y_8| + 1 = -1.$

If $y_0 \in (1, a_2) = \left(1, \frac{17}{16}\right)$ then $x_8 = 16y_0 - 17 < 0$.
Hence $P(1)$ is true. Now we suppose further that $P(k)$ is true for a positive integer k. We have
$x_{4k+4} = 2^{2k+2} y_0 - 4\delta_k + 3 < 0$ and $y_{4k+4} = 2^{2k+2} y_0 - 4\delta_k + 5 > 0$

for $y_0 \in (1, a_{k+1}) = \left(1, \frac{2^{2k+2}+1}{2^{2k+2}}\right)$. We determine the sign of x_{4k+4} by substituting 1 and a_{k+1} into y_0 of a linear function x_{4k+4} as follow:

$$x_{4k+4}(1) = 2^{2k+2}(1) - 4(2^{2k}+1) + 3 = 2^{2k+2} - 2^{2k+2} - 4 + 3 = -1 < 0,$$

$x_{4k+4}\left(\frac{2^{2k+2}+1}{2^{2k+2}}\right) = 2^{2k+2}\left(\frac{2^{2k+2}+1}{2^{2k+2}}\right) - 4(2^{2k}+1) + 3 = 2^{2k+2} + 1 - 2^{2k+2} - 4 + 3 = 0.$ Thus, we have $x_{4k+4} < 0$.

From now on, we will determine the sign of solutions by using this method. Then
$x_{4(k+1)+1} = x_{4k+5} = |x_{4k+4}| - y_{4k+4} - 3 = -2^{2(k+1)+1} y_0 + 8\delta_k - 11 = -2^{2(k+1)+1} y_0 + 2\delta_{k+1} - 5 < 0.$ We note that

$$8\delta_k - 11 = 8(2^{2k}+1) - 11 = 2^{2k+3} + 8 - 11 = 2^{2k+3} - 3 = 2\delta_{k+1} - 5.$$

$$y_{4(k+1)+1} = y_{4k+5} = x_{4k+4} - |y_{4k+4}| + 1 = 2^{2k+2} y_0 - 4\delta_k + 3 - 2^{2k+2} y_0 + 4\delta_k - 5 + 1 = -1,$$
$x_{4(k+1)+2} = x_{4k+6} = 2^{2k+3} y_0 - 2\delta_{k+1} + 3 > 0$ and $y_{4(k+1)+2} = y_{4k+6} = -2^{2k+3} y_0 + 2\delta_{k+1} - 5 < 0,$
$x_{4(k+1)+3} = x_{4k+7} = 2^{2k+4} y_0 - 4\delta_{k+1} + 5 > 0$ and $y_{4(k+1)+3} = y_{4k+7} = -1,$
$x_{4(k+1)+4} = x_{4k+8} = 2^{2k+4} y_0 - 4\delta_{k+1} + 3$ and $y_{4(k+1)+4} = y_{4k+8} = 2^{2k+4} y_0 - 4\delta_{k+1} + 5 > 0.$

If $y_0 \in [a_{k+2}, a_{k+1}) = \left[\frac{2^{2k+4}+1}{2^{2k+4}}, \frac{2^{2k+2}+1}{2^{2k+2}}\right)$ then $x_{4k+8} = 2^{2k+4} y_0 - 4\delta_{k+1} + 3 \geq 0$

and so

$x_{4(k+1)+5} = x_{4k+9} = -5$ and $y_{4(k+1)+5} = y_{4k+9} = -1.$

Thus $(x_{4(k+1)+5}, y_{4(k+1)+5}) = (-5,-1).$

If $y_0 \in (1, a_{k+2}) = \left(1, \dfrac{2^{2k+4}+1}{2^{2k+4}}\right)$ then $x_{4k+8} = 2^{2k+4} y_0 - 4\delta_{k+1} + 3 < 0.$ Hence $P(k+1)$ is true. By the mathematical induction we conclude that $P(n)$ is true for every positive integer n. We note that $\lim_{n\to\infty} a_n = 1$ and $(-5,-1)$ is a member of $P_{4.1}$. By the inductive statement $P(n)$, we can conclude that the solution is an eventually 4-cycle ($P_{4.1}$ or $P_{4.2}$).

By the above lemmas, we immediately have the following theorem.

Theorem 1 *Let* $(0, y_0) \in l_1$ *be an initial condition. Then the solution of the system* (3) *is eventually* 4-cycle $P_{4.1}$ or $P_{4.2}$.

Next, we will calculate the first iteration for $(x_0, y_0) \in l_2$:

$x_1 = |x_0| - y_0 - 3 = -y_0 - 3 < 0$ and $y_1 = x_0 - |y_0| + 1 = -y_0 + 1 > 0$ (8)

$x_2 = 2y_0 - 1$ and $y_2 = -3$. (9)

If $(0, y_0)$ is in the region for which $y_0 \le \frac{1}{2}$ then $x_2 \le 0$ and so

$x_3 = -2y_0 + 1 \le 0$ and $y_3 = 2y_0 - 3 < 0$,

$x_4 = -1$ and $y_4 = -1$.

