บทความวิจัย

ON QUASI-PRIME Γ-IDEALS IN ORDERED Γ-AG-GROUPOIDS

Pairote Yiarayong* and Supanit Jarernsuk

Department of Mathematics, Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanuloke 65000, Thailand

*Email: pairote0027@hotmail.com

ABSTRACT

The aim of this short note is to introduce the concepts of quasi prime and prime Γ-ideals in ordered Γ-AG-groupoids with left identity. These concepts are related to the concepts of prime and quasi-prime Γ-ideals, and play an important role in studying the structure of some ordered Γ-AG-groupoids, so it seems to be interesting to study them.

Keywords: ordered Γ-AG-groupoid, left (right) Γ-ideal, quasi-prime, prime

INTRODUCTION

Let S be a non empty set. If there exists a mapping $S \times S \to S$ written as (a,b) by ab then S is called an Abel-Grassmann's groupoid (AG-groupoid) if $ab \in S$ such that the following left invertive law holds for all $a,b,c \in S$;

$$(a\gamma b)\alpha c \;=\; (c\gamma b)\alpha a.$$

Naseeruddin (1970) introduced the notion of Abel-Grassmann's groupoids. The fundamentals of this non associative algebraic structure were first discovered by Kazim and Naseeruddin (1977). An AG-groupoid is a non-associative algebraic structure mid way between a groupoid and a commutative semigroup. This structure is closely related to a commutative semigroup because if an AG-groupoid contains a right identity, then it becomes a commutative monoid (Mushtaq & Yousuf, 1978). It is a mid structure between a groupoid and a commutative semigroup with wide range of applications in the theory of flocks (Naseeruddin, 1970). The Ideals in AG-groupoids have been discussed in (Mushtaq & Yousuf, 1978) and (Mushtaq & Yousuf, 1988). In 1981 the notion of Γ-semigroups was introduced by Sen (Sen & Saha, 1986). Let S and Γ be any non empty sets. If there exists a mapping $S \times \Gamma \times S \to S$ written as (a,γ,b) by $a\gamma b$ then S is called an Γ-Abel-Grassmann's groupoid (Γ-AG-groupoid) (Khan et al., 2010) if $a\gamma b \in S$ such that the following Γ-left invertive law holds for all $a,b,c \in S$ and $\gamma, \alpha \in \Gamma$;

$$(a\gamma b)\alpha c \;=\; (c\gamma b)\alpha a.$$

This structure is also known as left almost semigroup (LA-semigroup). The concept of an ordered AG-groupoid was first given by Faisal, Naveed Yaqoob and Kostaq Hila in (Faisal & Hila, 2012) which is in fact a generalization of an ordered semigroup. In this paper we characterize the ordered Γ-AG-groupoid. We study prime and quasi-prime Γ-ideals in ordered Γ-AG-groupoids with left identity.

METHODS

In this section we refer to (Faisal & Hila, 2012; Yiarayong, 2014) for some elementary aspects and quote few definitions and examples which are essential to this study. For more details we refer to the papers in the references.

Definition 2.1 (Faisal & Hila, 2012) Let S be a nonempty set, $\cdot$ a binary operation on S and ≤ a relation on S. $(S,\Gamma,\leq)$ is called an ordered Γ-AG-groupoid if $(S,\cdot)$ is a Γ-AG-groupoid, $(S,\leq)$ is a partially ordered set and for all $a,\ b,\ c\in S,\ a\leq b$ implies that $a\gamma c\leq b\gamma c$ and $c\gamma a\leq c\gamma b,$ where $\gamma\in\Gamma.$

Definition 2.2. (Yiarayong, 2014) A nonempty subset A of an ordered Γ-AG-groupoid S is called a Γ-AG-subgroupoid of S if $A\Gamma A\subseteq A.$

Let $(S,\Gamma,\leq)$ be an ordered Γ-AG-groupoid. For $A\subseteq S,$, let $(A]=\{x\in S: x\leq a$ for some $a\in A\}.$ The following lemma are similar to the case of ordered Γ-AG-groupoids.

Lemma 2.3 (Yiarayong, 2014) Let $(S,\Gamma,\leq)$ be an ordered Γ-AG-groupoid with left identity and A,B subsets of $S.$ The following statements hold:

(1) If $A\subseteq B,$ then $(A]\subseteq(B].$

(2) $(A]\Gamma(B]\subseteq(A\Gamma B].$

(3) $((A]\Gamma(B]]\subseteq(A\Gamma B].$

(4) $A\subseteq(A].$

(5) $((A]]=(A].$

Proof. We leave the straightforward proof to the reader.

Definition 2.4 (Faisal & Hila, 2012) A nonempty subset A of an ordered Γ-AG-groupoid S is called a left Γ-ideal of S if $(A]\subseteq A$ and $S\Gamma A\subseteq A$ and is called a right Γ-ideal of S if $(A]\subseteq A$ and $A\Gamma S\subseteq A.$ A nonempty subset A of S is called a Γ-ideal of S if A is both left and right Γ-ideal of $S.$

Lemma 2.5 (Yiarayong, 2014) Let $(S,\Gamma,\leq)$ be an ordered Γ-AG-groupoid with left identity. Then every right Γ-ideal of S is a left Γ-ideal of $S.$

Proof. We leave the straightforward proof to the reader.

