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Irrationality of some Series with Rational Terms

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ABSTRACT

An irrationality criterion due to Diananda and Oppenheim states that a Cantor series of rational terms is rational except possibly when the rational terms are of certain special shapes. In the first part, this excepted case is analyzed and conclusions are drawn for two specific classes of series. Badea in 1993 established very strong irrationality tests for series of positive rational terms and applied them to settle some previous open problems. Later Brown, Pei and Shiue extended Badea's applications to those series whose terms satisfy linear recurrence relations. Extensions of these results are derived in the second part.

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INTRODUCTION

It is known, see e.g. Theorem 1.6 p. 7 of Niven (1967) that any real number x is expressible as a Cantor series, i.e. in the form

$$x = a_0 + \sum_{i=1}^{\infty} \frac{a_i}{b_1 b_2 \ldots b_i} = a_0 + \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \frac{a_3}{b_1 b_2 b_3} + \ldots,$$

where $b_i > 1$ are integers, a_i are integers satisfying $0 \le a_i \le b_i - 1$ for all $i \ge 1$, and $a_i < b_i - 1$ for infinitely many i. Diananda and Oppenheim (1955) proved the following result giving a criterion for irrationality of certain numbers written as Cantor series: If every limit of $c_i = a_i/b_i$ is a rational number h/k where $0 < h < k$, $(h,k) = 1$, then x is irrational except possibly when $a_i = [hb_i/k]$ for all large i in the subsequence for which $c_i \to h/k$. In the excepted case x may be rational or irrational. Diananda and Oppenheim gave two examples illustrating that in the excepted case both possibilities do exist. Our first objective is to determine explicitly some excepted cases embracing these examples.

Badea (1993) proved some very strong theorems yielding irrationality criteria for series of the form $\Sigma\, b_n/a_n$, where a_n and b_n are positive integers and applied them to settle certain irrationality assertions of Erdos and Graham regarding series of the forms $\sum 1/F_{2^n+1}$ and $\sum 1/L_{2^n}$ where F_n and L_n are Fibonacci and Lucas numbers, respectively, as well as those of Sierpinski regarding series of the form $\Sigma(-1)^{n+1}/a_n$. Later Brown *et al* (1995) employed one of Badea's criteria to establish sufficient irrationality conditions for series of the form $\Sigma 1/H(f(n))$, where (H(k)) is a sequence of integers, positive from some point on, satisfying a homogeneous linear recurrence relation with integer coefficients and f is a strictly increasing function from the set of positive integers to the set of nonnegative integers. Our second objective is to extend a number of these results in the direction of irrationality criteria and their applications.

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MATERIALS AND METHODS

I. Cantor series

Let the notation pertaining Cantor series, save that the elements a_n are also allowed to be negative, be as expounded in the introduction. The following two lemmas were proved in Oppenheim (1954).

Lemma 1. Given $b_i \geq 2$, $0 \leq a_i \leq b_i - 1$. Then $x = a_0 + \sum_{i=1}^{\infty} \frac{a_i}{b_1 \ldots b_i}$ is irrational if $a_i > 0$ infinitely often and if there is a subsequence (i_n) such that $c_{i_n} = a_{i_n}/b_{i_n} \to 0$ and $b_{i_n} \to \infty$ $(n \to \infty)$.

Lemma 2. Given $b_i \geq 2$, $|a_i| \leq b_i - 1$ and $a_m a_n < 0$ for some $m > i$, $n > i$ when i is any assigned integer. Then $x = a_0 + \sum_{i=1}^{\infty} \frac{a_i}{b_1 \ldots b_i}$ is irrational if there exists a subsequence (i_n) such that one of the following two conditions holds: 1) $c_{i_n} = a_{i_n}/b_{i_n} \to 0$ and $b_{i_n} \to \infty$, or 2) $c_{i_n} = a_{i_n}/b_{i_n} \to 1$ or -1 $(n \to \infty)$.

II. Badea's criteria

Let (a_n) and (b_n) $(n \geq 1)$ be two sequences of positive integers and let $N = (n(k);\ k \geq 1)$ be an increasing sequence of positive integers. Define $d(k) := n(k+1) - n(k)$, $S_k(N) := a_{n(k)+1} \ldots a_{n(k+1)}$,

$$R_k(N) := \sum_{j=1}^{d(k)} \frac{b_{n(k)+j}}{a_{n(k)+j}} S_k(N).$$

The next two lemmas were proved in Badea (1993).

