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Some Estimates Involving Density of Algebraic Numbers and Integer Polynomials

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ABSTRACT

Nymann in 1970 derived an asymptotic formula for the probability that k positive integers, chosen at random from the first n natural numbers, are relatively prime. In 1996, Arno *et al.* introduced a new concept called the denominator of an integer polynomial. Using this concept, Arno *et al.* proved theorems establishing formulae for determining the denominator of any algebraic number and the density of algebraic numbers whose denominators are equal to the leading coefficients in their minimal polynomials. The proofs of Arno *et al.* made use of the result of Nymann . The first part of this paper is an extension of the work of Nymann done by relaxing the condition that the chosen numbers are relatively prime. In the second part, the formulae derived in the first part are employed to find asymptotic estimates and the density of the set of integer polynomials refining the work of Arno *et al.*

Key words: algebraic numbers, integer polynomials, denominators, asymptotic value

INTRODUCTION

Nymann (1970) derived an asymptotic formula for the probability of k integers chosen at random from the set $\{1, 2, \dots, n\}$ to be relatively prime. Nymann's result says that this probability is approximately equal to $1/\varsigma(k)$, where ς is the Riemann zeta function. An integer polynomial is a polynomial with integral coefficients. A complex number is an algebraic number if it is a root of a nonzero polynomial with rational coefficients. Among the polynomials with rational coefficients which have an algebraic number α as a root, the one which is monic, and has the least degree is called the minimal polynomial of α over $\mathbb{Q}$ and its degree is called the degree of α. Multiplying this minimal polynomial of α by the least common multiple of the denominators of its coefficients, we get what we call the **minimal polynomial of α over $\mathbb{Z}$.** If α is an algebraic number with p(x) as its minimal polynomial, then all the roots of p(x) are called conjugates of α. An algebraic integer is an algebraic number whose minimal polynomial over $\mathbb{Q}$ has all its coefficients integral. By the **denominator** of an algebraic number α, written **den(α),** we mean the least positive integer n such that $n\alpha$ is an algebraic integer. Denominators are useful in various approximation problems because they satisfy certain multiplicative and additive properties, yet their exact calculation is difficult. Arno *et al.*(1996), see also Laohakosol *et al*.(2000), introduced a new concept of the **denominator of an integer polynomial** A of degree d and with roots α_k $(1 \le k \le d)$, as the least positive integer n, written **den(A)**, for which $n\alpha_k$ is an algebraic integer for all such k. Working through this concept, Arno *et al.*(1996) established formulae for computing the denominator of an algebraic number and the density of those

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algebraic numbers whose denominators equal to the leading coefficients of their minimal polynomials. Their results say for example that this density is about 83%. The proofs of Arno *et al.*(1996) make essential use of the asymptotic estimates of Nymann (1970) mentioned above.

There are two main objectives in this paper. First, the work of Nymann (1970) is extended by relaxing the condition that the chosen numbers are relatively prime. Second, the formulae derived in the first part are used to find asymptotic estimates and the density of the set of integer polynomials refining the work of Arno *et al.* (1996). In the first part, it is found that the probability for k integers chosen at random from the first n natural numbers to have their greatest common divisor equal to g is approximately $1/g^k\varsigma(k)$. In the second part, it is found in particular that the probability that an integer polynomial, with its last two coefficients having their greatest common divisor equal to a square-free integer g, has its denominator equal to its leading coefficient is approximately equal to $|\mu(g)g|/\Pi(p+1)$, where μ is the Möbius function and the product extends over all primes p dividing g.

MATERIALS AND METHODS

Definition. Let α be an algebraic number whose minimal polynomial over $\mathbb{Z}$ is $A(x) = a_d x^d + \ldots + a_0 \in \mathbb{Z}[x]$. The **height** of a is defined as

$$H(\alpha) = \max\{|a_i| : 0 \le i \le d\}.$$

The following terminology will be kept standard throughout the entire paper.

$Z_k^{(g)}(t)$ the number of k-tuples $<m_1,\ldots,m_k>$ of integers such that all $|m_i| \le t$, $(m_1,\ldots,m_k) = g$ and $m_k \neq 0$

$Z_k^{(1,g)}(t)$ the number of k-tuples $<m_1,\ldots,m_k>$ of integers such that all $|m_i| \le t$, $(m_1,\ldots,m_k) = 1$, $(m_k,m_{k-1}) = g$ and $m_k \neq 0$

$\mathrm{Prob}_k^{(g)}(n)$ the probability that k integers chosen randomly from the set $\{0,\pm1,\pm2,\ldots,\pm n\}$ have $(m_1,\ldots,m_k) = g$ and $m_k \neq 0$

$\mathrm{Prob}_k^{(1,g)}(n)$ the probability that k integers chosen randomly from the set $\{0,\pm1,\pm2,\ldots,\pm n\}$ have $(m_1,\ldots,m_k) = 1$, $(m_k,m_{k-1}) = g$ and $m_k \neq 0$

$$P_d(H) = \{A(x) = \sum_{i=0}^{d} a_i x^i \in \mathbb{Z}[x] : a_d \neq 0, |a_i| \le H, (a_0,\ldots,a_d) = 1\}$$

$$P_d^{(g)}(H) = \{A(x) = \sum_{i=0}^{d} a_i x^i \in P_d(H) : (a_d,a_{d-1}) = g\}$$

$$\hat{P}_d(H) = \{A(x) = \sum_{i=0}^{d} a_i x^i \in P_d(H) : \mathrm{den}(A) = |a_d|\}$$

$$\hat{P}_d^{(g)}(H) = \{A(x) = \sum_{i=0}^{d} a_i x^i \in P_d^{(g)}(H) : \mathrm{den}(A) = |a_d|\}$$

$$S_r = \{A(x) \in P_d(H) : r^2 |a_d, r|a_{d-1}\}$$

$$S_r^{(g)} = \{A(x) = \sum_{i=0}^{d} a_i x^i \in P_d^{(g)}(H) : r^2 |a_d, r|a_{d-1}\}$$

$$A_d(H) = \{\alpha : \alpha \text{ is an algebraic number, } \deg(\alpha) = d,\ H(\alpha) \le H\}$$

$$\hat{A}_d(H) = \{\alpha \in A_d(H) : \mathrm{den}\ \alpha = \text{leading coefficient of } \alpha \text{ over } \mathbb{Z}\}$$

$$U_d(H) = \{A(x) = \sum_{i=0}^{d} a_i x^i \in \mathbb{Z}[x] : a_d \neq 0,\ |a_i| \le H\ (0 \le i \le d)\}$$