Lemma 2.6 (Yiarayong, 2014) Let $(S,\Gamma,\leq)$ be an ordered Γ-AG-groupoid with left identity and $A\subseteq S.$ Then $S\Gamma(S\Gamma A) = S\Gamma A$ and $S\Gamma(S\Gamma A]\subseteq(S\Gamma A].$

Proof. We leave the straightforward proof to the reader.

Lemma 2.7 (Yiarayong, 2014) If $(S,\Gamma,\leq)$ is an ordered Γ-AG-groupoid with left identity and let $a\in S,$ then $(S\Gamma a]$ a left Γ-ideal of $S.$

Proof. We leave the straightforward proof to the reader.

Lemma 2.8. (Yiarayong, 2014) Let $(S,\Gamma,\leq)$ be an ordered Γ-AG-groupoid with left identity and $a \in S$. Then $< a >= \left(S\Gamma a\right]$.

Proof. We leave the straightforward proof to the reader.

Lemma 2.9. (Yiarayong, 2014) If $(S,\Gamma,\leq)$ is an ordered Γ-AG-groupoid with left identity and let $a \in S$, then $\left(S\Gamma a^2\right]$ a Γ-ideal of S.

Proof. We leave the straightforward proof to the reader.

RESULTS

Let $\{(\mathrm{S}_i,\Gamma_i,\leq_i): i \in I\}$ be a nonempty family of ordered Γ_i-AG-groupoids. We consider the cartesian product $\prod_{i\in I} S_i$. Define a mapping

$$\prod_{i\in I} S_i \times \prod_{i\in I} \Gamma_i \times \prod_{i\in I} S_i \to \prod_{i\in I} S_i$$

written as $((x_i)_{i\in I},(\gamma_i)_{i\in I},(y_i)_{i\in I}) \mapsto (x_i)_{i\in I}(\gamma_i)_{i\in I}(y_i)_{i\in I}$,

by $(x_i)_{i\in I}(\gamma_i)_{i\in I}(y_i)_{i\in I} = (x_i\gamma_i y_i)_{i\in I}$.

Then $\prod_{i\in I} S_i$ is a $\prod_{i\in I}\Gamma_i$-AG-groupoid. Moreover, $\prod_{i\in I} S_i$ is an ordered $\prod_{i\in I}\Gamma_i$-AG-groupoid with the relation $\leq$ defined by $(x_i)_{i\in I} \leq (y_i)_{i\in I} \Leftrightarrow x_i \leq_i y_i$, for all $i \in I$. We consider the cartesian product of $\prod_{i\in I}\Gamma_i$-ideals.

Lemma 3.1 Let $\{(\mathrm{S}_i,\Gamma_i,\leq_i): i \in I\}$ be a nonempty family of ordered Γ_i-AG-groupoids. If A_i is a Γ_i-ideal of S_i for each $i \in I$, then the set $\prod_{i\in I} A_i$ is a $\prod_{i\in I}\Gamma_i$-ideal of $\prod_{i\in I} S_i$.

Proof. Since $A_i \neq \varnothing$, for all $i \in I$, we have $\prod_{i\in I} A_i \neq \varnothing$. By Lemma 2.3, we have

$$\left(\prod_{i\in I} A_i\right] = \prod_{i\in I}(A_i] = \prod_{i\in I} A_i.$$

Then
$$\begin{aligned}\prod_{i\in I} A_i \prod_{i\in I}\Gamma_i \prod_{i\in I} S_i &= \prod_{i\in I} A_i\Gamma_i S_i \\ &= \prod_{i\in I} A_i\end{aligned}$$

and
$$\prod_{i\in I} S_i \prod_{i\in I}\Gamma_i \prod_{i\in I} A_i = \prod_{i\in I} S_i\Gamma_i A_i = \prod_{i\in I} A_i.$$

Therefore $\prod_{i\in I} A_i$ is a $\prod_{i\in I}\Gamma_i$ -ideal of $\prod_{i\in I} S_i$.

Proposition 3.2. If $(\mathrm{S}_i,\Gamma_i,\leq_i)$ is an ordered Γ_i-AG-groupoid with left identity and $(a_i)_{i\in I} \in \prod_{i\in I} S_i$, then $\left((a_i)_{i\in I} \cup \prod_{i\in I} S_i\Gamma_i a_i\right]$ is a left $\prod_{i\in I}\Gamma_i$ -ideal of $\prod_{i\in I} S_i$.

