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SOME IDENTITIES OF (P,Q) - FIBONACCI NUMBERS BY MATRIX METHODS

Alongkot Suvarnamani

Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi (RMUTT), Pathumthani, 12110

E-mail: kotmaster2@rmutt.ac.th

ABSTRACT

In this paper, we consider the generalized Fibonacci sequence which is (p,q) - Fibonacci sequence. We used the matrix methods to show some properties of (p,q) - Fibonacci number. We get some generalized identities of (p,q) - Fibonacci number.

Keywords: (p,q) - Fibonacci sequence, (p,q) - Fibonacci number, matrix methods

INTRODUCTION

The Fibonacci numbers cover a wide range of interest in modern mathematics as they appear in the comprehensive works of Koshy (2001) and Vajda (1989). The Fibonacci numbers F_n are the terms of the sequence where each term is the sum of the two previous terms beginning with the initial values $F_0 = 0,\ F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1.$

Falcon & Plaza (2007) introduced the k - Fibonacci sequence $\{F_{k,n}\}$ which is defined as $F_{k,0} = 0,\ F_{k,1} = 1$ and $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$ for $n \geq 1,\ k \geq 1.$ If $k = 1$, we get the classical Fibonacci sequence $\{0,1,1,2,3,5,8,13,\ldots\}$. If $k = 2$, we get the Pell sequence $\{0,1,2,5,12,29,70,\ldots\}$.

The well-known Binet's formulas for k - Fibonacci numbers (Falcon & Plaza, 2007) are given by $F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}$ where $r_1 = \frac{k + \sqrt{k^2+4}}{2}$ and $r_2 = \frac{k - \sqrt{k^2+4}}{2}$ are roots of the characteristic equation $r^2 - kr - 1 = 0$.

Falco & Plaza (2007) studied the k - Fibonacci sequence and the Pascal 2 - triangle. Next, they considered the 3 - dimensional k - Fibonacci spiral in [3]. Next, Suvarnamani & Tatong (2015) showed some properties of (p,q) - Fibonacci numbers by using the Binet's formula. Then Suvarnamani (2016) proved some properties of (p,q) - Lucas numbers by using the Binet's formula. And Suvarnamani (2016) study on the odd and even terms of (p,q) - Fibonacci number and (p,q) - Lucas number by using the Binet's formulas.

In this paper, we find some properties of the (p,q) - Fibonacci numbers by using matrix methods.

The $(p,q)-$ Fibonacci Number

The $(p,q)-$ Fibonacci sequence $\{F_{p,q,n}\}$ is defined as $F_{p,q,0}=0, F_{p,q,1}=1$ and $F_{p,q,n}=pF_{p,q,n-1}+qF_{p,q,n-2}$ for $p\geq 1,\ q\geq 1$ and $n\geq 2.$ The Binet's formula for $(p,q)-$ Fibonacci numbers are given by $F_{p,q,n}=\frac{r_1^n-r_2^n}{r_1-r_2}$ where $r_1=\frac{p+\sqrt{p^2+4q}}{2}$ and $r_2=\frac{p-\sqrt{p^2+4q}}{2}$ are roots of the characteristic equation $r^2-pr-q=0.$ We note that $r_1+r_2=p,\quad r_1r_2=-q$ and $r_1-r_2=\sqrt{p^2+4q}.$

RESULTS

In this section, we establish some identities for $(p,q)-$ Fibonacci numbers by using matrix methods.

Lemma 3.1. Let p and q be positive integers. If $A=\begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}$ then $A^n=\begin{bmatrix} F_{p,q,n+1} & qF_{p,q,n} \\ F_{p,q,n} & qF_{p,q,n-1} \end{bmatrix}$ where n is a positive integer.

Proof. We will prove this theorem by mathematical induction.

Let P(n) : $A^n=\begin{bmatrix} F_{p,q,n+1} & qF_{p,q,n} \\ F_{p,q,n} & qF_{p,q,n-1} \end{bmatrix}$

For $n=1$, we get

$$
\begin{aligned}
A^1=A&=\begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}\\
&=\begin{bmatrix} p & q(1) \\ 1 & q(0) \end{bmatrix}\\
&=\begin{bmatrix} F_{p,q,2} & qF_{p,q,1} \\ F_{p,q,1} & qF_{p,q,0} \end{bmatrix}\\
&=\begin{bmatrix} F_{p,q,1+1} & qF_{p,q,1} \\ F_{p,q,1} & qF_{p,q,1-1} \end{bmatrix}.
\end{aligned}
$$

So, P(1) is true.