So, we have the following lemma.

Lemma 3 *Let* $(0, y_0) \in l_2$ *be an initial condition where* $y_0 \le \frac{1}{2}$. *Then the fourth iteration of the solution of the system* (3) *is equilibrium point* $(-1,-1)$.

If $(0, y_0)$ is in the region for which $\frac{1}{2} < y_0 \le 1$ then $x_2 > 0$ and so

$x_3 = 2y_0 - 1 > 0$ and $y_3 = 2y_0 - 3 < 0$, (10)

$x_4 = -1$ and $y_4 = 4y_0 - 3$, (11)

If $(0, y_0)$ is in the region for which $\frac{1}{2} < y_0 \le \frac{3}{4}$ then $y_4 = 4y_0 - 3 \le 0$ and so

$x_5 = -4y_0 + 1 < 0$ and $y_5 = 4y_0 - 3 \le 0$,

$x_6 = -1$ and $y_6 = -1$. So, we have the following lemma.

Lemma 4 *Let* $(0, y_0) \in l_2$ *be an initial condition where* $\frac{1}{2} < y_0 \le \frac{3}{4}$. *Then the sixth iteration of the solution of the system* (3) *is* $(-1,-1)$.

If $(0, y_0)$ is in the region for which $\frac{3}{4} < y_0 \le 1$ then $y_4 = 4y_0 - 3 > 0$ and so

$x_5 = -4y_0 + 1 < 0$ and $y_5 = -4y_0 + 3 < 0$, (12)

$x_6 = 8y_0 - 7$ and $y_6 = -8y_0 + 5 < 0$. (13)

If $(0, y_0)$ is in the region for which $\frac{3}{4} < y_0 \le \frac{7}{8}$ then $x_6 = 8y_0 - 7 \le 0$ and so

$x_7 = -1$ and $y_7 = -1$. So, we have the following lemma.

Lemma 5 *Let* $(0, y_0) \in l_2$ *be an initial condition where* $\frac{3}{4} < y_0 \le \frac{7}{8}$. *Then the seventh iteration of the solution of the system* (3) *is* $(-1,-1)$.

We will investigate the solution with the initial condition (x_0, y_0) belonging to the remain region for which $\frac{7}{8} < y_0 < 1$ as the following lemma.

Lemma 6 *Let* $(0, y_0) \in l_2$ *be an initial condition where* $y_0 \in \left(\frac{7}{8}, 1\right)$. *Then the solution of the system* (3) *is an eventually equilibrium point* $(-1,-1)$.

Proof. We have a closed form of the first six iterations in the form of equations (8) – (13) but $x_6 = 8y_0 - 7 > 0$. The following sequences are used in the inductive statement of this lemma: $b_n = \frac{2^{2n+1} - 1}{2^{2n+1}}, c_n = \frac{2^{2n+2} - 1}{2^{2n+2}}$ and $\gamma_n = 2^{2n+2} - 1$. Let $Q(n)$ be the following statement:

"for $y_0 \in (b_n, 1)$,

$x_{4n+3} = 2^{2n+2} y_0 - \gamma_n$ and $y_{4n+3} = -1.$

If $y_0 \in (b_n, c_n]$ then $x_{4n+3} \le 0$ and so

$x_{4n+4} = -2^{2n+2} y_0 + \gamma_n - 2 < 0$ and $y_{4n+4} = 2^{2n+2} y_0 - \gamma_n \le 0$,

$x_{4n+5} = -1$ and $y_{4n+5} = -1.$

If $y_0 \in (c_n, 1)$ then $x_{4n+3} > 0$ and so

$x_{4n+4} = 2^{2n+2} y - \gamma_n - 2 < 0$ and $y_{4n+4} = 2^{2n+2} y - \gamma_n > 0$,

$x_{4n+5} = -2^{2n+3} y_0 + 2\gamma_n - 1 < 0$ and $y_{4n+5} = -1$

$x_{4n+6} = 2^{2n+3} y_0 - 2\gamma_n - 1$ and $y_{4n+6} = -2^{2n+3} y_0 + 2\gamma_n - 1 < 0$.

If $y_0 \in (c_n, b_{n+1}]$ then $x_{4n+6} \le 0$ and so

$x_{4n+7} = -1$ and $y_{4n+7} = -1.$

If $y_0 \in (b_{n+1}, 1)$ then $x_{4n+6} > 0$".

The inductive proof for statement $Q(n)$ mirrors the methodology applied in proving $P(n)$. Consequently, the detailed proof for $Q(n)$ being true is omitted for brevity. It is noteworthy that selecting an initial condition within l_2, specifically y_0 in the interval $\left(\frac{7}{8},1\right)$, aligns the solution's closed form with that of $Q(n)$. As n increases, the sequences b_n and c_n are observed to converge towards 1, indicating that the solution asymptotically approaches the equilibrium point $(-1,-1)$.

By the Lemma 3 – Lemma 6, we immediately have the following theorem.