Proof. Assume that $(\mathrm{S}_i,\Gamma_i,\leq_i)$ is an ordered Γ_i-AG-groupoid with left identity. By Lemma 2.3, we have $\left((a_i)_{i\in I} \cup \prod_{i\in I} S_i\Gamma_i a_i\right] = \left(\left((a_i)_{i\in I} \cup \prod_{i\in I} S_i\Gamma_i a_i\right]\right]$. Then

$$\begin{aligned}
\prod_{i\in I} S_i \prod_{i\in I}\Gamma_i \left((a_i)_{i\in I} \cup \prod_{i\in I} S_i\Gamma_i a_i\right] &\subseteq \left(\prod_{i\in I} S_i \prod_{i\in I}\Gamma_i \left((a_i)_{i\in I} \cup \prod_{i\in I} S_i\Gamma_i a_i\right)\right] \\
&= \left(\prod_{i\in I} S_i \prod_{i\in I}\Gamma_i (a_i)_{i\in I} \cup \prod_{i\in I} S_i \prod_{i\in I}\Gamma_i \left(\prod_{i\in I} S_i\Gamma_i a_i\right)\right] \\
&= \left(\prod_{i\in I} S_i\Gamma_i a_i \cup \prod_{i\in I} S_i\Gamma_i (S_i\Gamma_i a_i)\right] \\
&= \left(\prod_{i\in I} S_i\Gamma_i a_i \cup \prod_{i\in I} (S_i\Gamma_i S_i)\Gamma_i (S_i\Gamma_i a_i)\right] \\
&= \left(\prod_{i\in I} S_i\Gamma_i a_i \cup \prod_{i\in I} (a_i\Gamma_i S_i)\Gamma_i (S_i\Gamma_i S_i)\right] \\
&= \left(\prod_{i\in I} S_i\Gamma_i a_i \cup \prod_{i\in I} (a_i\Gamma_i S_i)\Gamma_i S_i\right] \\
&= \left(\prod_{i\in I} S_i\Gamma_i a_i \cup \prod_{i\in I} (S_i\Gamma_i S_i)\Gamma_i a_i\right] \\
&= \left(\prod_{i\in I} S_i\Gamma_i a_i \cup \prod_{i\in I} S_i\Gamma_i a_i\right] \\
&= \left(\prod_{i\in I} S_i\Gamma_i a_i\right] \\
&\subseteq \left((a_i)_{i\in I} \cup \prod_{i\in I} S_i\Gamma_i a_i\right].
\end{aligned}$$

Therefore $\left((a_i)_{i\in I} \cup \prod_{i\in I} S_i\Gamma_i a_i \right]$ is a left $\prod_{i\in I}\Gamma_i$ -ideal of $\prod_{i\in I} S_i$.

Proposition 3.3. If $(S_i,\Gamma_i,\leq_i)$ is an ordered Γ_i-AG-groupoid with left identity and $(a_i)_{i\in I} \in \prod_{i\in I} S_i$, then $\left(\left(\prod_{i\in I} a_i\Gamma_i S_i \right) \cup \left(\prod_{i\in I} S_i\Gamma_i a_i \right) \right]$ is a $\prod_{i\in I}\Gamma_i$ -ideal of $\prod_{i\in I} S_i$.

Proof. Assume that $(S_i,\Gamma_i,\leq_i)$ is an ordered Γ_i-AG-groupoid with left identity. By Lemma 2.3, we have $\left(\left(\prod_{i\in I} a_i\Gamma_i S_i \right) \cup \left(\prod_{i\in I} S_i\Gamma_i a_i \right) \right] = \left(\left(\left(\prod_{i\in I} a_i\Gamma_i S_i \right) \cup \left(\prod_{i\in I} S_i\Gamma_i a_i \right) \right] \right]$. Then

$$
\begin{aligned}
\left(\left(\prod_{i\in I} a_i\Gamma_i S_i \right) \cup \left(\prod_{i\in I} S_i\Gamma_i a_i \right) \right] \prod_{i\in I}\Gamma_i \prod_{i\in I} S_i
&\subseteq \left(\begin{array}{l} \left(\prod_{i\in I} a_i\Gamma_i S_i \right) \prod_{i\in I}\Gamma_i \prod_{i\in I} S_i \cup \\ \left(\prod_{i\in I} S_i\Gamma_i a_i \right) \prod_{i\in I}\Gamma_i \prod_{i\in I} S_i \end{array} \right] \\
&= \left(\begin{array}{l} \left(\prod_{i\in I} (a_i\Gamma_i S_i)\Gamma_i S_i \right) \cup \\ \left(\prod_{i\in I} (S_i\Gamma_i a_i)\Gamma_i S_i \right) \end{array} \right] \\
&= \left(\begin{array}{l} \left(\prod_{i\in I} (S_i\Gamma_i S_i)\Gamma_i a_i \right) \cup \\ \left(\prod_{i\in I} (S_i\Gamma_i a_i)\Gamma_i (S_i\Gamma_i S_i) \right) \end{array} \right] \\
&= \left(\left(\prod_{i\in I} S_i\Gamma_i a_i \right) \cup \left(\prod_{i\in I} (S_i\Gamma_i S_i)\Gamma_i (a_i\Gamma_i S_i) \right) \right] \\
&= \left(\left(\prod_{i\in I} S_i\Gamma_i a_i \right) \cup \left(\prod_{i\in I} S_i\Gamma_i (a_i\Gamma_i S_i) \right) \right] \\
&= \left(\left(\prod_{i\in I} S_i\Gamma_i a_i \right) \cup \left(\prod_{i\in I} a_i\Gamma_i (S_i\Gamma_i S_i) \right) \right] \\
&= \left(\left(\prod_{i\in I} S_i\Gamma_i a_i \right) \cup \left(\prod_{i\in I} a_i\Gamma_i S_i \right) \right] \\
&= \left(\left(\prod_{i\in I} a_i\Gamma_i S_i \right) \cup \left(\prod_{i\in I} S_i\Gamma_i a_i \right) \right].
\end{aligned}
$$

