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Bessel Function in Galois Fields

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ABSTRACT

Analogues of Bessel functions in function fields of nonzero characteristic are investigated. Certain basic properties , are derived. It was found that

(i) their domain and range are contained in the whole universe,

(ii) basic recurrence relations hold,

(iii) they are not one - to - one,

and (iv) they are linear and have generating function.

Key words: Bessel funtion, Galois field, nonzero characteristic

INTRODUCTION

The following terminology and notation, see Goss (1996), are standard throughout the entire paper :

$F_q[x]$ the ring of polynomials over the Galois (finite) field of characteristic p with p being prime, q being a power of p.

$F_q(x)$ the quotient field of $F_q[x]$.

deg or $|\cdot|$ (or $|\cdot|_\infty$) the nonarchimedean valuation (at ∞) normalized so that $|x| = q^{\deg x} = q$.

$F_q(x_\infty)$ the completion of $F_q(x)$ (at ∞) with respect to $|\cdot|$, this complete field is isomorphic to $F_q\left(\frac{1}{x}\right)$, the field of formal Laurent series in $\frac{1}{x}$.

$F_q(x_\infty)^{clos}$ the algebraic closure of $F_q(x)_\infty$.

Ω the completion of $F_q(x)_\infty^{clos}$.

For $m \in N$, let $[m] = x^{q^m} - x$, $[0] = 0$

$L_0 = 1$, $L_m = [m]\,[m-1] \ldots [1]$

$F_0 = 1$, $F_m = [m]\,[m-1]^q \ldots [1]^{q^{m-1}}$

In the classical case, see eg. Boas (1983), Bessel's equation is a linear differential equation in the form

$$t^2y''(t)+ty'(t)+(t^2-n^2)y(t) = 0$$

where n is a real constant called the order of the function y(t) , which is a solution of this differential equation. It is found that the function

$$J_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\,\Gamma(r+n+1)} \left(\frac{t}{2}\right)^{2r+n} \qquad (1)$$

satisfies the Bessel's equation and is called the Bessel function of the first kind of order n. The domain and range of $J_n(t)$ in the real case is $\mathbb{R}$, the entire field of real numbers. Bessel's equation in the classical case have been studied extensively and whose solutions have been well tabulated. These functions arise naturally in the problems about electricity, heat, hydrodynamics, elastricity, mechanical motions, optics, etc.

The objectives of this work are

(i) to find analogy of these functions in the nonzero characteristic case in Ω

(ii) to derive difference equations satisfied by them

and (iii) to investigate their inverses if exist.

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MATERIALS AND METHODS

Carlitz (1960) defined the following series as an Ω-analogue of Bessel functions

$$J_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r t^{q^{n+r}}}{F_{n+r}F_r^{q^n}} \qquad (2)$$

Carlitz (1960) showed that a generating function of $J_n(t)$ is

$$\sum_{n=-\infty}^{\infty} u^{q^n} J_n(t) = \psi(tG(u))$$

where $J_{-n}(t) := (-1)^n J_n(t)^{q^{-n}}$

and $G(u) := \sum_{r=0}^{\infty} \frac{u^{q^{-r}}}{F_r{}^{q^{-r}}}$

I Valuation (see Geijsel (1979))

For , $E \in F_q[x]$, define deg E := degree of E, deg 0 = -∞.

Extend deg to by $\deg\left(\frac{E}{F}\right) = \deg E - \deg F$

where E, $F \in F_q[x]$, F ≠ 0.

It is known that the completion ($F_q(x)_\infty$, deg) of ($F_q(x)$, deg) exists uniquely together with a prolonged valuation deg. The field ($F_q(x)_\infty$, deg) can be uniquely extended to the algebraic closure, $\left(F_q(x)_\infty^{clos}, \deg\right)$ but the later field is not complete. The completion Ω of $\left(F_q(x)_\infty^{clos}, \deg\right)$ with respect to this valuation and their final field is both algebraically closed and complete and series as universe for all over arguments.