Next, we will show that if P(k) is true then P(k+1) is true.

Suppose that P(k) is true, i.e.,

$$A^k=\begin{bmatrix} F_{p,q,k+1} & qF_{p,q,k} \\ F_{p,q,k} & qF_{p,q,k-1} \end{bmatrix}.$$

Then

$$
\begin{aligned}
A^{k+1}&=A^kA\\
&=\begin{bmatrix} F_{p,q,k+1} & qF_{p,q,k} \\ F_{p,q,k} & qF_{p,q,k-1} \end{bmatrix}\begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}\\
&=\begin{bmatrix} pF_{p,q,k+1}+qF_{p,q,k} & qF_{p,q,k+1} \\ pF_{p,q,k}+qF_{p,q,k-1} & qF_{p,q,k} \end{bmatrix}\\
&=\begin{bmatrix} F_{p,q,k+2} & qF_{p,q,k+1} \\ F_{p,q,k+1} & qF_{p,q,k} \end{bmatrix}
\end{aligned}
$$

$$=\begin{bmatrix} F_{p,q,(k+1)+1} & qF_{p,q,k+1} \\ F_{p,q,k+1} & qF_{p,q,(k+1)-1} \end{bmatrix}.$$

So, P(k+1) is true.

Hence $A^n=\begin{bmatrix} F_{p,q,n+1} & qF_{p,q,n} \\ F_{p,q,n} & qF_{p,q,n-1} \end{bmatrix}$ where n is a positive integer.

Next, let us define the matrix A as in the following Lemma and by using this matrix, we obtain some identities for (p,q)– Fibonacci numbers.

Theorem 3.2. Let p, q and n be positive integers. Then $F_{p,q,n+1}F_{p,q,n-1}-F^2_{p,q,n}=(-1)^n q^{n-1}$.

Proof. Let p, q and n be positive integers.

We have $|A|=\begin{vmatrix} p & q \\ 1 & 0 \end{vmatrix}=-q$. Then $|A^n|=|A|^n=(-q)^n$.

Moreover, we have

$$|A^n|=\begin{vmatrix} F_{p,q,n+1} & qF_{p,q,n} \\ F_{p,q,n} & qF_{p,q,n-1} \end{vmatrix}$$

$$=F_{p,q,n+1}\left(qF_{p,q,n-1}\right)-qF_{p,q,n}\left(F_{p,q,n}\right)$$

$$=q\left(F_{p,q,n+1}F_{p,q,n-1}-\left(F_{p,q,n}\right)^2\right)$$

So, $q\left(F_{p,q,n+1}F_{p,q,n-1}-\left(F_{p,q,n}\right)^2\right)=(-q)^n$

$$F_{p,q,n+1}F_{p,q,n-1}-\left(F_{p,q,n}\right)^2=(-1)^n q^{n-1}.$$

□

Remark 3.3. From Theorem 3.2, if $p=1$ and $q=1$ then the Cassini's identity is obtained, i.e., $F_{1,1,n+1}F_{1,1,n-1}-F^2_{1,1,n}=(-1)^n$. It is similarly as $F_{n+1}F_{n-1}-F^2_n=(-1)^n$.

Theorem 3.4. Let p, q and n be positive integers. Then $F_{p,q,n-2}F_{p,q,n+1}-F_{p,q,n-1}F_{p,q,n}=(-1)^{n-1}pq^{n-2}$.

Proof. Let p, q and n be positive integers.

We have $|A|=\begin{vmatrix} p & q \\ 1 & 0 \end{vmatrix}=-q$. Then $|A^{n-1}|=|A|^{n-1}=(-q)^{n-1}$.

$$F_{p,q,n-2}F_{p,q,n+1}-F_{p,q,n-1}F_{p,q,n}=F_{p,q,n-2}\left(pF_{p,q,n}+qF_{p,q,n-1}\right)-F_{p,q,n-1}\left(pF_{p,q,n-1}+qF_{p,q,n-2}\right)$$

$$=pF_{p,q,n-2}F_{p,q,n}+qF_{p,q,n-2}F_{p,q,n-1}-pF_{p,q,n-1}F_{p,q,n-1}-qF_{p,q,n-1}F_{p,q,n-2}$$