$\hat{U}_d(H) = \{A(x) = \sum_{i=0}^{d} a_i x^i \in U_d(H) : \text{den}(A) = |a_d|\}$

Lemma 1. Let $g \geq 1$ be fixed. For $t \geq 1$, we have

$$Z_k^{(g)}(t) = \frac{(2t)^k}{g^k \varsigma(k)} + O(\frac{t^{k-1}}{g^{k-1}}) \quad (k \geq 3), \text{ and } \quad Z_2^{(g)}(t) = \frac{(2t)^2}{g^2 \varsigma(2)} + O(\frac{t}{g} \log \frac{t}{g}).$$

Proof. We first treat the case g = 1. Observe that

$$Z_k^{(1)}(t) = \sum_{\substack{(m_1,\ldots,m_k)=1 \\ -t \leq m_i \leq t, m_k \neq 0}} 1 \tag{1}$$

and

$$2[t](2[t]+1)^{k-1} = \sum_{\substack{-t \leq m_i \leq t, m_k \neq 0 \\ i=1,2,\ldots,k}} 1 = \sum_{1 \leq d \leq t} \sum_{\substack{(m_1,\ldots,m_k)=d \\ -t \leq m_i \leq t, m_k \neq 0}} 1 \tag{2}$$

Since $(m_1,\ldots,m_k) = d$ if and only if $(m_1/d,\ldots,m_k/d) = 1$, then there is a one-to-one correspondence between the k-tuples $<m_1,\ldots,m_k>$ with $(m_1,\ldots,m_k) = d$, $-t \leq m_i \leq t$, $m_k \neq 0$ and the k-tuples $<m_1',\ldots,m_k'>$ with $(m_1',\ldots,m_k') = 1$, $-t/d \leq m_i' \leq t/d$, $m_k' \neq 0$. By definition, the number of such k-tuples $<m_1',\ldots,m_k'>$ is $Z_k^{(1)}(\frac{t}{d})$. From (1) and (2) we get

$$2[t](2[t]+1)^{k-1} = \sum_{1 \leq d \leq t} Z_k^{(1)}\left(\frac{t}{d}\right). \tag{3}$$

Applying the Möbius inversion formula to (3) yields

$$Z_k^{(1)}(t) = \sum_{1 \leq d \leq t} \mu(d)\{2[t/d]+1\}^{k-1} 2[t/d] = \sum_{1 \leq d \leq t} \mu(d)\{2t/d + O(1)\}^k$$

$$= (2t)^k \sum_{1 \leq d \leq t} \frac{\mu(d)}{d^k} + (2t)^{k-1} O(\sum_{1 \leq d \leq t} \frac{\mu(d)}{d^{k-1}}) + \ldots + (2t) O(\sum_{1 \leq d \leq t} \frac{\mu(d)}{d}) + O(\sum_{1 \leq d \leq t} 1) \tag{4}$$

From Apostol (1976), we know that

$$\sum_{1 \leq d \leq t} \frac{\mu(d)}{d^k} = \frac{1}{\varsigma(k)} + O(\frac{1}{t^{k-1}}), \tag{5}$$

and then the first term on the right-hand side of equation (4) is equal to $\frac{(2t)^k}{\varsigma(k)} + O(2^k t)$. From equation (5), we have

$$\sum_{1 \leq d \leq t} \frac{\mu(d)}{d^i} = O(1) \qquad (2 \leq i \leq k-1), \tag{6}$$

while, see Apostol (1976),

$$|\sum_{1 \leq d \leq t} \frac{\mu(d)}{d}| \leq \sum_{1 \leq d \leq t} \frac{1}{d} = \log t + \gamma + O(1/t), \tag{7}$$

where γ is the Euler's constant, and

$$\sum_{1 \leq d \leq t} 1 = [t] = O(t). \tag{8}$$

Using equations (4)-(8), we arrive at

$$Z_k^{(1)}(t) = \frac{(2t)^k}{\varsigma(k)} + O(t) + O(t^{k-1}) + O(t^{k-2}) + \dots + O(t^2) + O(t\ \log t) + O(t).$$

$$= \begin{cases} \frac{(2t)^k}{\varsigma(k)} + O(t^{k-1}), & k \ge 3 \\ \frac{(2t)^2}{\varsigma(2)} + O(t \log t), & k = 2 \end{cases}.$$

Next, we observe that $Z_k^{(g)}(t) = \sum_{\substack{(m_1,\dots,m_k)=g \\ -t \le m_i \le t, m_k \ne 0}} 1 = \sum_{\substack{(\frac{m_1}{g},\dots,\frac{m_k}{g})=1 \\ -\frac{t}{g} \le \frac{m_i}{g} \le \frac{t}{g}, \frac{m_k}{g} \ne 0}} 1 = \sum_{\substack{(m_1',\dots,m_k')=1 \\ -\frac{t}{g} \le m_i' \le \frac{t}{g}, m_k' \ne 0}} 1..$

Replacing t by t/g in the preceding discussion, we get

$$Z_k^{(g)}(t) = Z_k^{(1)}(\frac{1}{g}) = \begin{cases} \frac{(2t)^k}{(g)^k \varsigma(k)} + O(\frac{t^{k-1}}{g^{k-1}}), k \ge 3 \\ \frac{(2t)^k}{(g)^2 \varsigma(2)} + O(\frac{t}{g} \log \frac{t}{g}), k = 2 \end{cases}$$

as to be proved. Q.E.D.

Lemma 2. Let g ≥ 1 be fixed. For t ≥ 1, we have

$$Z_k^{(1,g)}(t) = \frac{(2t)^k}{(g)^2 \varsigma(2)} \prod_{p|g} (1 - \frac{1}{p^{k-2}}) + O(\frac{t^{k-1}}{g} \log \frac{t}{g}) \qquad (k \ge 3), \text{ and}$$

$$Z_2^{(1,g)}(t) = \frac{(2t)^2}{\varsigma(2)} + O(t \log t).$$

Proof. Observe that

$$Z_k^{(1,g)}(t) \sum_{\substack{(m_1,\dots,m_k)=1 \\ (m_k, m_{k-1})=g \\ -t \le m_i \le t, m_k \ne 0}} 1, \tag{9}$$

and, applying the case k = 2 of Lemma 1 to the last two coordinates, we have

$$\{\frac{(2t)^2}{g^2 \varsigma(2)} + O(\frac{t}{g} \log \frac{t}{g})\}(2[t]+1)^{k-2} = \sum_{\substack{(m_k, m_{k-1})=g \\ -t \le m_i \le t, m_k \ne 0 \\ i=1,2,\dots,k}} 1 = \sum_{1 \le d \le t} \sum_{\substack{(m_1,\dots m_k)=d \\ (m_k, m_{k-1})=g \\ -t \le m_i \le t, m_k \ne 0}} 1. \tag{10}$$

