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Denominators of Algebraic Numbers

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ABSTRACT

Formulae for the denominators of algebraic numbers are derived using recent results of Arno, Robinson and Wheeler concerning denominators of integer polynomials. The remaining 16.8 % of algebraic numbers whose denominators are not equal to the leading coefficients in their minimal polynomials is quantitatively analyzed and it is found that the majority of them are numbers of high degrees.

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INTRODUCTION

The denominator of an algebraic number α, denoted by den α, is the least positive integer n such that nα is an algebraic integer. Being constant depending only on the numbers, the denominators of algebraic numbers play important roles in various estimates appeared in the theory of transcendental numbers, see e.g. Lang (1966), Waldschmidt (1974,1979) . Though useful in the theoretical sense, given an algebraic number α, the determination of den α is not an easy task and relatively few results are known. Recently, Arno *et al*.(1996) proved some very interesting and important results about the denominators of algebraic numbers via the introduction of a new concept called the denominator of an integer polynomial. Recall that an integer polynomial is a polynomial having all of its coefficients being rational integers. Given an integer polynomial A(x) all of whose roots are α_k, the denominator of A, also denoted by den A, is the least positive integer n for which all of nα_k are algebraic integers. By the minimal polynomial of an algebraic number α over the ring of rational integers, we mean an integer polynomial with root α which is irreducible over $\mathbb{Z}[x]$ and has positive leading coefficient. Note that we employ here a slightly different definition of minimal polynomial, cf. p. 44 of Pollard and Diamond (1975).It clearly follows from this definition that if $A(x) = \sum_{i=0}^{t} a_i x^i$ is the minimal polynomial of α over ⊠, then den A = den α. It is also well-known, see e.g. Waldschmidt (1974,1979) or Pollard and Diamond (1975), that $a_t\alpha$ is an algebraic integer and so

$$\text{den } \alpha \leq a_t \qquad (1)$$

Arno et al. (1996) addressed the following two questions:

Question 1: For what proportion of the algebraic numbers of degree t, does equality hold in (1) ?

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Question 2: For what proportion of all algebraic numbers does equality hold in (1) ?

They gave precise quantitative results to both questions. Particularly of interest to us is the estimate that equality holds for approximately 83.2% of the total algebraic numbers. Not only did they give a number of useful estimates, they also proved a simple yet significant theorem enabling us to compute denominators of integer polynomials.

The objectives of this work first is to apply this latter theorem of Arno, Robinson and Wheeler to derive formulae for the denominators of algebraic numbers, and second is to analyze the remaining 16.8% of all algebraic numbers for which equality does not hold in (1). It was proved that algebraic numbers of high degrees constitute most of them. This is confirmed by explicit numerical computations.

MATERIALS AND METHODS

Formulae

Lemma 1. Let $A(x) := a_t x^t + \ldots + a_1 x + a_0 \in \mathbb{Z}[x]$ be a nonconstant polynomial of degree t. Then den $A \mid a_t$ and

(i) den $A = \min \{ n \geq 1 : n \mid a_t , (n^t/a_t) A(x/n) \in \mathbb{Z}[x] \}$,

(ii) den $A = \min \{ n \geq 1 : n \mid a_t , a_t / (a_t , a_{t-j}) \mid n^j \ (1 \leq j \leq t) \}$,

(iii) $$\text{den } A = \prod_{p \mid a_t} p^{n_p}, \text{ where } n_p = \max \{0, \max_{1\leq j\leq t} [(v_p(a_t)-v_p(a_{t-j}))/j] \},$$

and $v_p(m)$ denotes the exponent of the largest power of the prime p dividing m.

Proof. This is Theorem 1 of Arno *et al*. (1996).

Now use Lemma 1 to obtain the following formulae for the denominators of algebraic numbers.

Theorem 1. Let α be an algebraic number of degree t having $A(x) := a_t x^t + \ldots + a_1 x + a_0 \in \mathbb{Z}[x]$ as its minimal polynomial over $\mathbb{Z}$.

(i) For t = 1, we have den $\alpha = a_1$.

(ii) For $t \geq 2$, we have

(ii.1) if $(a_t , a_{t-1}) = 1$, then den $\alpha = a_t$,

(ii.2) if $(a_t , a_{t-1}) \neq 1$, then

$$\text{den } \alpha = \prod_{p \mid a_t} p^{n_p}, \text{ where } n_p = \max \{0, \max_{1\leq j\leq t} [(v_p(a_t)-v_p(a_{t-j}))/j] \},$$

and $v_p(m)$ denotes the exponent of the largest power of the prime p dividing m.