Therefore $\left(\left(\prod_{i\in I} a_i\Gamma_i S_i\right)\cup\left(\prod_{i\in I} S_i\Gamma_i a_i\right)\right]$ is a right $\prod_{i\in I}\Gamma_i$ -ideal of $\prod_{i\in I} S_i$. By Lemma 2.5, we have $\left(\left(\prod_{i\in I} a_i\Gamma_i S_i\right)\cup\left(\prod_{i\in I} S_i\Gamma_i a_i\right)\right]$ is a $\prod_{i\in I}\Gamma_i$ -ideal of $\prod_{i\in I} S_i$.

Lemma 3.4. If $(\mathrm{S}_\mathrm{i},\Gamma_i,\leq_i)$ is an ordered Γ_i-AG-groupoid with left identity and $(a_i)_{i\in I}\in\prod_{i\in I} S_i$, then $\left((a_i)_{i\in I}\cup\left(\prod_{i\in I} a_i\Gamma_i S_i\right)\cup\left(\prod_{i\in I} S_i\Gamma_i a_i\right)\right]$ is a $\prod_{i\in I}\Gamma_i$ -ideal of $\prod_{i\in I} S_i$.

Proof. Assume that $(\mathrm{S}_\mathrm{i},\Gamma_i,\leq_i)$ is an ordered Γ_i-AG-groupoid with left identity. By Lemma 2.3, we have

$$\left((a_i)_{i\in I}\cup\left(\prod_{i\in I} a_i\Gamma_i S_i\right)\cup\left(\prod_{i\in I} S_i\Gamma_i a_i\right)\right]=\left(\left((a_i)_{i\in I}\cup\left(\prod_{i\in I} a_i\Gamma_i S_i\right)\cup\left(\prod_{i\in I} S_i\Gamma_i a_i\right)\right]\right].$$

Then

$$\begin{aligned}
\left(\begin{matrix}(a_i)_{i\in I}\cup\\ \left(\prod_{i\in I} a_i\Gamma_i S_i\right)\cup\\ \left(\prod_{i\in I} S_i\Gamma_i a_i\right)\end{matrix}\right]\prod_{i\in I}\Gamma_i\prod_{i\in I} S_i &\subseteq \left(\begin{matrix}\left((a_i)_{i\in I}\prod_{i\in I}\Gamma_i\prod_{i\in I} S_i\right)\cup\left(\prod_{i\in I} a_i\Gamma_i S_i\right)\prod_{i\in I}\Gamma_i\prod_{i\in I} S_i\cup\\ \left(\prod_{i\in I} S_i\Gamma_i a_i\right)\prod_{i\in I}\Gamma_i\prod_{i\in I} S_i\end{matrix}\right]\\
&= \left(\begin{matrix}\left(\prod_{i\in I} a_i\Gamma_i S_i\right)\cup\left(\prod_{i\in I}(a_i\Gamma_i S_i)\Gamma_i S_i\right)\cup\\ \left(\prod_{i\in I}(S_i\Gamma_i a_i)\Gamma_i S_i\right)\end{matrix}\right]\\
&= \left(\begin{matrix}\left(\prod_{i\in I} a_i\Gamma_i S_i\right)\cup\left(\prod_{i\in I}(S_i\Gamma_i S_i)\Gamma_i a_i\right)\cup\\ \left(\prod_{i\in I}(S_i\Gamma_i a_i)\Gamma_i(S_i\Gamma_i S_i)\right)\end{matrix}\right]\\
&= \left(\begin{matrix}\left(\prod_{i\in I} a_i\Gamma_i S_i\right)\cup\left(\prod_{i\in I} S_i\Gamma_i a_i\right)\cup\\ \left(\prod_{i\in I}(S_i\Gamma_i S_i)\Gamma_i(a_i\Gamma_i S_i)\right)\end{matrix}\right]
\end{aligned}$$

$$= \left(\left(\left(\prod_{i\in I} a_i\Gamma_i S_i\right)\cup\left(\prod_{i\in I} S_i\Gamma_i a_i\right)\cup\left(\prod_{i\in I} S_i\Gamma_i\left(a_i\Gamma_i S_i\right)\right)\right]$$

$$= \left(\left(\left(\prod_{i\in I} a_i\Gamma_i S_i\right)\cup\left(\prod_{i\in I} S_i\Gamma_i a_i\right)\cup\left(\prod_{i\in I} a_i\Gamma_i\left(S_i\Gamma_i S_i\right)\right)\right]$$

$$= \left(\left(\left(\prod_{i\in I} a_i\Gamma_i S_i\right)\cup\left(\prod_{i\in I} S_i\Gamma_i a_i\right)\cup\left(\prod_{i\in I} a_i\Gamma_i S_i\right)\right]$$

$$= \left(\left(\left(\prod_{i\in I} a_i\Gamma_i S_i\right)\cup\left(\prod_{i\in I} S_i\Gamma_i a_i\right)\right]$$

$$\subseteq \left(\left(a_i\right)_{i\in I}\cup\left(\prod_{i\in I} a_i\Gamma_i S_i\right)\cup\left(\prod_{i\in I} S_i\Gamma_i a_i\right)\right].$$