Since $(m_1,\dots,m_k) = d$ if and only if $(m_1/d,\dots,m_k/d) = 1$, then there is a one-to-one correspondence between the k-tuples $<m_1,\dots,m_k>$ with $(m_1,\dots,m_k) = d$, $-t \le m_i \le t$, $m_k \ne 0$, $(m_k, m_{k-1}) = g$ and the k-tuples $<m_1',\dots,m_k'>$ with $(m_1',\dots,m_k') = 1$, $-t/d \le m_i' \le t/d$, $m_k' \ne 0$, $(m_k', m_{k-1}') = g/d$, $d \mid g$. By definition, the number of such k-tuples $<m_1',\dots,m_k'>$ is equal to $Z_k^{(1,\frac{g}{d})}(\frac{t}{d})$. By equations (9) and (10), we obtain

$$\{\frac{(2t)^2}{g^2\varsigma(2)}+O(\frac{t}{g}\log\frac{t}{g})\}(2[t]+1)^{k-2}=\sum_{\substack{1\le d\le t\\ d|g}} Z_k^{(1,\frac{g}{d})}(\frac{t}{d}). \quad (11)$$

Applying the Möbius inversion formula to equation (11), we deduce that

$$Z_k^{(1,g)}(t)=\sum_{\substack{1\le d\le t\\ d|g}}\mu(d)\{2[t/d]+1\}^{k-2}\{\frac{1}{\varsigma(2)}(\frac{2(t/d)}{g/d})^2+O(\frac{t/d}{g/d}\log\frac{t/d}{g/d})\}$$

$$=\quad \{(2t)^{k-2}\sum_{\substack{1\le d\le t\\ d|g}}\frac{\mu(d)}{d^{k-2}}+(2t)^{k-3}O(\sum_{\substack{1\le d\le t\\ d|g}}\frac{\mu(d)}{d^{k-3}})+\ldots+2tO(\sum_{\substack{1\le d\le t\\ d|g}}\frac{\mu(d)}{d})+O(\sum_{\substack{1\le d\le t\\ d|g}}1)\}\times$$

$$\times\{\frac{1}{\varsigma(2)}(\frac{2t}{g})^2+O(\frac{t}{g}\log\frac{t}{g})\}\cdot \quad (12)$$

Since, see Apostol (1976),

$$\sum_{\substack{1\le d\le t\\ d|g}}\frac{\mu(d)}{d^{k-2}}=\prod_{p|g}(1-\frac{1}{p^{k-2}}), \quad (13)$$

$$\sum_{\substack{1\le d\le t\\ d|g}}\frac{\mu(d)}{d^{i}} = O(1) \qquad (2\le i\le k\text{-}3), \quad (14)$$

$$\sum_{\substack{1\le d\le t\\ d|g}}\frac{\mu(d)}{d}=O(\log t), \quad (15)$$

and

$$\sum_{\substack{1\le d\le t\\ d|g}}1=[t/g] = O(t/g), \quad (16)$$

then from equations (12)-(16), we have for $k\ge 3$

$$Z_k^{(1,g)}(t)=\frac{(2t)^k}{g^2\varsigma(2)}\prod_{p|g}(1-\frac{1}{p^{k-2}})+O(\frac{t^{k-1}}{g^2})+\ldots+O(\frac{t^3}{g^2}\log t)+O(\frac{t^3}{g^3})+O(\frac{t^{k-1}}{g}\log\frac{t}{g})$$

$$+O(\frac{t^{k-2}}{g}\log\frac{t}{g})+\ldots+O(\frac{t^2}{g}\log\frac{t}{g}\log t)+O(\frac{t^2}{g^2}\log\frac{t}{g})$$

$$=\frac{(2t)^k}{(g)^2\varsigma(2)}\prod_{p|g}(1-\frac{1}{p^{k-2}})+O(\frac{t^{k-1}}{g}\log\frac{t}{g}),$$

while for $k = 2$, from equation (12), we have

$$Z_2^{(1,g)}(t)=\sum_{\substack{1\le d\le t\\ d|g}}\mu(d)(\frac{1}{\varsigma(2)}(\frac{2t}{g})^2+O(\frac{t}{g}\log\frac{t}{g}))=\frac{(2t)^2}{g^2\varsigma(2)}\sum_{\substack{1\le d\le t\\ d|g}}\mu(d)+O(\frac{t}{g}\log\frac{t}{g}),$$

which can be rewritten by using $\sum_{\substack{1\le d\le t\\ d|g}} \mu(d) = [1/g]$ as

$$Z_2^{(1,g)}(t) = \begin{cases} 0, & g>1 \\ \dfrac{(2t)^2}{\varsigma(2)} + O(t\log t), & g=1 \end{cases}. \qquad \text{Q.E.D.}$$

Lemma 3. Let $d \ge 2$ and g be fixed integers. Then

$$\left| P_d^{(g)}(H) \right| = \frac{(2H)^{d+1}}{\varsigma(2)g^2} \prod_{p|g} \left(1-\frac{1}{p^{d-1}}\right) + O_d\left(\frac{H^d}{g}\log\frac{H}{g}\right).$$

Proof. From its definition, see also Pólya and Szegö (1976), $P_d^{(g)}(H) = Z_k^{(1,g)}(t)$, with $t = H$, $d = k-1$.Using Lemma 2, we get

$$\left| P_d^{(g)}(H) \right| = \frac{(2H)^{d+1}}{\varsigma(2)g^2} \prod_{p|g} \left(1-\frac{1}{p^{d-1}}\right) + O_d\left(\frac{H^d}{g}\log\frac{H}{g}\right). \qquad \text{Q.E.D.}$$

Lemma 4. Let $d \ge 2$, $g \ge 1$ be fixed integers, and $1 \le r \le H$. Then for $u = (r, g/mr)$, we have

$$\left| S_r^{(g)} \right| = \frac{(2H)^{d+1}u}{\varsigma(2)g^2 r} \prod_{p|\frac{r}{u}} \left(\frac{1}{1+\frac{1}{p}}\right) \prod_{p|g} \left(1-\frac{1}{p^{d-1}}\right) + O_d\left(\frac{H^d u}{gr}\log\frac{H}{g}\right).$$