Proof. (i) For t = 1, α is an algebraic number of degree 1, i.e. a rational number with $A(x) = a_1 x + a_0$ as its minimal polynomial over $\mathbb{Z}$. By Lemma 1(ii) and the fact that $(a_1,a_0) =1$, it can be deduced that

den α = den $A = \min \{ n \geq 1 : n \mid a_1, a_1 / (a_1,a_0) \mid n \} = a_1$.

(ii) For $t \geq 2$, separate into two cases.

(ii.1) If $(a_t , a_{t-1}) = 1$, then $v_p(a_{t-1}) = 0$ for each prime factor p of a_t. From Lemma 1(iii),

$n_p = \max \{0, \max_{1\leq j\leq t} [(v_p(a_t)-v_p(a_{t-j}))/j] \}= v_p(a_t)$,

and so

$$\text{den } \alpha = \text{den } A = \prod_{p \mid a_t} p^{n_p} = a_t.$$

(ii.2) If $(a_t, a_{t-1}) \neq 1$, then the result in this case follows immediately from Lemma 1(iii).

Density

Let α be an algebraic number having $A(x) := a_t x^t + \ldots + a_1 x + a_0 \in \mathbb{Z}[x]$ as its minimal polynomial over $\mathbb{Z}$. Then the **height** of α, denoted by $H(\alpha)$ is defined to be $H(\alpha) := \max\{|a_i|; 0 \le i \le t\}$. The following notation is used throughout this section.

$$P_t(H) := \{ A(x) = \sum_{i=0}^{t} a_i x^i \in \mathbb{Z}[x] \,;\, a_t \neq 0, |a_i| \le H, A(x) \text{ is primitive} \}$$

$$P_t^*(H) := \{ A(x) \in P_t(H) \,;\, \text{den } A = |a_t| \}$$

$$I_t(H) := \{ A(x) \in P_t(H) \,;\, A(x) \text{ is irreducible over } \mathbb{Z}[x] \}$$

$$I_t^*(H) := \{ A(x) \in I_t(H) \,;\, \text{den } A = |a_t| \}$$

$$N_t(H) := \{ \alpha \,;\, \alpha \text{ is an algebraic number with deg } \alpha = t,\ H(\alpha) \le H \}$$

$$N_t^*(H) := \{ \alpha \in N_t(H) \,;\, \text{den } \alpha = \text{the leading coefficient of its minimal polynomial over } \mathbb{Z} \}.$$

Those algebraic numbers whose denominators equal the leading coefficients of their minimal polynomials over $\mathbb{Z}$ are called **good** numbers, and the rest as **peculiar** numbers.In Arno *et al.* (1996) the following lemma is proved.

Lemma 2. Let $t \ge 1$ be fixed. Uniformly for $H \ge 2$, we have

(i) $$\frac{2}{t}|N_t(H)| = |I_t(H)| = \frac{(2H)^{t+1}}{\varsigma(t+1)}\left(1 + O_t\left(\frac{\log^2 H}{H}\right)\right),$$

(ii) $$\frac{2}{t}|N_t^*(H)| = |I_t^*(H)| = \frac{(2H)^{t+1}}{\varsigma(t+1)}\prod_p\left(1 - \frac{1}{p^3}\frac{1-\frac{1}{p^{t-1}}}{1-\frac{1}{p^{t+1}}}\right) + O_t(H^t \log^2 H).$$

As mentioned in the introduction, Arno, Robinson and Wheeler have also discovered that the ratio of algebraic numbers of degree t whose denominators are not the leading coefficients of their minimal polynomials (over $\mathbb{Z}$) to all algebraic numbers of degree t is equal to $r(t,H) = \frac{|N_t(H)| - |N_t^*(H)|}{|N_t(H)|}$. Our task now is to analyze this particular quantity.

Theorem 2. Let H be a large but fixed positive number.Let α be an algebraic number having $A(x) := a_t x^t + \ldots + a_1 x + a_0 \in \mathbb{Z}[x]$ as its minimal polynomial over $\mathbb{Z}$.

(i) If $t = 1$, then $r(1,H) = 0$.

(ii) If $t \ge 2$, then $r(t,H) = 1 - \pi(t) + O_t((\log^2 H)/H)$,

where $$\pi(t) := \prod_p\left(1 - \frac{1}{p^3}\frac{1-\frac{1}{p^{t-1}}}{1-\frac{1}{p^{t+1}}}\right).$$

Proof. (i) If t = 1, by Theorem 1(i), den $\alpha = a_1$, the leading coefficient of its minimal polynomial over $\mathbb{Z}$ in the minimal polynomial and so r(1,H) = 0.