Therefore $\left(\left(a_i\right)_{i\in I}\cup\left(\prod_{i\in I} a_i\Gamma_i S_i\right)\cup\left(\prod_{i\in I} S_i\Gamma_i a_i\right)\right]$ is a right $\prod_{i\in I}\Gamma_i$ -ideal of $\prod_{i\in I} S_i$. By Proposition 3.3, we have $\left(\left(a_i\right)_{i\in I}\cup\left(\prod_{i\in I} a_i\Gamma_i S_i\right)\cup\left(\prod_{i\in I} S_i\Gamma_i a_i\right)\right]$ is a $\prod_{i\in I}\Gamma_i$ -ideal of $\prod_{i\in I} S_i$.

CONCLUSIONS AND DISCUSSIONS

We consider the product of prime $\prod_{i\in I}\Gamma_i$ -ideals in ordered $\prod_{i\in I}\Gamma_i$ -AG-groupoids.

Definition 4.1. Let $\prod_{i\in I} S_i$ be an ordered $\prod_{i\in I}\Gamma_i$ -AG-groupoid. A $\prod_{i\in I}\Gamma_i$ -ideal $\prod_{i\in I} P_i$ of $\prod_{i\in I} S_i$ is called prime if for every $\prod_{i\in I}\Gamma_i$ -ideals $\prod_{i\in I} A_i, \prod_{i\in I} B_i$ of $\prod_{i\in I} S_i$ such that $\prod_{i\in I} A_i \prod_{i\in I}\Gamma_i \prod_{i\in I} B_i \subseteq \prod_{i\in I} P_i$, we have $\prod_{i\in I} A_i \subseteq \prod_{i\in I} P_i$, or $\prod_{i\in I} B_i \subseteq \prod_{i\in I} P_i$.

Definition 4.2. Let $\prod_{i\in I} S_i$ be an ordered $\prod_{i\in I}\Gamma_i$ -AG-groupoid. A left $\prod_{i\in I}\Gamma_i$ -ideal $\prod_{i\in I} P_i$ of $\prod_{i\in I} S_i$ is called quasi-prime if for every left $\prod_{i\in I}\Gamma_i$ -ideals $\prod_{i\in I} A_i, \prod_{i\in I} B_i$ of $\prod_{i\in I} S_i$ such that $\prod_{i\in I} A_i \prod_{i\in I}\Gamma_i \prod_{i\in I} B_i \subseteq \prod_{i\in I} P_i$, we have $\prod_{i\in I} A_i \subseteq \prod_{i\in I} P_i$, or $\prod_{i\in I} B_i \subseteq \prod_{i\in I} P_i$.

Remark It is easy to see that every quasi-$\prod_{i\in I}\Gamma_i$ -prime ideal is $\prod_{i\in I}\Gamma_i$ -prime.

Theorem 4.3. Let $(\mathrm{S_i},\Gamma_i,\leq_i)$ be an ordered Γ_i-AG-groupoid with left identity and let P_i be a Γ_i-ideal of Si. Then a $\prod_{i\in I}\Gamma_i$ - ideal $\prod_{i\in I}P_i$ of $\prod_{i\in I}S_i$ is quasi-prime if and only if $\prod_{i\in I}\left(a_i\Gamma_i\left(S_i\Gamma_i b_i\right)\right)\subseteq\prod_{i\in I}P_i$ implies that $\left(a_i\right)_{i\in I}\in\prod_{i\in I}P_i$ or $\left(b_i\right)_{i\in I}\in\prod_{i\in I}P_i,$ where $\left(a_i\right)_{i\in I},\left(b_i\right)_{i\in I}\in\prod_{i\in I}S_i.$

Proof. $\Rightarrow$ Assume that $(\mathrm{S_i},\Gamma_i,\leq_i)$ is an ordered Γ_i-AG-groupoid with left identity. Then