Proof. First consider the case $r = 1$. Since $S_1^{(g)} = P_d^{(g)}(H)$, then the theorem is true in this case. Assume $r \ge 2$. For brevity, we omit writing the hypothesis $0 \le i \le d$ underneath the summation sign. Thus

$$\left| S_r^{(g)} \right| = \sum_{\substack{|a_i|\le H, a_d\ne 0\\ r^2|a_d, r|a_{d-1}\\ (a_d,\ldots,a_0)=1\\ (a_d,a_{d-1})=g}} 1 = \sum_{\substack{|a_i|\le H, a_d\ne 0\\ r^2|a_d, r|a_{d-1}\\ (a_d,a_{d-1})=g}} \sum_{k|(a_d,\ldots,a_0)} \mu(k) = \sum_{\substack{|a_i|\le H, a_d\ne 0\\ r^2|a_d, r|a_{d-1}\\ (a_d,a_{d-1})=g}} \sum_{k|a_i(\forall i)} \mu(k)$$

$$= \sum_{\substack{k\le H\\ k|g}} \mu(k) \sum_{\substack{|b_i|\le \frac{H}{k}, b_d\ne 0\\ r^2|kb_d, r|kb_{d-1}\\ (b_d,b_{d-1})=\frac{g}{k}}} 1 \quad , \quad (\text{writing } a_i = kb_i)$$

$$= \sum_{s|r} \sum_{\substack{k\le H\\ (k,r)=s\\ k|g}} \mu(k) \sum_{\substack{|b_i|\le \frac{H}{k}, b_d\ne 0\\ r^2|kb_d, r|kb_{d-1}\\ (b_d,b_{d-1})=\frac{g}{k}}} 1 = \sum_{s|r} \sum_{\substack{m\le \frac{H}{s}\\ (m,\frac{r}{s})=1\\ m|g}} \mu(ms) \sum_{\substack{|b_i|\le \frac{H}{ms}, b_d\ne 0\\ \frac{r^2}{s}|mb_d, \frac{r}{s}|mb_{d-1}\\ (b_d,b_{d-1})=\frac{g}{ms}}} 1 \quad ,$$

with $k = ms$. Note that the term $\mu(ms)$ ensures that $(m,s) = 1$. This together with $(m,r/s) = 1$ is equivalent to $(m,r) = 1$, and so $(r^2/s)|b_d$ and $(r/s)|b_{d-1}$. Put $(r^2/s)c_d = b_d$, $(r/s)c_{d-1} = b_{d-1}$ and $c_i = b_i$ $(0\le i \le d-2)$. Thus,

$$\left| S_r^{(g)} \right| = \sum_{s|r} \mu(s) \sum_{\substack{m\le \frac{H}{s} \\ (m,r)=1 \\ m|g}} \mu(m) \sum_{\substack{|c_i|\le \frac{H}{ms}\ (0\le i\le d-2) \\ 0<|c_d|\le \frac{H}{mr^2}\ |c_{d-1}|\le \frac{H}{mr} \\ (rc_d, c_{d-1}) = \frac{g}{mr}}} 1. \qquad (17)$$

Claim 1: For $x,y \in \mathbb{Z}$, $r,g \in \mathbb{Z}$, $r \ge 2$, with $0<|x|\le C_1$ and $|y|\le C_2$, the number of ordered pairs <x,y> satisfying

$$(rx,y) = g \text{ is } \frac{4C_1C_2u}{\varsigma(2)g^2} \prod_{p|\frac{r}{u}} \left(\frac{1}{1+1/p}\right) + O\left(\frac{C_2}{g}, \frac{C_1u}{g}\log(C_2/g)\right),$$

where $u = (r,g)$ and $O(\lambda_1, \lambda_2) = \max(O(\lambda_1), O(\lambda_2))$.

To see this, consider x,y for which (rx ,y) = g. There are two possible cases.

Case 1: (r, g) = 1.

If $y = \pm kg$ and $(k,r) = 1$, then, see Pólya and Szegö (1976), the total number of possible x's is $2\left[\frac{\varphi(k)}{k}\frac{C_1}{g}\right]$

$= 2\left(\frac{\varphi(k)}{k}\frac{C_1}{g}\right) + O(1)$, and possible values of k are $\pm 1,\ldots,\pm[C_2/g]$. Thus, the number of possible ordered

$$\text{pairs } \langle x,y\rangle \text{ is } \quad 2\sum_{\substack{k=1 \\ (k,r)=1}}^{[\frac{C_2}{g}]} \left(2\frac{\varphi(k)}{k}\frac{C_1}{g} + O(1)\right)$$

$$= 4\left(\frac{C_1}{g}\right)\sum_{\substack{k=1 \\ (k,r)=1}}^{[\frac{C_2}{g}]} \frac{\varphi(k)}{k} + O\left(\frac{C_2}{g}\right) = 4\left(\frac{C_1C_2}{\varsigma(2)g^2}\right)\prod_{p|r}\left(\frac{1}{1+1/p}\right) + O\left(\frac{C_2}{g}, \frac{C_1}{g}\log\frac{C_2}{g}\right).$$

Case 2: $u = (r,g) > 1$.

Let $r = uR$ and $g = uG$ where $R,G \in \mathbb{Z}$, $(R,G) = 1$. Similar to the arguments used in Case 1, the number of

$$\text{possible ordered pairs } \langle x,y\rangle \text{ is } \quad 2\sum_{k=1}^{[C_2/g]} \left\{2\frac{\varphi(k)}{k}\frac{C_1}{g} + O(1)\right\}$$

$$= (4C_2/G)\sum_{\substack{k=1 \\ (k,R)=1}}^{[\frac{C_2}{g}]} \frac{\varphi(k)}{k} + O(C_2/g) = 4\left(\frac{C_1C_2u}{\varsigma(2)g^2}\right)\prod_{p|\frac{r}{u}}\left(\frac{1}{1+1/p}\right) + O\left(\frac{C_2}{g}, \frac{C_1u}{g}\log\frac{C_2}{g}\right).$$