II Functions

Carlitz (1935) introduced the so - called Carlitz ψ - function as a Ω- analogue of exponential function

$$\psi(t) = \sum_{r=0}^{\infty} \frac{(-1)^r t^{q^r}}{F_r} = t\prod_{E}\left(1 - \frac{t}{E\xi}\right)$$

where the product extends over all nonzero $E \in F_q[x]$. The function ψ(t) is entire and linear.

RESULTS AND DISCUSSION

(i) Analogue domain and range

The function defined in (2) is a true Ω - analogue of Bessel's function as defined in (1). The domain of $J_n(t)$ in (2) is Ω , the whole universe, which is the same as in the classical case, while the range is contained in Ω.

Proof From $J_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r t^{q^{n+r}}}{F_{n+r}F_r{}^{q^n}}$,

Since $\deg\left(\frac{(-1)^r t^{q^{n+r}}}{F_{n+r}F_r{}^{q^n}}\right) = q^{n+r}(\deg t - n - 2r)$

$\to -\infty$, for each fixed t, as $\to \infty$,

and so the domain of $J_n(t)$ is Ω. The fact that $J_n(t)$ is linear follows immediately from

$$\frac{(-1)^r (t_1 + t_2)^{q^{n+r}}}{F_{n+r}F_r^{q^n}} = \frac{(-1)^r}{F_{n+r}F_r^{q^n}}\left(t_1^{q^{n+r}} + t_2^{q^{n+r}}\right)$$

(ii) Recurrence relations

The following table give a comparison of recurrence relations in the classical and Ω- analogue cases

Proof of recurrent relations

Since $J_{-n}(t) = \sum_{r=0}^{\infty} \frac{(-1)^r t^{q^{-n+r}}}{F_{-n+r}F_r^{q^{-n}}} = \sum_{s=0}^{\infty} \frac{(-1)^{n+s} t^{q^s}}{F_s F_{n+s}^{q^{-n}}}$

$= J_n{}^{q^{-n}}(t)$

this proves (1).

Since $J_{-n}(xt) - xJ_{-n}(t) = \sum_{r=0}^{\infty} \frac{(-1)^r t^{q^{n+r}}}{F_{n+r}F_r^{q^n}}\left(x^{q^{n+r}} - x\right)$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r t^{q^{n+r}}}{F_{n+r-1}^{q} F_r^{q^n}} = J_{n-r}^{q}(t)$$

Table 1 Comparison of recurrence relations in the classical and Ω- analogue cases.

Classical	Ω- analogue
(1) $J_{-n}(t) = (-1)^n J_n(t)$	(1) $J_{-n}(t) = (-1)^n J_n(t)\, q^{-n}$
(2) $J_{n-1}(t)+J_{n+1}(t) = \frac{2n}{t} J_n(t)$	(2) $J_{n+1}(t) - \left(x^{q^n} - x\right) J_n(t) + J^q_{n-1}(t) = 0$
$J_{n-1}(t)-J_{n+1}(t) = 2J'_n(t)$	$J_n\left(x^2 t\right) - \left(x^{q^n} + x\right) J_n(xt) + x^{q^{n+1}} J_n(t) = J_n^q(t)$
(3) $(t^n J_n(t))' = t^n J_{n-1}(t)$ $(t^n J_n(t))' = -t^{-n} J_{n+1}(t)$	(3) $\Delta^r J_n(t) = J^{q^r}_{n-s}(t)$
(4) $J'_n(t) = -J_{n+1}(t) + \frac{n}{t} J_n(t)$	(4) $J_n(xt) - x^{q^n} J_n(t) = -J_{n+1}(t)$
(5) $J'_n(t) = J_{n-1}(t) - \frac{n}{t} J_n(t)$	(5) $J_n(xt) - xJ_n(t) = J^q_{n-1}(t)$

which is (5).