$$
\begin{aligned}
\left(a_i\right)_{i\in I}\left(\gamma_i\right)_{i\in I}\left(b_i\right)_{i\in I}\in\left(\prod_{i\in I}S_i\Gamma_i a_i\right)\left(\prod_{i\in I}\Gamma_i\right)\left(\prod_{i\in I}S_i\Gamma_i b_i\right)= &\left(\prod_{i\in I}\left(S_i\Gamma_i a_i\right)\Gamma_i\left(S_i\Gamma_i b_i\right)\right)\\
= &\left(\prod_{i\in I}\left(\left(S_i\Gamma_i b_i\right)\Gamma_i a_i\right)\Gamma_i S_i\right)\\
= &\left(\prod_{i\in I}\left(\left(S_i\Gamma_i b_i\right)\Gamma_i a_i\right)\Gamma_i\left(S_i\Gamma_i S_i\right)\right)\\
= &\left(\prod_{i\in I}\left(S_i\Gamma_i S_i\right)\Gamma_i\left(a_i\Gamma_i\left(S_i\Gamma_i b_i\right)\right)\right)\\
= &\left(\prod_{i\in I}S_i\Gamma_i\left(a_i\Gamma_i\left(S_i\Gamma_i b_i\right)\right)\right)\\
= &\left(\prod_{i\in I}S_i\right)\left(\prod_{i\in I}\Gamma_i\right)\left(\prod_{i\in I}\left(a_i\Gamma_i\left(S_i\Gamma_i b_i\right)\right)\right)\\
\subseteq &\left(\prod_{i\in I}S_i\right)\left(\prod_{i\in I}\Gamma_i\right)\left(\prod_{i\in I}P_i\right)\\
\subseteq &\prod_{i\in I}P_i.
\end{aligned}
$$

By Lemma 2.8, we have $\left(a_i\right)_{i\in I}\in\prod_{i\in I}P_i$ or $\left(b_i\right)_{i\in I}\in\prod_{i\in I}P_i.$

$\Leftarrow$ We leave the straightforward proof to the reader.

Lemma 4.4 Let $(\mathrm{S_i},\Gamma_i,\leq_i)$ be an ordered Γ_i-AG-groupoid with left identity. Then a $\prod_{i\in I}\Gamma_i$ -ideal $\prod_{i\in I}P_i$ of $\prod_{i\in I}P_i$ is quasi-prime if and only if $\left(a_i\right)_{i\in I}\left(\gamma_i\right)_{i\in I}\left(b_i\right)_{i\in I}\in\prod_{i\in I}P_i$ implies that $\left(a_i\right)_{i\in I}\in\prod_{i\in I}P_i$ or $\left(b_i\right)_{i\in I}\in\prod_{i\in I}P_i,$ where $\left(a_i\right)_{i\in I},\left(b_i\right)_{i\in I}\in\prod_{i\in I}S_i.$

Proof. $\Rightarrow$ Assume that $(S_i,\Gamma_i,\leq_i)$ is an ordered Γ_i-AG-groupoid with left identity. Then by hypothesis, $(a_i\gamma_i b_i)_{i\in I}=(a_i)_{i\in I}(\gamma_i)_{i\in I}(b_i)_{i\in I}\in\prod_{i\in I}P_i$, for any $(a_i)_{i\in I}$, $(b_i)_{i\in I}\in\prod_{i\in I}S_i$ and $(\gamma_i)_{i\in I}\in\prod_{i\in I}\Gamma_i$. Then

$$
\begin{aligned}
&\left((a_i)_{i\in I}\cup\prod_{i\in I}S_i\Gamma_i a_i\right]\prod_{i\in I}\Gamma_i\left((b_i)_{i\in I}\cup\prod_{i\in I}S_i\Gamma_i b_i\right] \subseteq \left(\left((a_i)_{i\in I}\cup\prod_{i\in I}S_i\Gamma_i a_i\right)\prod_{i\in I}\Gamma_i\left((b_i)_{i\in I}\cup\prod_{i\in I}S_i\Gamma_i b_i\right)\right]\\
&= \left(\left((a_i)_{i\in I}\prod_{i\in I}\Gamma_i\left((b_i)_{i\in I}\cup\prod_{i\in I}S_i\Gamma_i b_i\right)\right)\cup\left(\left(\prod_{i\in I}S_i\Gamma_i a_i\right)\prod_{i\in I}\Gamma_i\left((b_i)_{i\in I}\cup\prod_{i\in I}S_i\Gamma_i b_i\right)\right)\right]\\
&= \left((a_i)_{i\in I}\prod_{i\in I}\Gamma_i(b_i)_{i\in I}\cup (a_i)_{i\in I}\prod_{i\in I}\Gamma_i\left(\prod_{i\in I}S_i\Gamma_i b_i\right)\cup\left(\prod_{i\in I}S_i\Gamma_i a_i\right)\prod_{i\in I}\Gamma_i(b_i)_{i\in I}\cup\left(\prod_{i\in I}S_i\Gamma_i a_i\right)\prod_{i\in I}\Gamma_i\left(\prod_{i\in I}S_i\Gamma_i b_i\right)\right]\\
&= \left((a_i\gamma_i b_i)_{i\in I}\cup\prod_{i\in I}a_i\Gamma_i(S_i\Gamma_i b_i)\cup\prod_{i\in I}(S_i\Gamma_i a_i)\Gamma_i b_i\cup\prod_{i\in I}(S_i\Gamma_i a_i)\Gamma_i(S_i\Gamma_i b_i)\right]\\
&\subseteq \left(\prod_{i\in I}P_i\cup\prod_{i\in I}S_i\Gamma_i(a_i\Gamma_i b_i)\cup\prod_{i\in I}(b_i\Gamma_i a_i)\Gamma_i S_i\cup\prod_{i\in I}S_i\Gamma_i((S_i\Gamma_i a_i)\Gamma_i b_i)\right]\\
&= \left(\prod_{i\in I}P_i\cup\prod_{i\in I}S_i\prod_{i\in I}\Gamma_i(a_i\gamma_i b_i)_{i\in I}\cup\prod_{i\in I}(b_i\Gamma_i a_i)\Gamma_i(S_i\Gamma_i S_i)\cup\prod_{i\in I}(b_i\Gamma_i a_i)\Gamma_i(S_i\Gamma_i S_i)\right]\\
&= \left(\prod_{i\in I}P_i\cup\prod_{i\in I}S_i\prod_{i\in I}\Gamma_i\prod_{i\in I}P_i\cup\prod_{i\in I}(S_i\Gamma_i S_i)\Gamma_i(a_i\Gamma_i b_i)\cup\prod_{i\in I}(S_i\Gamma_i S_i)\Gamma_i(a_i\Gamma_i b_i)\right]\\
&\subseteq \left(\prod_{i\in I}P_i\cup\prod_{i\in I}P_i\cup\prod_{i\in I}S_i\Gamma_i(a_i\Gamma_i b_i)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\prod_{i\in I} P_i \cup \prod_{i\in I} S_i \prod_{i\in I} \Gamma_i (a_i\gamma_i b_i)_{i\in I}\right] \\
&\subseteq \left(\prod_{i\in I} P_i \cup \prod_{i\in I} S_i \prod_{i\in I} \Gamma_i \prod_{i\in I} P_i\right] \\
&\subseteq \left(\prod_{i\in I} P_i \cup \prod_{i\in I} P_i\right] \\
&= \left(\prod_{i\in I} P_i\right] \\
&= \prod_{i\in I} P_i.
\end{aligned}
$$