From Claim 1, let $x = c_d$ and $y = c_{d-1}$. Then the number of ordered pairs $\langle c_d, c_{d-1}\rangle$ with $(rc_d, c_{d-1}) = g/mr$, $0<|c_d|\le H/mr^2$ and $|c_{d-1}|\le H/mr$ is equal to

$$\frac{4}{\varsigma(2)}\frac{H^2u}{rg^2}\prod_{p|\frac{r}{u}}\left(\frac{1}{1+1/p}\right) + O\left(\frac{H}{g}, \frac{Hu}{rg}\log\frac{H}{g}\right) = \frac{4}{\varsigma(2)}\frac{H^2u}{rg^2}\prod_{p|\frac{r}{u}}\left(\frac{1}{1+1/p}\right) + O\left(\frac{Hu}{rg}\log\frac{H}{g}\right),$$

with $u = (r, g/mr)$, and since $s \le r$ and $g \ge s$, we have

$$\sum_{\substack{|c_i|\le \frac{H}{ms}\,(0\le i\le d-2)\\ 0<|c_d|\le \frac{H}{mr^2},|c_{d-1}|\le \frac{H}{mr}\\ (rc_d,c_{d-1})=\frac{g}{mr}}} 1 = \left(\frac{4}{\varsigma(2)}\frac{Hu^2}{rg^2}\prod_{p|\frac{r}{u}}\left(\frac{1}{1+1/p}\right)+O\left(\frac{Hu}{rg}\log\frac{H}{g}\right)\right)\left(\frac{2H}{ms}+O(1)\right)^{d-1}.$$

$$= \frac{(2H)^{d+1}u}{\varsigma(2)rg^2(ms)^{d-1}}\prod_{p|\frac{r}{u}}\left(\frac{1}{1+1/p}\right)+O_d\left(\frac{H^d u}{rg(ms)^{d-1}}\log\frac{H}{g}\right).$$

Substituting into equation (17) and separate into two terms $M_r^{(g)}$ and $R_r^{(g)}$, we have

$$\left|S_r^{(g)}\right| = \sum_{s|r}\mu(s)\sum_{\substack{m\le\frac{H}{s}\\(m,r)=1\\m|g}}\mu(m)\left\{\frac{(2H)^{d+1}u}{\varsigma(2)rg^2(ms)^{d-1}}\prod_{p|\frac{r}{u}}\left(\frac{1}{1+1/p}\right)+O_d\left(\frac{H^d u}{rg(ms)^{d-1}}\log\frac{H}{g}\right)\right\}$$

$$= \frac{(2H)^{d+1}u}{\varsigma(2)rg^2}\prod_{p|\frac{r}{u}}\left(\frac{1}{1+1/p}\right)\sum_{s|r}\frac{\mu(s)}{s^{d-1}}\sum_{\substack{m\le H/s\\(m,r)=1\\m|g}}\frac{\mu(m)}{m^{d-1}} + \sum_{s|r}\mu(s)\sum_{\substack{m\le H/s\\(m,r)=1\\m|g}}\mu(m)\,O_d\left(\frac{H^d u}{rg(ms)^{d-1}}\log\frac{H}{g}\right).$$

$= M_r^{(g)} + R_r^{(g)}$,
where

$$M_r^{(g)} = \frac{(2H)^{d+1}u}{g^2\varsigma(2)r}\prod_{p|\frac{r}{u}}\left(\frac{1}{1+1/p}\right)\sum_{s|r}\frac{\mu(s)}{s^{d-1}}\sum_{\substack{m\le H/s\\(m,r)=1\\m|g}}\frac{\mu(m)}{m^{d-1}}$$

$$= \frac{(2H)^{d+1}u}{g^2\varsigma(2)r}\prod_{p|\frac{r}{u}}\left(\frac{1}{1+1/p}\right)\sum_{s|r}\frac{\mu(s)}{s^{d-1}}\sum_{\substack{m=1\\(m,r)=1\\m|g}}^{\infty}\frac{\mu(m)}{m^{d-1}} - \frac{(2H)^{d+1}u}{g^2\varsigma(2)r}\prod_{p|\frac{r}{u}}\left(\frac{1}{1+1/p}\right)\sum_{s|r}\frac{\mu(s)}{s^{d-1}}\sum_{\substack{m>H/s\\(m,r)=1\\m|g}}\frac{\mu(m)}{m^{d-1}}.$$

Claim 2: $\displaystyle\sum_{\substack{m>H/s\\(m,r)=1\\m|g}}\frac{\mu(m)}{m^{d-1}} = O\left(g\left(\frac{s}{H}\right)^{d-1}\right).$

To see this, put $g = p_1^{\alpha_1}\dots p_t^{\alpha_t}C(r)$. As $r|g$, $(p_i, r) = 1$, $C(r)$ is the factor of g relatively prime to r. Let $r = q^{r_1}\dots q_s^{r_s}$. Since $r|g$, then $g = q_1^{t_1}\dots q_s^{t_s}\;p_1^{\alpha_1}\dots p_t^{\alpha_t}$, $r_i \le t_i$, and so $C(r) = q_1^{t_1}\dots q_s^{t_s}$. If $(m, r) = 1$ and $m|g$, then $m = p_1^{\beta_1}\dots p_t^{\beta_t}$ where $0 \le \beta_i \le \alpha_i$. Now

$$\left|\sum_{\substack{m>H/s\\(m,r)=1,m|g}}\frac{\mu(m)}{m^{d-1}}\right| = \left|\sum_{\substack{0\le\beta_i\le\alpha_i\\p_1^{\beta_1}\dots p_t^{\beta_t}>H/s}}\frac{\mu(p_1^{\beta_1}\dots p_t^{\beta_t})}{(p_1^{\beta_1}\dots p_t^{\beta_t})^{d-1}}\right| \le \left(\frac{s}{H}\right)^{d-1}\sum_{\beta_1=0}^{1}\dots\sum_{\beta_t=0}^{1}1 \le g(s/H)^{d-1},$$

as claimed. Using $\sum_{s|r} \frac{\mu(s)}{s^{d-1}} = \prod_{p|r} (1-1/p^{d-1})$ and $\sum_{\substack{m=1 \\ (m,r)=1 \\ m|g}}^{\infty} \frac{\mu(m)}{m^{d-1}} = \frac{\prod_{p|g} (1-1/p^{d-1})}{\prod_{p|r} (1-1/p^{d-1})}$, we have

$$\sum_{s|r} \frac{\mu(s)}{s^{d-1}} \sum_{\substack{m>\frac{H}{s} \\ (m,r)=1 \\ m|g}} \frac{\mu(m)}{m^{d-1}} = \sum_{s|r} \frac{\mu(s)}{s^{d-1}} \; O(g\,(\frac{s}{H})^{d-1}) = O(\frac{g}{H^{d-1}}),\text{ and so}$$

$$M_r^{(g)} = \frac{(2H)^{d+1}u}{\varsigma(2)g^2 r} \prod_{p|\frac{r}{u}} (\frac{1}{1+1/p}) \prod_{p|g} (1-1/p^{d-1}) + O_d(\frac{H^2 u}{gr}).$$