Since $\Delta J_n(t) = J_n(xt) - xJ_n(t) = J^q_{n-1}(t)$

then by induction

$$\begin{aligned}\Delta^r J_n(t) &:= \Delta^{r-1} J_n(xt) - x^{q^{r-1}} \Delta^{r-1} J_n(t)\\ &= J_n^{q^{r-1}}(xt) - x^{q^{r-1}} J^{q^{r-1}}_{n-r+1}(t)\\ &= (J_{n-r+1}(xt) - xJ_{n-r+1}(t))q^{r-1}\\ &= J^{q^r}_{n-s}(t)\end{aligned}$$

which is (3).

Since $J_{-n}(xt) - x^{q^n} J_n(t)$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r t^{q^{n+r}}}{F_{n+r} F_r^{q^n}} \left(x^{q^{n+r}} - x\right)$$

$$= \sum_{r=1}^{\infty} \frac{(-1)^r t^{q^{n+r}}}{F_{n+r} F_{r-1}^{q^{n+1}}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^{r+1} t^{q^{n+r+1}}}{F_{n+r+1} F_r^{q^{n+1}}} = -J_{n+1}(t)$$

which is (4).

Subtracting (4) from (5), we get

$$J_{n+1}(t) - \left(x^{q^n} - x\right) J_n(t) + J^q_{n-1}(t) = 0$$

which is the first equation of (2).

Since $(J_n(x^2t) - xJ_n(xt)) - x^{q^n}(J_n(xt) - xJ_n(t))$

$$\begin{aligned}&= J^q_{n-1}(xt) - x^{q^n} J^q_{n-1}(t) && \text{(by (5))}\\ &= -J_n^q(t) && \text{(by(4))}\end{aligned}$$

which is the second equation of (2) and is also the difference equation by $J_n(t)$.

(iii) Inverses

Both in the classical and Ω- analogue cases, the Bessel's functions do not have inverses because they are not one to one.

(iv) Other

Table 2 gives a comparison of three other properties

Proof of the generating formula

Since $\psi(tG(u))$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{F_r} (tG(u))^{q^r}$$

Table 2 Comparison of three properties in the classical and Ω-analogue cases.

Classical	Ω-analogue
(1) Linearity : not true	(1) linearity : true
(2) Differential equation	(2) Difference equation
$t^2J''_n(t)+tJ'_n(t)+(t^2-n^2)J_n(t) = 0$	$J_n(x^2t)-(x^{q^n}+x)J_n(xt)+x^{q^n}J_n(t)=-J_n^q(t)$
(3) Generating function	(3) Generating function
$\exp\left(\frac{1}{2}t\left(u-\frac{1}{u}\right)\right) = \sum_{n=-\infty}^{\infty} u^n J_n(t)$	$\psi(tG(u)) = \sum_{n=-\infty}^{\infty} u^{q^n} J_n(t)$

$$\psi(tG(u)) = \sum_{r=0}^{\infty} \frac{(-1)^r t^{q^r}}{F_r} \left(\sum_{s=0}^{\infty} \frac{u^{q^{-s}}}{F_s^{q^{-s}}} \right)^{q^r}$$

$$= \sum_{s=0}^{\infty} \sum_{n=-s}^{\infty} \frac{(-1)^{s+n} t^{q^{s+n}} u^{q^n}}{F_{s+n} F_s^{q^n}}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n u^{q^n} \sum_{s=0}^{\infty} \frac{(-1)^s t^{q^{s+n}}}{F_{s+n} F_s^{q^n}} =$$

$$\sum_{n=-\infty}^{\infty} (-1)^n u^n J_n(t)$$

CONCLUSION

Analogues of Bessel functions are constructed and certain properties are derived. It was found that

(i) their domain and range are contained in the whole universe,

(ii) basic recurrence relations hold,

(iii) they are not one - to - one,

and (iv) they are linear and have generating function.

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