Lemma 3. Assume that $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is rational and that

$$S_{k+1}(N) \geq \frac{R_{k+1}(N)}{R_k(N)} S_k(N)\{S_k(N) - 1\} + 1.$$

Then $S_{k+1}(N) = \frac{R_{k+1}(N)}{R_k(N)} S_k(N)\{S_k(N) - 1\} + 1$ when k is sufficiently large.

Lemma 4. Assume that $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is rational and that

$$a_{n+1} \geq \frac{b_{n+1}}{b_n} a_n^2 - \frac{b_{n+1}}{b_n} a_n + 1.$$

Then $a_{n+1} = \frac{b_{n+1}}{b_n} a_n^2 - \frac{b_{n+1}}{b_n} a_n + 1$ when n is sufficiently large.

Finally, we list here some more notation to be used in the last part.

$$F(k) = \frac{b_{n(k)+1}}{a_{n(k)+1}} + \frac{b_{n(k)+2}}{a_{n(k)+2}} + \ldots + \frac{b_{n(k+1)}}{a_{n(k+1)}},$$

$$F_o(2,k) = \frac{b_{2n(k)+1}}{a_{2n(k)+1}} + \frac{b_{2n(k)+3}}{a_{2n(k)+3}} + \ldots + \frac{b_{2n(k+1)-1}}{a_{2n(k+1)-1}},$$

$$F_e(2,k) = \frac{b_{2n(k)+2}}{a_{2n(k)+2}} + \frac{b_{2n(k)+4}}{a_{2n(k)+4}} + \ldots + \frac{b_{2n(k+1)}}{a_{2n(k+1)}},$$

$$\Pi(k) = a_{n(k)+1} \ldots a_{n(k+1)},\ \Pi(2,k) = a_{2n(k)+1} \ldots a_{2n(k+1)}.$$

RESULTS

Theorem 1. Let x be a real number whose Cantor series is $\sum_{i=1}^{\infty} \frac{a_i}{b_1 b_2 \ldots b_i}$, $b_i \geq 2$, $0 \leq a_i \leq b_i - 1$. Assume that every limit of (a_i/b_i) is a rational number. Let h/k where $0 < h < k$, $(h,k) = 1$ be one such limit corresponding to a subsequence (j), called (h/k)-subsequence. Let $ka_i/h - b_i + 1 = \xi_i$ $(i \geq 1)$. Assume further that $b_j \to \infty$ $(j \to \infty)$ in the (h/k)-subsequence.

A) If i) $|\xi_i| \leq b_i - 1$ $(i \geq 1)$,

ii) $\xi_m \xi_n < 0$ for some $m > i$, $n > i$, when i is any assigned positive integer

and iii) $\xi_j/b_j \to 0$ as $j \to \infty$ in the (h/k)-subsequence, then x is irrational.

B) Suppose that ξ_i is constant, equal to a rational ξ for all sufficiently large i. Then x is rational if and only if $\xi = 0$.

Proof. Write $x = \sum_{i=1}^{\infty} \frac{a_i}{b_1 b_2 \ldots b_i} = \frac{h}{k} \sum_{i=1}^{\infty} \frac{b_i - 1 + \xi_i}{b_1 b_2 \ldots b_i}$

$$= \frac{h}{k}\left(\left(\frac{b_1 - 1}{b_1} + \frac{b_2 - 1}{b_1 b_2} + \frac{b_3 - 1}{b_1 b_2 b_3} + \ldots\right) + \sum_{i=1}^{\infty} \frac{\xi_i}{b_1 b_2 \ldots b_i}\right) = \frac{h}{k}\left(1 + \sum_{i=1}^{\infty} \frac{\xi_i}{b_1 b_2 \ldots b_i}\right).$$

To prove A), note that by Lemma 2, $\sum_{i=1}^{\infty}\frac{\xi_i}{b_1b_2\ldots b_i}$ is irrational and so is x.

To prove B), since there is an index i_0 such that $\xi_i = \xi$ $(i \geq i_0)$, then

$$x = \frac{h}{k}\left(\alpha + \xi\sum_{i=i_0}^{\infty}\frac{1}{b_1b_2\ldots b_i}\right),$$

for some rational α. Now $\sum_{i=i}^{\infty}\frac{1}{b_1b_2\ldots b_i}$ is irrational by Lemma 1 and this immediately yields the conclusion.

Theorem 2. Let the notation be as set out in Part II of the last section. If $1 > \frac{F(k+1)}{F(k)}\{\prod(k) - 1\} + \frac{1}{\prod(k+1)}$ for sufficiently large k, then $\sum_{n=1}^{\infty}\frac{b_n}{a_n}$ is irrational.