Similarly for $d \geq 2$, we get

$$R_r^{(g)} = \sum_{s|r} \mu(s) \sum_{\substack{m\leq H/s \\ (m,r)=1 \\ m|g}} \mu(m)\, O(\frac{H^d u}{gr(ms)^{d-1}} \log(H/g)) = O_d(\frac{H^d u}{gr} \log(H/g) \sum_{s|r} \frac{\mu(s)}{s^{d-1}} \sum_{\substack{m\leq H/s \\ (m,r)=1 \\ m|g}} \frac{\mu(m)}{m^{d-1}})$$

$$= O_d(\frac{H^d u}{gr} \log(H/g) \prod_{p|g} (1-1/p^{d-1}) + \frac{H^d u}{gr} \cdot \log \frac{H}{g} \cdot \frac{g}{H^{d-1}}) = O_d(\frac{H^d u}{rg} \cdot \log \frac{H}{g}),$$

and the desired result follows. Q.E.D.

Lemma 5. If $A(x) = a_d x^d + \ldots + a_0 \in \mathbb{Z}[x]$ has content $c > 1$, then $\text{den}(A) \neq |a_d|$ and $\hat{U}_d(H) = \hat{P}_d(H)$.

Proof. Since $A(x)$, $(1/c)A(x) \in \mathbb{Z}[x]$ both have the same set of roots and $(1/c)A(x)$ is primitive, then den(A) = den((1/c)A). By Theorem 1 of Arno *et al.* (1996), den((1/c)A) divides a_d/c and $\text{den}(A) = \text{den}((1/c)A) \leq |a_d|/c < |a_d|$ implying that $\text{den}(A) \neq |a_d|$. Next, let $A(x) \in \mathbb{Z}[x]$ have c as its content. Assume that $A(x) \in \hat{U}_d(H)$. If $c > 1$, then $\text{den}(A) \neq |a_d|$, and so $A(x) \notin \hat{U}_d(H)$, which is a contradiction. Thus, $c = 1$ and so $A(x) \in \hat{P}_d(H)$ yielding $\hat{U}_d(H) \subseteq \hat{P}_d(H)$. Finally, we assume that $A(x) \in \hat{P}_d(H)$. Thus, $A(x) \in \hat{U}_d(H)$ which gives $\hat{P}_d(H) \subseteq \hat{U}_d(H)$. Hence, $\hat{U}_d(H) = \hat{P}_d(H)$. Q.E.D.

RESULTS

Theorem 1. Let $g \geq 1$ be fixed. Then for $n \in \mathbb{N}$, we have

(i) $\text{Prob}_k^{(g)}(n) = \frac{1}{g^k \varsigma(k)} (\frac{2n}{2n+1})^k + O(\frac{n^{k-1}}{g^{k-1}(2n+1)^k})$, when $k \geq 3$,

(ii) $\text{Prob}_2^{(g)}(n) = \frac{1}{g^2 \varsigma(2)} (\frac{2n}{2n+1})^2 + O(\frac{n}{g(2n+1)^2} \log \frac{n}{g})$, and

(iii) $\lim_{n\to\infty} \text{Prob}_k^{(g)}(n) = \frac{1}{g^k \varsigma(k)}$.

Proof. Since $\mathrm{Prob}_k^{(g)}(n) = \dfrac{Z_k^{(g)}(n)}{(2n+1)^k}$, then from Lemma 1, for k ≥ 3, we have

$$\mathrm{Prob}_k^{(g)}(n) = \frac{\dfrac{(2n)^k}{g^2\varsigma(k)} + O(\dfrac{n^{k-1}}{g^{k-1}})}{(2n+1)^k} = \frac{1}{g^k\varsigma(k)}(\frac{2n}{2n+1})^k + O(\frac{n^{k-1}}{g^{k-1}(2n+1)^k}),$$

while for k = 2, we have

$$\mathrm{Prob}_2^{(g)}(n) = \frac{\dfrac{(2n)^2}{g^2\varsigma(2)} + O(\dfrac{n}{g}\log\dfrac{n}{g})}{(2n+1)^2} = \frac{1}{g^2\varsigma(2)}(\frac{2n}{2n+1})^2 + O(\frac{n}{g(2n+1)^2}\log\frac{n}{g}),$$

and (iii) follows immediately from (i) and (ii). Q.E.D.

Theorem 2. Let g ≥ 1 be fixed and p be a prime. Then for n ∈ ℕ, we have

(i) $\mathrm{Prob}_k^{(1,g)}(n) = \dfrac{1}{g^2\varsigma(2)}(\dfrac{2n}{2n+1})^k \prod_{p|g}(1-\dfrac{1}{p^{k-2}}) + O(\dfrac{n^{k-1}}{g(2n+1)^k}\log\dfrac{n}{g})$, when k ≥ 3,

(ii) $\mathrm{Prob}_2^{(1,g)}(n) = \dfrac{1}{\varsigma(2)}(\dfrac{2n}{2n+1})^2 + O(\dfrac{n}{(2n+1)^2}\log n)$,

(iii) $\lim\limits_{n\to\infty}\mathrm{Prob}_k^{(1,g)}(n) = \dfrac{1}{g^2\varsigma(2)}\prod_{p|g}(1-\dfrac{1}{p^{k-2}})$, when k ≥ 3, and

(iv) $\lim\limits_{n\to\infty}\mathrm{Prob}_2^{(1,g)}(n) = \dfrac{1}{\varsigma(2)}$.

Proof. Since $\mathrm{Prob}_k^{(1,g)}(n) = \dfrac{Z_k^{(1,g)}(n)}{(2n+1)^k}$, then from Lemma 2 for the case k ≥ 3, we get

$$\mathrm{Prob}_k^{(1,g)}(n) = \frac{\dfrac{(2n)^k}{g^2\varsigma(2)}\prod_{p|g}(1-\dfrac{1}{p^{k-2}}) + O(\dfrac{n^{k-1}}{g}\log\dfrac{n}{g})}{(2n+1)^k}$$

$$= \frac{1}{g^2\varsigma(2)}(\frac{2n}{2n+1})^k\prod_{p|g}(1-\frac{1}{p^{k-2}}) + O(\frac{n^{k-1}}{g(2n+1)^k}\log\frac{n}{g}),$$

while for the case k = 2, we get

$$\mathrm{Prob}_2^{(1,g)}(n) = \frac{\dfrac{(2n)^2}{\varsigma(2)} + O(n\ \log n)}{(2n+1)^2} = \frac{1}{\varsigma(2)}(\frac{2n}{2n+1})^2 + O(\frac{n}{(2n+1)^2}\log\ n).$$

Lastly , (iii) and (iv) follow directly from (i) and (ii). Q.E.D.