By hypothesis, $\left((a_i)_{i\in I} \cup \prod_{i\in I} S_i\Gamma_i a_i\right] \subseteq \prod_{i\in I} P_i$ or $\left((b_i)_{i\in I} \cup \prod_{i\in I} S_i\Gamma_i b_i\right] \subseteq \prod_{i\in I} P_i$ and so that $(a_i)_{i\in I} \in \prod_{i\in I} P_i$ or $(b_i)_{i\in I} \in \prod_{i\in I} P_i$.

$\Leftarrow$ We leave the straightforward proof to the reader.

Theorem 4.5. Let $(\mathrm{S}_\mathrm{i}, \Gamma_i, \leq_i)$ be an ordered Γ_i-AG-groupoid with left identity. Then a $\prod_{i\in I}\Gamma_i$-ideal $\prod_{i\in I} P_i$ of $\prod_{i\in I} S_i$ is prime if and only if $(a_i^2)_{i\in I}(\gamma_i)_{i\in I}(b_i^2)_{i\in I} \in \prod_{i\in I} P_i$ implies that $(a_i^2)_{i\in I} \in \prod_{i\in I} P_i$ or $(b_i^2)_{i\in I} \in \prod_{i\in I} P_i$, where $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I} S_i$.

Proof. We leave the straightforward proof to the reader.

Theorem 4.6. Let $(\mathrm{S}_\mathrm{i}, \Gamma_i, \leq_i)$ be an ordered Γ_i-AG-groupoid with left identity and let $\prod_{i\in I} P_i$ be a $\prod_{i\in I}\Gamma_i$-ideal of $\prod_{i\in I} S_i$. Then $\prod_{i\in I}\Gamma_i$-ideal $\prod_{i\in I} P_i$ of $\prod_{i\in I} S_i$. is prime if and only if $\prod_{i\in I}\left(a_i^2\Gamma_i\left(S_i\Gamma_i b_i^2\right)\right] \subseteq \prod_{i\in I} P_i$ implies that $(a_i^2)_{i\in I} \in \prod_{i\in I} P_i$ or $(b_i^2)_{i\in I} \in \prod_{i\in I} P_i$, where $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I} S_i$.

Proof. We leave the straightforward proof to the reader.

Definition 4.7. An ordered $\prod_{i\in I}\Gamma_i$-AG-groupoid $\prod_{i\in I} S_i$. is called ordered $\prod_{i\in I}\Gamma_i$-AG-3-band if its every element satisfies

$$(a_i)_{i\in I}(\gamma_i)_{i\in I}\left((a_i)_{i\in I}(\alpha_i)_{i\in I}(a_i)_{i\in I}\right) = \left((a_i)_{i\in I}(\gamma_i)_{i\in I}(a_i)_{i\in I}\right)(\alpha_i)_{i\in I}(a_i)_{i\in I} = (a_i)_{i\in I}.$$

Proposition 4.8. Every left identity in an ordered $\prod_{i\in I}\Gamma_i$ -AG-3-band is a right identity.