(ii) If t ≥ 2, by Lemma 2,

$$r(t,H) = \frac{\frac{t}{2}\frac{(2H)^{t+1}}{\varsigma(t+1)}\left(1+O_t\left(\frac{\log^2 H}{H}\right)\right) - \frac{t}{2}\frac{(2H)^{t+1}}{\varsigma(t+1)}\pi(t) + O_t(H^t \log^2 H)}{\frac{t}{2}\frac{(2H)^{t+1}}{\varsigma(t+1)}\left(1+O_t\left(\frac{\log^2 H}{H}\right)\right)}$$

$$= \frac{\frac{(2H)^{t+1}}{\varsigma(t+1)}(1-\pi(t)+O_t(H^t \log^2 H))}{\frac{(2H)^{t+1}}{\varsigma(t+1)}\left(1+O_t\left(\frac{\log^2 H}{H}\right)\right)} = 1 - \pi(t) + O_t((\log^2 H)/H), \text{ as desired.}$$

Remark. It follows immediately from Theorem 2 that there is no peculiar numbers of degree 1 and the ratio of peculiar numbers of degree t (≥2) to all algebraic numbers of degree t is $1 - \pi(t) + O_t((\log^2 H)/H)$.

The main term in this last expression is now analyzed.

Lemma 3. The sequence (π(t)) is a sequence of positive numbers strictly decreasing with t with limiting value $1/\zeta(3) \approx 0.831\,907\,372\,580\,811$.

Proof. Let $s_t := \dfrac{1-\frac{1}{p^{t-1}}}{1-\frac{1}{p^{t+1}}}$. It is easy to see that (s_t) is a strictly increasing sequence of positive numbers whose limit is 1, and so $(1 - s_t/p^3)$ is a strictly decreasing sequence of positive numbers whose limit is $1-1/p^3$. Consequently, (π(t)) is a strictly decreasing sequence of positive numbers whose limit is, see e.g. Apostol (1980),

$$\lim_{t\to\infty}\prod_p\left(1-\frac{s_t}{p^3}\right) = \frac{1}{\varsigma(3)}, \text{ as to be proved.}$$

Remarks. An immediate consequence of Lemma 3 is the fact that the ratio of peculiar numbers of degree t to all algebraic numbers of degree t is an increasing function of t whose limiting value is $1-1/\zeta(3) \approx 0.168\,092\,627\,419\,290$. This implies that the majority of peculiar numbers comes from algebraic numbers of high degrees.

The question of how high the degrees should be is now answered by estimating the difference π(t) - π(1+t).

Theorem 3. We have $\pi(t) - \pi(1+t) = O(1/2^{t+2}) \quad (t \to \infty)$.

Proof. Using the same notation for s_t as in Lemma 3, we get

$$s_{t+1} - s_t = \frac{1}{p^{t-1}}\left(1-\frac{1}{p}\right) - \frac{1}{p^{t+1}}\left(1-\frac{1}{p}\right) + \frac{1}{p^{2t}}\left(1-\frac{1}{p^2}\right) - \frac{1}{p^{2t+2}}\left(1-\frac{1}{p^2}\right)$$

$$+\frac{1}{p^{3t+1}}\left(1-\frac{1}{p^3}\right)-\frac{1}{p^{3t+3}}\left(1-\frac{1}{p^3}\right)+\dots$$

$$<\left(1-\frac{1}{p^2}\right)\left(\frac{1}{p^{t-1}}+\frac{1}{p^{2t}}+\frac{1}{p^{3t+1}}+\dots\right)\le\frac{1}{p^{t-1}},$$

and so

$$0 < s_{t+1} < s_t + \frac{1}{p^{t-1}}.$$

Thus when $n \ge 2$, we get

$$0 < s_{t+1}^n < \sum_{k=0}^{n}\binom{n}{k}\frac{s_t^{n-k}}{p^{(t-1)k}} < s_t^n + \frac{2^n}{p^{t-1}}. \quad (2)$$

Since

$$\log \pi(t) = \sum_p \log\left(1-\frac{s_t}{p^3}\right) = -\sum_p\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{s_t}{p^3}\right)^n,$$

using (2) , we have

$$\log\{\pi(t)/\pi(1+t)\} = \sum_p\sum_{n=1}^{\infty}\frac{1}{n}\frac{s_{t+1}^n - s_t^n}{p^{3n}}$$

$$< \sum_p \frac{1}{p^{t+2}}\sum_{n=1}^{\infty}\frac{2^n}{np^{3n-3}} = O(1/2^{t+2}),$$

and so

$$\pi(t)/\pi(1+t) = 1 + O(1/2^{t+2}) \quad (t \to \infty).$$

This together with Lemma 3 yield

$$\pi(t) - \pi(1+t) = \pi(1+t)\{\pi(t)/\pi(1+t) - 1\} = O(1/2^{t+2}),$$

as desired.