Proof. Note first that

$$R_k(N) = \sum_{j=1}^{d(k)} S_k(N)\frac{b_{n(k)+j}}{a_{n(k)+j}} = \prod(k)F(k),$$

and $S_k(N) = a_{n(k)+1}\ldots a_{n(k+1)} = \prod(k)$.

From $1 > \frac{F(k+1)}{F(k)}\{\prod(k)-1\}+\frac{1}{\prod(k+1)}$, we get

$\prod(k+1) > \frac{F(k+1)}{F(k)}\prod(k+1)\{\prod(k)-1\}+1$,

i.e. $S_{k+1}(N) > \frac{R_{k+1}(N)}{R_k(N)}S_k(N)\{S_k(N) - 1\} +1$ for sufficiently large k. By Lemma 3, $\sum_{n=1}^{\infty}\frac{b_n}{a_n}$ is irrational.

Theorem 3. If $\lim\limits_{k\to\infty}\frac{F(k+1)}{F(k)}\prod(k) < 1$ and $\lim\limits_{k\to\infty}\prod(k) = \infty$, then $\sum_{n=1}^{\infty}\frac{b_n}{a_n}$ is irrational.

Proof. Let $\lim\limits_{k\to\infty}\frac{F(k+1)}{F(k)}\prod(k) = \alpha < 1, \varepsilon = \frac{1-\alpha}{2} > 0$. Then for sufficiently large k, we have

$$0 < \frac{F(k+1)}{F(k)}\prod(k) - \frac{F(k+1)}{F(k)} < \varepsilon+\alpha - \frac{\alpha-\varepsilon}{\prod(k)} \leq \alpha + 2\varepsilon = 1.$$

Since $\lim\limits_{k\to\infty}\prod(k) = \infty$, then $0 < \frac{F(k+1)}{F(k)}\prod(k) - \frac{F(k+1)}{F(k)} + \frac{1}{\prod(k+1)} < 1$ for sufficiently large k, and Theorem 2 yields the desired result.

Theorem 4. Let $(n(k) ; k \geq 1)$ and (x_n) be increasing sequences of positive integers.

Assume that 1) $n(k+1) \geq 2n(k) - 1$ for all k sufficiently large, and 2) $x_n \sim A\beta^n$ where $\beta > 1$, $A > 0$ and $A\beta \leq 1$.

Then $\sum_{k=1}^{\infty}\frac{1}{x_{n(k)}}$ is irrational.

Proof. From $\frac{x_{2n-1}}{x_n^2} \sim \frac{1}{A\beta} \geq 1$, we get $x_{2n-1} \geq x_n^2$, and so $x_{n(k+1)} \geq x_{2n(k)-1} \geq x_{n(k)}^2$ when k is sufficiently large. Since $x_{n(k)} > 1$, then $x_{n(k+1)} > x_{n(k)}^2 - x_{n(k)} + 1$ when k is sufficiently large. Considering $x_{n(k)}$ as a_k when k is sufficiently large, the result now follows from Lemma 4.

Theorem 5. Let $(n(k) ; k \geq 1)$ and (x_n) be increasing sequences of positive integers. Let $y_n = x_{n+r} +\ldots+ x_{n+3} + x_{n+1} + x_{n-1} + x_{n-3} +\ldots+ x_{n-r}$, $r = 2m + 1$ for some nonnegative integer m and $n \geq r$. Assume that

1) $n(k+1) \geq 2n(k)$ for all k sufficiently large
2) $x_n \sim A\beta^n$ where $\beta > 1$, $A > 0$, and
3) $A\beta^r\left\{1+\frac{1}{\beta^2}+\ldots+\frac{1}{\beta^{2r}}\right\} \leq 1$.

Then $\sum_{k=1}^{\infty}\frac{1}{y_{n(k)}}$ is irrational.