Theorem 2. *Let* $(0, y_0) \in l_2$ *be an initial condition. Then the solution of the system* (3) *is eventually equilibrium point* $(-1,-1)$.

As indicated in the articles by Krisuk et al. (2022) and Tikjha & Piasu (2020), the solution is eventually an equilibrium point or a 4-cycle in certain regions of $\mathbf{R}^2$. Proving the global behavior of map (3) is challenging, as it requires us to set an initial condition in each region of $\mathbf{R}^2$ and verify that the solution converges to certain invariant sets. As we know, the invariant sets of the map (3) consist of an equilibrium point $(-1,-1)$ and two 4-cycles. We believe that our results will serve as a tool for studying this system with other initial conditions in the $\mathbf{R}^2$ region. We conjecture that the solution to system (3) will be confined to either the equilibrium point or 4-cycles.

Conclusions

We divide the positive y-axis into two segments, l_1 and l_2. For the first segment, l_1, when the initial condition belongs to l_1 and $y_0 \geq \frac{5}{4}$, the solution is $(-5,-1)$, which is a member of $P_{4.1}$ within 5 iterations. For y_0 in the interval $\left(1,\frac{5}{4}\right)$, the solution eventually becomes a 4-cycle ($P_{4.1}$) by utilizing the first inductive statement. Specifically, if $y_0 = 1$, then the solution transitions to a 4-cycle ($P_{4.2}$) within 3 iterations.

For the second segment, l_2, when the initial condition $y_0 \leq \frac{1}{2}$, the solution converges to the equilibrium point within 4 iterations. Selecting $\frac{1}{2} < y_0 \leq \frac{3}{4}$ results in the solution reaching the equilibrium point within 6 iterations. Furthermore, for $\frac{3}{4} < y_0 \leq \frac{7}{8}$, the solution becomes equilibrium within 7 iterations. By applying the second inductive statement, we deduce that for $\frac{7}{8} < y_0 < 1$, the solution eventually reaches an equilibrium point.

Acknowledgement

The authors are supported by the National Research Council of Thailand and Pibulsongkram Rajabhat University.

References

Grove, E. A., & Ladas, G. (2005). **Periodicities in Nonlinear Difference Equations**. New York: Chapman Hall.

Grove, E. A., Lapierre, E., & Tikjha, W. (2012). On the Global Behavior of $x_{n+1} = |x_n| - y_n - 1$ and $y_{n+1} = x_n + |y_n|$. **Cubo A Mathematical Journal**, **14**(2), 125–166.

Krisuk, S., Soprom, K., Pantain, J., & Tikjha, W. (2022). Equilibrium Solution on a Two-dimensional Piecewise Linear Map. **KKU Science Journal, 50**(3), 223-230.

Krinket, S., & Tikjha, W. (2015). **Prime period solution of cartain piecewise linear system of difference equation**. In P. Supavititpatana, (Ed.), Proceedings of the Pibulsongkram Research (76-83). Pibulsongkram Rajabhat University Press. (In Thai)

Lozi, R. (1978). Un attracteur étrange du type attracteur de Hénon. **Journal de Physique Colloques, 39**(C5), 9-10.

Simpson, D. J. W. (2010). **Bifurcations in piecewise-smooth continuous systems**. Singapore: World Scientific Publishing.

Simpson, D. J. W. (2014a). Sequences of Periodic Solutions and Infinitely Many Coexisting Attractors in the Border-Collision Normal Form. **International Journal of Bifurcation and Chaos, 24**(6), 1430018.

Simpson, D. J. W. (2014b). Scaling Laws for Large Numbers of Coexisting Attracting Periodic Solutions in the Border-Collision Normal Form. **International Journal of Bifurcation and Chaos, 24**(9), 1450118.

Tikjha, W., & Lapierre, E. (2020). Periodic solutions of a system of piecewise linear difference equations. **Kyungpook Mathematical Journal, 60**(2), 401-413.

Tikjha, W., Lapierre, E. G., & Sitthiwirattham, T. (2017). The stable equilibrium of a system of piecewise linear difference equations. **Advances in Difference Equations, 2017,** 67.

Tikjha, W., Lenbury, Y., & Lapierre, E. G. (2010). On the Global Character of the System of Piecewise Linear Difference Equations $x_{n+1} = |x_n| - y_n - 1$ and $y_{n+1} = x_n - |y_n|$. **Advances in Difference Equations, 2010**, 573281.

Tikjha, W., Lapierre, E. G., & Lenbury, Y. (2015). Periodic solutions of a generalized system of piecewise linear difference equations. **Advances in Difference Equations, 2015,** 248.

Tikjha, W., & Piasu, K. (2020). A necessary condition for eventually equilibrium or periodic to a system of difference equations. **Journal of Computational Analysis and Applications, 28**(2), 254-261.

Zhusubaliyev, Z. T., Mosekilde, E., & Banerjee, S. (2008). Multiple-attractor bifurcations and quasiperiodicity in piecewise-smooth maps. **International Journal of Bifurcation and Chaos, 18**(6), 1775-1789.