Proof. Let $(e_i)_{i\in I}$ be a left identity and let $(a_i)_{i\in I}$ be any element in an ordered $\prod_{i\in I}\Gamma_i$ -AG-3-band S. Then

$$\begin{aligned}(a_i)_{i\in I}(\gamma_i)_{i\in I}(e_i)_{i\in I} &= \left((a_i)_{i\in I}(\beta_i)_{i\in I}\left((a_i)_{i\in I}(\alpha_i)_{i\in I}(a_i)_{i\in I}\right)\right)(\gamma_i)_{i\in I}(e_i)_{i\in I}\\ &= \left((e_i)_{i\in I}(\beta_i)_{i\in I}\left((a_i)_{i\in I}(\alpha_i)_{i\in I}(a_i)_{i\in I}\right)\right)(\gamma_i)_{i\in I}(a_i)_{i\in I}\\ &= \left((a_i)_{i\in I}(\alpha_i)_{i\in I}(a_i)_{i\in I}\right)(\gamma_i)_{i\in I}(a_i)_{i\in I}\\ &= (a_i)_{i\in I}\end{aligned}$$

for all $(a_i)_{i\in I}\in\prod_{i\in I}S_i$ and $(\gamma_i)_{i\in I},(\alpha_i)_{i\in I},(\beta_i)_{i\in I}\in\prod_{i\in I}\Gamma_i$. Hence $(e_i)_{i\in I}$ is right identity.

Lemma 4.9. If an ordered $\prod_{i\in I}\Gamma_i$ -AG-band S has a left identity, then every left $\prod_{i\in I}\Gamma_i$ -ideal is a $\prod_{i\in I}\Gamma_i$ -ideal.

Proof. Let $\prod_{i\in I}S_i$ be an ordered $\prod_{i\in I}\Gamma_i$ -AG-3-band, and let $\prod_{i\in I}A_i$ be a left $\prod_{i\in I}\Gamma_i$ -ideal. Then

$$\begin{aligned}(a_i)_{i\in I}(\gamma_i)_{i\in I}(s_i)_{i\in I} &= \left(\left((a_i)_{i\in I}(\alpha_i)_{i\in I}(a_i)_{i\in I}\right)(\beta_i)_{i\in I}(a_i)_{i\in I}\right)(\gamma_i)_{i\in I}(s_i)_{i\in I}\\ &= \left((s_i)_{i\in I}(\beta_i)_{i\in I}(a_i)_{i\in I}\right)(\gamma_i)_{i\in I}\left((a_i)_{i\in I}(\alpha_i)_{i\in I}(a_i)_{i\in I}\right)\\ &\in \left(\prod_{i\in I}S_i\prod_{i\in I}\Gamma_i\prod_{i\in I}A_i\right)\prod_{i\in I}\Gamma_i\left(\prod_{i\in I}A_i\prod_{i\in I}\Gamma_i\prod_{i\in I}A_i\right)\\ &= \prod_{i\in I}A_i\prod_{i\in I}\Gamma_i\prod_{i\in I}A_i\\ &\subseteq \prod_{i\in I}A_i\end{aligned}$$

for all $(\gamma_i)_{i\in I},(\alpha_i)_{i\in I},(\beta_i)_{i\in I}\in\prod_{i\in\Gamma}\Gamma_i$, $(a_i)_{i\in I}\in\prod_{i\in I}A_i$ and $(s_i)_{i\in I}\in\prod_{i\in I}S_i$. Hence $\prod_{i\in I}A_i$ is a $\prod_{i\in I}\Gamma_i$ -ideal.

Theorem 4.10. Let $\prod_{i\in I}S_i$ be an ordered $\prod_{i\in I}\Gamma_i$ -AG-3-band with left identity. Then $\prod_{i\in I}P_i$ is a quasi-prime $\prod_{i\in I}\Gamma_i$ -ideal in $\prod_{i\in I}S_i$ if and only if $\prod_{i\in I}P_i$ is a prime $\prod_{i\in I}\Gamma_i$ -ideal in $\prod_{i\in I}S_i$.

Proof. We leave the straightforward proof to the reader.

ACKNOWLEDGEMENT

The authors are very grateful to the anonymous referee for stimulating comments and improving presentation of the paper.

REFERENCES

Faisal, N. & Hila, K. (2012). On fuzzy (2,2)-regular ordered Γ-AG**-groupoids. **Series A Applied Mathematics and Physics.** 74(2), 87-104.

Kazim, M.A. & Naseeruddin, M. (1977). On almost semigroups. **Portugaliae mathematica.** 2, 1-7.

Mushtaq, Q. & Yousuf, S.M. (1978). On LA-semigroups. **The Aligarh Bulletin of Mathematics.** 8, 65-70.

Naseeruddin, M. (1970). **Some studies in almost semigroups and flocks.** Ph.D. thesis: Aligarh Muslim University: Aligarh: India.

Sen, M.K. & Saha, N.K. (1986). On Γ-semigroups I. **Bulletin of Calcutta Mathematical Society.** 78, 180-186.

Yiarayong, P. (2014). On m-systems and weak m-systems in ordered Γ-AG-groupoids. **Proceedings of the Jangjeon mathematical society.** 17(2), 237-246.