Proof. We have $y_{2n} = x_{2n+r} +\ldots+ x_{2n+3} + x_{2n+1} + x_{2n-1} + x_{2n-3}+\ldots+ x_{2n-r}$
$\sim A\beta^{2n+r} +\ldots+ A\beta^{2n+1} + A\beta^{2n-1} +\ldots+ A\beta^{2n-r} = A\beta^{2n+r}\left(1+\frac{1}{\beta^2}+\ldots+\frac{1}{\beta^{2r}}\right)$,

and similarly, $y_n \sim A\beta^{n+r}\left(1+\frac{1}{\beta^2}+\ldots+\frac{1}{\beta^{2r}}\right)$. Thus

$$\frac{y_{2n}}{y_n^2} \sim \frac{1}{A\beta^r\left(1+\frac{1}{\beta^2}+\ldots+\frac{1}{\beta^{2r}}\right)} \geq 1,$$

and so $y_{2n} \geq y_n^2$ for n sufficiently large. Since $n(k+1) \geq 2n(k)$, then

$y_{n(k+1)} \geq y_{2n(k)} \geq y_{n(k)}^2$ for k sufficiently large.

As $y_{n(k)} > 1$, we get $y_{n(k+1)} > y_{n(k)}^2 - y_{n(k)} + 1$. The result again follows from Lemma 4.

Theorem 6. Let $(b_n/a_n\ ;\ n \geq 1)$ be a decreasing sequence of positive rationals satisfying

$$1 > \frac{F_o(2,k+1)-F_e(2,k+1)}{F_o(2,k)-F_e(2,k)}\{\Pi(2,k)-1\} + \frac{1}{\Pi(2,k+1)}. \qquad (*)$$

If $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{b_n}{a_n}$ is rational, then (*) becomes equality when k is sufficiently large.

Proof. Put $\frac{B_n}{A_n} = \frac{b_{2n-1}}{a_{2n-1}} - \frac{b_{2n}}{a_{2n}}$, with $A_n := a_{2n}a_{2n-1}$. Now

$$F_o(2,k) - F_e(2,k) = \left(\frac{b_{2n(k)+1}}{a_{2n(k)+1}} - \frac{b_{2n(k)+2}}{a_{2n(k)+2}}\right) + \ldots + \left(\frac{b_{2n(k+1)-1}}{a_{2n(k+1)-1}} - \frac{b_{2n(k+1)}}{a_{2n(k+1)}}\right) = \frac{B_{n(k)+1}}{A_{n(k)+1}} + \ldots + \frac{B_{n(k+1)}}{A_{n(k+1)}},$$

and $\Pi(2,k) = a_{2n(k)+1}\ldots a_{2n(k+1)} = A_{n(k)+1}\ldots A_{n(k+1)}$. From the hypotheses, we deduce that

$$1 > \frac{\frac{B_{n(k+1)+1}}{A_{n(k+1)+1}} + \ldots + \frac{B_{n(k+2)}}{A_{n(k+2)}}}{\frac{B_{n(k)+1}}{A_{n(k)+1}} + \ldots + \frac{B_{n(k+1)}}{A_{n(k+1)}}}\{A_{n(k)+1}\ldots A_{n(k+1)} - 1\} + \frac{1}{A_{n(k+1)+1}\ldots A_{n(k+2)}}.$$

Since (b_n/a_n) is a decreasing sequence of positive rationals, then (B_n/A_n) is a sequence of positive rationals and by Theorem 2, $\sum_{n=1}^{\infty}\frac{B_n}{A_n} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{b_n}{a_n}$ is irrational.

Theorem 7. Let f be a function from the positive integers to the nonnegative integers, G and H be functions from the nonnegative integers to the positive integers. Assume that

1) f is strictly increasing, and
2) there are real numbers $A > 0$, $c \geq 0$ and $\beta > 1$ such that $\lim_{k\to\infty}\frac{H(k)}{G(k)k^c\beta^k} = A$, and

$$\lim_{k\to\infty}\frac{G(f(k)f(k)^{2c}}{f(k+1)^c\beta^{f(k+1)-2f(k)}} = 0.$$

Then $\sum_{n=1}^{\infty}\frac{G(f(n))}{H(f(n))}$ is irrational.

Proof. By Lemma 4, we must show that for sufficiently large n,

$$H(f(n+1)) > \frac{G(f(n+1))}{G(f(n))}\{H(f(n))^2 - H(f(n))\} + 1.$$

From $\lim_{n\to\infty}\frac{H(f(n+1))}{G(f(n+1))f(n+1)^c\beta^{f(n+1)}} = A > 0$, $c \geq 0$ and $\beta > 1$, we deduce that

$$\frac{H(f(n+1))}{G(f(n+1))f(n+1)^c\beta^{f(n+1)}} > \varepsilon A,$$

for fixed $0 < \varepsilon < 1$ and sufficiently large n. Now

$$\frac{\frac{H(f(n))^2 - H(f(n))}{G(f(n))} + \frac{1}{G(f(n+1))}}{f(n+1)^c\beta^{f(n+1)}} =$$

$$\frac{G(f(n))f(n)^{2c}}{f(n+1)^c\beta^{f(n+1)-2f(n)}}\left(\frac{H(f(n))^2}{G(f(n))^2 f(n)^{2c}\beta^{2f(n)}} - \frac{H(f(n))}{G(f(n))^2 f(n)^{2c}\beta^{2f(n)}}\right) + \frac{1}{G(f(n+1))f(n+1)^c\beta^{f(n+1)}}.$$