$$=pF_{p,q,n-2}F_{p,q,n}-pF_{p,q,n-1}F_{p,q,n-1}$$

$$=p\left(F_{p,q,n-2}F_{p,q,n}-F_{p,q,n-1}F_{p,q,n-1}\right)$$

$$=\frac{p}{q}\left(qF_{p,q,n-2}F_{p,q,n}-qF_{p,q,n-1}F_{p,q,n-1}\right)$$

$$=\frac{p}{q}\begin{vmatrix} F_{p,q,n} & qF_{p,q,n-1} \\ F_{p,q,n-1} & qF_{p,q,n-2} \end{vmatrix}$$

$$=\frac{p}{q}\left|A^{n-1}\right|$$

$$=\frac{p}{q}(-q)^{n-1}$$

$$=(-1)^{n-1}pq^{n-2}.$$

□

Remark 3.5. From Theorem 3.4, if $p=1$ and $q=1$ then we get $F_{1,1,n-2}F_{1,1,n+1}-F_{1,1,n-1}F_{1,1,n}=(-1)^{n-1}$. It is similarly as $F_{n-2}F_{n+1}-F_{n-1}F_n=(-1)^{n-1}$.

Theorem 3.6. Let p, q, m and n be positive integers. Then $F_{p,q,m+n}=F_{p,q,m}F_{p,q,n+1}+qF_{p,q,m-1}F_{p,q,n}$.

Proof. Let p, q and n be positive integers. We have

$$A^{m+n}=\begin{bmatrix} F_{p,q,m+n+1} & qF_{p,q,m+n} \\ F_{p,q,m+n} & qF_{p,q,m+n-1} \end{bmatrix}$$

and

$$\begin{aligned} A^{m+n} &= A^m A^n \\ &= \begin{bmatrix} F_{p,q,m+1} & qF_{p,q,m} \\ F_{p,q,m} & qF_{p,q,m-1} \end{bmatrix}\begin{bmatrix} F_{p,q,n+1} & qF_{p,q,n} \\ F_{p,q,n} & qF_{p,q,n-1} \end{bmatrix} \\ &= \begin{bmatrix} F_{p,q,m+1}F_{p,q,n+1}+qF_{p,q,m}F_{p,q,n} & qF_{p,q,m+1}F_{p,q,n}+q^2F_{p,q,m}F_{p,q,n-1} \\ F_{p,q,m}F_{p,q,n+1}+qF_{p,q,m-1}F_{p,q,n} & qF_{p,q,m}F_{p,q,n}+q^2F_{p,q,m-1}F_{p,q,n-1} \end{bmatrix} \end{aligned}$$

So, we get $F_{p,q,m+n}=F_{p,q,m}F_{p,q,n+1}+qF_{p,q,m-1}F_{p,q,n}$.

□

Remark 3.7. From Theorem 3.6, if $p=1$ and $q=1$ then the shifting property is obtained, i.e., $F_{1,1,m+n}=F_{1,1,m}F_{1,1,n+1}+F_{1,1,m-1}F_{1,1,n}$. It is similarly as $F_{m+n}=F_mF_{n+1}+F_{m-1}F_n$.

APPLICATIONS

In this section we give the solutions of some Diophantine equations by applying Theorem 3.2 and Theorem 3.4.

Theorem 4.1. If p, q and n are positive integers then $(x,y,z)=(F_{p,q,n+1},F_{p,q,n-1},F_{p,q,n})$ and $(x,y,z)=(F_{p,q,n-1},F_{p,q,n+1},F_{p,q,n})$ are solutions of Diophantine equation $xy-z^2=(-1)^n q^{n-1}$.

Proof. The result follows immediately from Theorem 3.2.

□

Theorem 4.2. Let p, q and n are positive integers. If $(w,x,y,z)\in A\cup B$ then (w,x,y,z) is the solution of Diophantine equation $wx-yz=(-1)^{n-1}pq^{n-2}$ where $A=\{(F_{p,q,n-2},F_{p,q,n+1},F_{p,q,n-1},F_{p,q,n}),(F_{p,q,n-2},F_{p,q,n+1},F_{p,q,n},F_{p,q,n-1})\}$ and $B=\{(F_{p,q,n+1},F_{p,q,n-2},F_{p,q,n-1},F_{p,q,n}),(F_{p,q,n+1},F_{p,q,n-2},F_{p,q,n},F_{p,q,n-1})\}$

Proof. The result follows immediately from Theorem 3.4.

□

CONCLUSIONS

In this paper, some identities for (p,q) Fibonacci numbers are established by using matrix methods and the solutions of some Diophantine equations are presented by applying these identities.

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