Theorem 3. Let d ≥ 1 and g ≥ 1 be fixed. For H ≥ 2, we have

$$\left|\hat{P}_d^{(g)}(H)\right| = \frac{|\mu(g)|(2H)^{d+1}}{\varsigma(2)g^2}\prod_{p|g}(1-\frac{1}{p^{d-1}})(1-\frac{1}{p+1}) + O_d(\frac{H^d}{g}\log\frac{H}{g}).$$

Proof. For d = 1, we have $\hat{P}_1^{(g)}(H) = P_1^{(g)}(H)$. Therefore, the theorem is true in this case. Assume $d \geq 2$. Using Theorem 1 (i) and the proof of Theorem 2 of Arno et al. (1996), we get den (A) $\neq |a_d| \Leftrightarrow$ there is a prime p such that $p \mid a_d$ and $n = a_d/p$ satisfies

$$(n^d/a_d)A(x/n) \in \mathbb{Z}[x]$$

$\Leftrightarrow$ there is a prime $p \mid a_d$ and $(1, a_{d-1}/p, a_{d-2}a_d/p^2, a_{d-3}a_d^2/p^3, \ldots, a_0a_d^{d-1}/p^d) \in \mathbb{Z}^{d+1}$.
Since A(x) is primitive, then den (A) $\neq |a_d|$ is equivalent to $p^2|a_d$ and $p|a_{d-1}$. Thus ,

$$\{A(x) = \sum_{i=0}^{d} a_i x^i \in P_d^{(g)}(H) : \text{den}(A) \neq |a_d|\} = \bigcup_{p\leq H} S_p^{(g)},$$

where $S_p^{(g)} = \{A(x) = \sum_{i=0}^{d} a_i x^i \in P_d^{(g)}(H)\text{: } p^2|a_d, p|a_{d-1}\}$. Observe from its definition that

$$S_r^{(g)} = \{A(x) = \sum_{i=0}^{d} a_i x^i \in P_d^{(g)}(H)\text{: } r^2|a_d, r|a_{d-1}\}.$$

Thus, $S_1^{(g)} = P_d^{(g)}(H)$, $S_r^{(g)} = \bigcap_{p|r} S_p^{(g)}$, with $r \geq 2$, r square-free, and $S_r^{(g)} = \varnothing$ when

$r > \sqrt{H}$. Since $\hat{P}_d^{(g)}(H) = S_1^{(g)} - \bigcup_{p\leq H} S_p^{(g)}$, then by the inclusion-exclusion principle,

$$|\hat{P}_d^{(g)}(H)| = |S_1^{(g)} - \bigcup_{p\leq H} S_p^{(g)}| = |S_1^{(g)}| + \sum_{2\leq r\leq H}\mu(r)|\bigcap_{p|r} S_r^{(g)}| = \sum_{1\leq r\leq H}\mu(r)|S_r^{(g)}|.$$

Apply Lemma 4 and separate the sum into two parts M_g and R_g to get

$$|\hat{P}_d^{(g)}(H)| = \sum_{\substack{1\leq r\leq H\\ r|g}}\mu(r)(\frac{(2H)^{d+1}u}{\varsigma(2)g^2r}\prod_{p|g}(1-\frac{1}{p^{d-1}})\prod_{p|g}(1-\frac{1}{p^{d-1}}) + O_d(\frac{H^d u}{gr}\log\frac{H}{g}))$$

$$= M_g + R_g, \quad \text{where } M_g = \sum_{\substack{1\leq r\leq H\\ r|g}}\mu(r)\{\frac{(2H)^{d+1}u}{\varsigma(2)g^2r}\prod_{p|\frac{r}{u}}(\frac{1}{1+1/p})\prod_{p|g}(\frac{1}{p^{d-1}})\}$$

$$= \frac{(2H)^{d+1}}{\varsigma(2)g^2}\prod_{p|g}(1-\frac{1}{p^{d-1}})\sum_{\substack{1\leq r\leq H\\ r|g}}\frac{\mu(r)u}{r}\prod_{p|\frac{r}{u}}(\frac{1}{1+1/p}).$$

Set $$F_g^{(r)} = \sum_{\substack{1\leq r\leq H\\ r|g}}\frac{\mu(r)u}{r}\pi(\frac{r}{u}) \quad\text{and}\quad \pi(\frac{r}{u}) = \prod_{p|\frac{r}{u}}(\frac{1}{1+1/p}).$$

Let $g = p_1^{a_1}\cdots p_k^{a_k}$. As r|g, then $r = p_1^{b_1}\cdots p_k^{b_k}$, $0 \leq b_i \leq a_i$. Thus,

$$F_g^{(r)} = \sum_{b_1=0}^{1}\frac{\mu(p_1^{b_1})}{p_1^{b_1}}(p_1^{a_1-b_1}, P_1^{b_1})\cdots\sum_{b_k=0}^{1}\frac{\mu(p_k^{b_k})}{p_k^{bk}}(p_k^{a_k-b_k}, p_k^{b_k})\pi(\frac{p_1^{b_1}\cdots p_k^{b_k}}{(p_1^{a_1-b_1}\cdots p_k^{a_k-b_k}, p_1^{b_1}\cdots p_k^{b_k})}).$$

$$= \begin{cases} 0, & \exists a_i > 1 \Leftrightarrow g \text{ not square-free} \Leftrightarrow \mu(g) = 0 \\ \prod_{p|g}(1 - \frac{1}{p+1}), \forall a_i = 1 \Leftrightarrow g \text{ square-free} \Leftrightarrow |\mu(g)| = 1 \end{cases}$$

$$= |\mu(g)| \prod_{p|g}(1\frac{1}{p+1}).$$

Thus, $M_g = \frac{|\mu(g)|(2H)^{d+1}}{\varsigma(2)g^2} \prod_{p|g}(1 - \frac{1}{p^{d-1}})(1 - \frac{1}{p+1})$, and

$$R_g = \sum_{\substack{1 \le r \le H \\ r|g}} \mu(r) O_d(\frac{H^d u}{gr} \log \frac{H}{g}) = O_d(\frac{H^d}{g} \log \frac{H}{g}),$$

using $\frac{H^d u}{g} \log \frac{H}{g} \sum_{\substack{1 \le r \le H \\ r|g}} \frac{\mu(r)}{r} = O_d(\frac{H^d}{g} \log \frac{H}{g})$, and the result follows. Q.E.D.