Using

$$\lim_{n\to\infty}\frac{G(f(n))f(n)^{2c}}{f(n+1)^c\beta^{f(n+1)-2f(n)}} = 0,$$

$$\lim_{n\to\infty}\frac{H(f(n))^2}{G(f(n))^2 f(n)^{2c}\beta^{2f(n)}} = A^2,$$

we get for sufficiently large n,

$$\frac{\frac{H(f(n))^2 - H(f(n))}{G(f(n))} + \frac{1}{G(f(n+1))}}{f(n+1)^c \beta^{f(n+1)}} < \varepsilon A/2, \text{ i.e.}$$

$$\frac{H(f(n+1))/G(f(n+1))}{\frac{H(f(n))^2 - H(f(n))}{G(f(n))} + \frac{1}{G(f(n+1))}} > \frac{\varepsilon A}{\varepsilon A/2} = 2 > 1.$$

Hence, for sufficiently large n, H(f(n+1)) > $\frac{G(f(n+1))}{G(f(n))}$ {$H(f(n))^2$ – H(f(n))} + 1, yielding the desired result.

The following examples illustrate a wide applicability of Theorem 7.

Examples. Take G(0) = 1 ,G(1) = 1 , G(k+2) = G(k+1) + G(k) , H(0) = H(1) = 1, H(k+2) = 3H(k+1) + 4H(k). Then G(k) $\sim \frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k$, H(k) $\sim \frac{2}{5}4^k$, and so $\frac{H(k)}{G(k)} \sim \frac{4}{5+\sqrt{5}}\left(\frac{8}{1+\sqrt{5}}\right)^k$. Taking $\beta = \frac{8}{1+\sqrt{5}}$, c = 0, we get $\lim_{k\to\infty} H(k)/G(k)k^c\beta^k = \frac{4}{5+\sqrt{5}}$, and I(k) = $\frac{G(f(k))f(k)^{2c}}{f(k+1)^c\beta^{f(k+1)-2f(k)}} \sim$

$$\frac{\left((1+\sqrt{5}/2)\right)^{f(k)+1}/\sqrt{5}}{\left(8/(1+\sqrt{5})\right)^{f(k+1)-2f(k)}}.$$

1) For $f(k) = 3^k$, we see that $I(k) \to 0$ $(k\to\infty)$, and so $\sum_{k=1}^{\infty}\frac{G(3^k)}{H(3^k)}$ is irrational.

2) For f(k) = k!, we conclude similarly that $\sum_{k=1}^{\infty}\frac{G(k!)}{H(k!)}$ is irrational.

DISCUSSION AND CONCLUSION

The two examples of Dirananda and Oppenheim mentioned in the introduction are special cases of our Theorem 1 above. The first example corresponds to $a_i = i$, $b_i = 2i + 1$, which gives $a_i/b_i \to h/k = {}^1/_2$, $\xi_i = 2i - 2i - 1 + 1 = 0$ and so x is rational. The second example corresponds to $a_i = 1$, $b_i = 3i + 2$, which gives $a_i/b_i \to h/k = 1/3$, $\xi_i = 3i - 3i - 2 + 1 \neq 0$ and so x is irrational.

Our Theorem 2 is a modification of Theorem 2.1' in Badea (1993), while our Theorem 3 gives a simplification of Theorem 2. Theorems 4, 5, 6 are extensions of Corollaries 3.2, 3.4, 3.5 and 3.6 of Badea (1993).

Theorem 1 of Brown *et al.* (1995) is a special case with G(k) = 1 of our Theorem7,because $\lim_{k\to\infty}\{f(k+1) - 2f(k)\} = \infty$ implies $\lim_{k\to\infty}\frac{1}{\beta^{f(k+1)-2f(k)}} = 0$ and $f(k+1) \geq f(k)^2$ implies $\lim_{k\to\infty}\frac{f(k)^{2c}}{f(k+1)^c\beta^{f(k+1)-2f(k)}} = 0$.

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