RESULTS AND DISCUSSIONS

From the results in Theorem 2, Lemma 3 and Theorem 3, we deduce that the majority of peculiar numbers, which amount to approximately 16.8% of all algebraic numbers, comes from numbers of high degrees. Quantitatively, the ratio of peculiar numbers of degree t to all algebraic numbers of degree t approaches its limiting value of $1-1/\zeta(3) \approx 0.168\,092\,627\,419\,290$ quite rapidly as t increases. Indeed, for $t \ge 25$, the error term $O(1/2^{t+2})$ tells us that this ratio should differ from its limit, $1-1/\zeta(3)$, by at least 10^{-8}. This is confirmed by our numerical computation, given in Table 1, where we take for approximation the first 2,500 primes and the values of t in the range 1 to 60. The table clearly shows that from t = 25 onwards, the values of $1-\pi(t)$, which represents the ratio asymptotically, agrees with $1-1/\zeta(3)$ up to at least 8 decimal places.

Table 1 Value of π(t) and 1 – π(t).

$$\text{Where } \pi(t) = \prod_p \left(1 - \frac{1}{p^3} \frac{1 - \frac{1}{p^{t-1}}}{1 - \frac{1}{p^{t-1}}} \right), \text{ using the first 2500 primes (p)}$$

deg t	π(t)	1 – π(t)
t = 2	0.895166360432376000	0.104833639567624000
t = 3	0.859292678315528000	0.140707321684472000
t = 4	0.844540831074330000	0.155459168925670000
t = 5	0.837932043118304000	0.162067956881696000
t = 6	0.834834412281501000	0.165165587718499000
t = 7	0.833345052154378000	0.166654947845622000
t = 8	0.832618184713964000	0.167381815286036000
t = 9	0.832260238811366000	0.167739761188634000
t =10	0.832082990684385000	0.167917009315615000
t =11	0.831994916755863000	0.168005083244137000
t =12	0.831951057192022000	0.168048942807978000
t =13	0.831929185088860000	0.168070814911140000
t =14	0.831918267907114000	0.168081732092886000
t =15	0.831912815518448000	0.168087184481552000
t =16	0.831910091369841000	0.168089908630159000
t =17	0.831908729972076000	0.168091270027924000
t =18	0.831908049497366000	0.168091950502634000
t =19	0.831907709334411000	0.168092290665589000
t =20	0.831907539277645000	0.168092460722355000
t =21	0.831907454257483000	0.168092545742517000
t =22	0.831907411750134000	0.168092588249866000
t =23	0.831907390497377000	0.168092609502623000
t =24	0.831907379871283000	0.168092620128717000
t =25	0.831907374558359000	0.168092625440641000
t = 26	0.831907371901919000	0.168092628098081000
t = 27	0.831907370573714000	0.168092629426286000
t = 28	0.831907369909618000	0.168092630090382000
t = 29	0.831907369577567000	0.168092630422433000
t = 30	0.831907369411543000	0.168092630588457000
t = 31	0.831907369328528000	0.168092630771472000
t = 32	0.831907369287021000	0.168092630712979000
t = 33	0.831907369266271000	0.168092630733729000
t = 34	0.831907369255895000	0.168092630744105000
t = 35	0.831907369250698000	0.168092630749302000
t = 36	0.831907369248108000	0.168092630751892000
t = 37	0.831907369246814000	0.168092630753186000

deg t	π(t)	1 – π(t)
t = 38	0.831907369246170000	0.168092630753830000
t = 39	0.831907369245847000	0.168092630754153000
t = 40	0.831907369245678000	0.168092630754322000
t = 41	0.831907369245594000	0.168092630754406000
t = 42	0.831907369245559000	0.168092630754441000
t = 43	0.831907369245533000	0.168092630754467000
t = 44	0.831907369245524000	0.168092630754476000
t = 45	0.831907369245527000	0.168092630754473000
t = 46	0.831907369245519000	0.168092630754481000
t = 47	0.831907369245517000	0.168092630754483000
t = 48	0.831907369245521000	0.168092630754479000
t = 49	0.831907369245525000	0.168092630754475000
t = 50	0.831907369245520000	0.168092630754480000
t = 51	0.831907369245515000	0.168092630754485000
t = 52	0.831907369245519000	0.168092630754481000
t = 53	0.831907369245518000	0.168092630754482000
t = 54	0.831907369245519000	0.168092630754481000
t = 55	0.831907369245519000	0.168092630754481000
t = 56	0.831907369245519000	0.168092630754481000
t = 57	0.831907369245521000	0.168092630754479000
t = 58	0.831907369245517000	0.168092630754483000
t = 59	0.831907369245518000	0.168092630754482000
t = 60	0.831907369245518000	0.168092630754482000

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