Theorem 4. Let $d \ge 1$ and $g \ge 1$ be fixed. For $H \ge 2$, we have $\frac{|\hat{P}_1^{(g)}(H)|}{|P_1^{(g)}(H)|} = 1$, and when $d \ge 2$, $\frac{|\hat{P}_d^{(g)}(H)|}{|P_d^{(g)}(H)|}$ $= |\mu(g)| g \prod_{p|g}(\frac{1}{p+1}) + O_d(\frac{g}{H} \log \frac{H}{g})$, where g is square-free.

Proof. For d = 1, we have $\hat{P}_1^{(g)}(H) = P_1^{(g)}(H)$, i.e. the result is true in this case.

Assume $d \ge 2$. From Theorem 3, and Lemma 3 we see that

$$\frac{|\hat{P}_d^{(g)}(H)|}{|P_d^{(g)}(H)|} = \frac{\frac{\varsigma(2)g^2}{(2H)^{d+1}}}{\prod_{p|g}(1 - \frac{1}{p^{d-1}})} \{\frac{|\mu(g)|(2H)^{d+1}}{\varsigma(2)g^2} \prod_{p|g}(1 - \frac{1}{p^{d-1}})(1 - \frac{1}{p+1}) + O_d(\frac{H^d}{g} \log \frac{H}{g})\} \times$$

$$\times \{\frac{1}{1 + O_d(\frac{1}{H} \log \frac{H}{g})}\}$$

$$= |\mu(g)| \prod_{p|g}(\frac{p}{p+1}) + O_d(\frac{g}{H} \log \frac{H}{g}) = |\mu(g)| g \prod_{p|g}(\frac{1}{p+1}) + O_d(\frac{g}{H} \log \frac{H}{g}),$$

as g is square-free. Q.E.D.

Theorem 5. Let $d \ge 1$, $H \ge 2$ and p be prime. Then $\lim_{d \to \infty} \lim_{H \to \infty} \frac{|\hat{U}_d(H)|}{|U_d(H)|} = \frac{1}{\varsigma(3)}$.

Proof. From Lemma 5, $\hat{U}_d(H) = \hat{P}_d(H)$, and from Theorem 3,

$$|\hat{P}_d(H)| = \frac{(2H)^{d+1}}{\varsigma(d+1)} \prod_p (1 - \frac{1}{p^3} \frac{(1 - 1/p^{d-1})}{(1 - 1/p^{d+1})} + O_d(H^d \log^2 H).$$

Using $|U_d(H)| = (2H)^{d+1} + O_d(H^d)$, we deduce that

$$\frac{|\hat{U}_d(H)|}{|U_d(H)|} = \frac{1}{(2H)^{d+1}}\{\frac{(2H)^{d+1}}{\varsigma(d+1)}\prod_p(1-\frac{1}{p^3}\frac{(1-1/p^{d-1})}{(1-1/p^{d+1})})+O_d(H^d\log^2 H)\}\frac{1}{1+O_d(1/H)}$$

$$= \frac{1}{\varsigma(d+1)}\prod_p(1-\frac{1}{p^3}\frac{(1-1/p^{d-1})}{(1-1/p^{d+1})})+O_d(\frac{\log^2 H}{H}),$$

and so $\lim_{H\to\infty}\frac{|\hat{U}_d(H)|}{|U_d(H)|} = \frac{1}{\varsigma(d+1)}\prod_p(1-\frac{1}{p^3}\frac{(1-1/p^{d-1})}{(1-1/p^{d+1})})$

$$= \prod_p(1-\frac{1}{p^{d+1}})\prod_p(1-\frac{1}{p^3}\frac{(1-1/p^{d-1})}{(1-1/p^{d+1})}) = \prod_p(1-\frac{1}{p^3}-\frac{1}{p^{d+1}}+\frac{1}{p^{d+2}}).$$

Consequently, $\lim_{d\to\infty}\lim_{H\to\infty}\frac{|\hat{U}_d(H)|}{|U_d(H)|} = \prod_p(1-\frac{1}{p^3}) = \frac{1}{\varsigma(3)}.$ Q.E.D.

DISCUSSION

The first group of results obtained in this paper is

(I) $\text{Prob}_k^{(g)}(n) = \frac{1}{g^k\varsigma(k)}(\frac{2n}{2n+1})^k + O(\frac{n^{k-1}}{g^{k-1}(2n+1)^k}),\ k \ge 3.$

(II) $\text{Prob}_2^{(g)}(n) = \frac{1}{g^2\varsigma(2)}(\frac{2n}{2n+1})^2 + O(\frac{n}{g(2n+1)^2}\log\frac{n}{g}).$

(III) $\lim_{n\to\infty}\text{Prob}_k^{(g)}(n) = \frac{1}{g^k\varsigma(k)}.$

This extends the results of Nymann (1970) which correspond to the case g = 1.
The second group of results obtained in this work is

(1) $\frac{|\hat{P}_d^{(g)}(H)|}{|P_d^{(g)}(H)|} = |\mu(g)|\, g\prod_{p|g}(\frac{1}{p+1})+O_d(\frac{g}{H}\log\frac{H}{g})$ for $d \ge 2$.

(2) $\lim_{d\to\infty}\lim_{H\to\infty}\frac{|\hat{U}_d(H)|}{|U_d(H)|} = \prod_p(1-\frac{1}{p^3}) = \frac{1}{\varsigma(3)}$;

both provide refinements to the following results of Arno *et al.* (1996)

(i) $\frac{|\hat{P}_d(H)|}{|P_d(H)|} = \prod_p\{1-\frac{1}{p^3}\frac{(1-1/p^{d-1})}{(1-1/p^{d+1})}\}+O_d(\frac{\log^2 H)}{H}),\ d \ge 2.$

(ii) $\lim_{d\to\infty}\lim_{H\to\infty}\frac{\left|\bigcup_{k\le d}\hat{A}_k(H)\right|}{\left|\bigcup_{k\le d}A_k(H)\right|} = \frac{1}{\varsigma(3)}.$

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