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Integration in Finite Terms which Includes Exponential Integral

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ABSTRACT

This paper generalizes the Liouville's theorem on integration in finite terms by extending the class of fields to an extension, called Ei extension, which includes the elementary extension and contains exponential integral.

Key words : Liouville's theorem, integration in finite terms

INTRODUCTION

The problem of integration in finite terms is that given a γ in a differential field F with derivation D, we ask when a solution of $D(y) = \gamma$ can be expressed in certain special forms. The answer is given by Liouville's theorem: Let F be a differential field of characteristic zero and $\gamma \in F$. If $D(y) = \gamma$ has a solution in an elementary extension E of F having the same subfield of constants, then there are constants $c_1, c_2, \ldots, c_n$ in F and elements $u_1, u_2, \ldots, u_n, v$ in F such that

$$\gamma = D(v) + \sum_{i=1}^{n} c_i \frac{D(u_i)}{u_i}.$$

M. Rosenlicht gave a completely algebraic proof of Liouville's theorem as described in his series of papers (Rosenlicht , 1968,1972,1976).

In this paper we give a generalization of Liouville's theorem called Ei extension, by extending the class of fields from elementary to another class of fields containing strategically the exponential integral. As an application, we give a sufficient condition for certain functions to be Ei integrable.

In section 2 we state a result of Rosenlicht(1976) and Rothstien and Caviness (1979) that is used in the proof of the main result and define our extended class of fields.

In section 3 we give the main result of this paper and prove the main theorem.

All fields are assumed to be of characteristic zero. $\boldsymbol{Q}$, $\boldsymbol{Z}$ and $\boldsymbol{Z}+$ stand for the set of rational numbers, the set of integers and the set of positive integers, respectively.

MATERIALS AND METHODS

In this section we state a result from Rosenlicht (1976) and Rothstein and Caviness (1979) that will be used later.

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Lemma 2.1 (Rosenlicht, 1976). Let F be a differential field, K a differential extension field of F with the same subfield of constants, with K algebraic over $F(t)$ for some given $t \in K$. Suppose that $c_1,...,c_n$ are constants of F that are linearly independent over Q, that $u_1,...,u_n$, v are elements of K, with $u_1,...,u_n$ nonzero, and that for each given derivation D of K we have

$$\sum_{i=1}^{n} c_i D(u_i)/u_i + D(v) \in F.$$

If for each given derivation D of K we have $D(t) \in F$, then $u_1,...,u_n$ are algebraic over F and there exists a constant c of F such that $v - ct$ is algebraic over F. If for each given derivation D of K we have $D(t)/t \in F$, then v is algebraic over F and there are integers $m_0, m_1, ..., m_n$, with $m_0 \neq 0$, such that each $u_i^{m_0} t^{m_i}$ $(i = 1,...,n)$ is algebraic over F.

Definition. Let F be a differential field with a derivation D and E a differential extension of F. We say that $t \neq 0$ is an *exponential* over F if $\frac{D(t)}{t} = D(a)$ for some a in F, we write $t = exp(a)$. We call t an *integral* over F if $D(t) = a$ for some a in F; in this case we write $t = \int a$ If $D(t) = \frac{D(u)}{u}$ for for some nonzero element u in F, we write $t = log(u)$, and call t *logarithmic* over F. We say that t is *simple logarithmic* over F if there exist $u_1, u_2, ..., u_m \in F$ such that for some constant c,

$t + c \in F(log(u_1),...,log(u_m))$. We say that t is *nonsimple* if it is not simple logarithmic over F.

E is a *generalized log-explicit extension* of F if there exists a finite tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n = E$ such that for each $i = 1,...,n$, $F_i = F_{i-1}(t_i)$ and one of the following holds :

(i) $t_i = exp(u_i)$ for some u_i in F_{i-1},
(ii) t_i is integral and nonsimple over F_{i-1},
(ii) $t_i = log(u_i)$ for some nonzero element u_i in F_{i-1},
(iv) t_i is algebraic over F_{i-1}.

Lemma 2.2 (Rothstein and Caviness , 1979) . Let $F = C(t_1,...,t_n)$ be a generalized log-explicit extension field of C, where C is the field of constants of F. If $u \neq 0$ and v are members of F such that $\frac{D(u)}{u} = D(v)$, then there exist rational numbers r_i and a constant c such that

$$v = c + \sum_{i \in L} r_i t_i + \sum_{i \in E} r_i a_i,$$

where

$$L = \{ i / t_i = log(a_i), a_i \in F_{i-1} - \{0\}, 1 \le i \le n \},$$

$$E = \{ i / t_i = exp(a_i), a_i \in F_{i-1}, 1 \le i \le n \},$$

and $t_i \;=\; exp(a_i)$ for $i \in E$.

We now define our special class of functions.

Definition. Let F be a differential field with derivation D and C its subfield of constants . We say that a $\gamma \in F$ is an *Ei element* over F if there exist

(i) $a_i \in C$, v_0 algebraic over F and nonzero elements v_i algebraic over F for all $i \in I$,

(ii) $b_i \in C$, nonzero elements w_i,and x_i algebraic over F for all $i \in J$, such that

$$\gamma \;=\; D(v_0) + \sum_{i \in I} a_i \frac{D(v_i)}{v_i} + \sum_{i \in J} b_i \frac{D(w_i)}{w_i} x_i ,$$

where I and J are all finite indexing sets and $\dfrac{D(x_i)}{x_i} \;=\; D(w_i)$ for all $i \in J$.

We say that a differential extension E of F is an *Ei extension* of F if there exists a finite tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n = E$ such that for each $i = 1,\ldots,n$, $F_i = F_{i-1}(t_i)$ and one of the following holds:

(i) t_i is algebraic over F_{i-1},

(ii) $D(t_i)$ or $D(t_i)/t_i$ is an Ei element over F_{i-1}.

Remarks. 1. Every elementary extension of F is an Ei extension of F (Rosenlicht,1968,1972,1976) .

2. The Ei extension contains the exponential integral which is defined by

$$Ei(u) \;=\; \int \frac{D(u)}{u}\, exp(u).$$

Example. Let C be the field of complex numbers and let $F = C(x)$ be the set of rational functions with coefficients in C. Then F is a differential field under the usual derivation $D = d/dx$. Hence

$F(exp(x), log(x), \int \dfrac{(exp(x)+1)}{x} D(x))$ is an Ei extension of F.

RESULTS

Theorem 3.1. Let F be a differential field with derivation D and subfield of constants C. Let $\gamma \in F$. Assume that there exist an Ei extension E of F whose subfield of constants is C and $y \in E$ such that $D(y) = \gamma$. Then γ is an Ei element over F.

The proof of the Theorem is by induction on the transcendence degree of E over F and it suffices to consider only transcendental extension of degree 1 corresponding to each of the adjoined elements. These are done in Lemmas 3.2 and 3.3.

Before proving the lemmas, it will be convenient to define the following term:

Definition. If f and g are polynomials over a field F, and $g \neq 0$, then there exist unique polynomials $q(X) = a_0 + a_1 X + \cdots + a_n X^n$ and $r(X)$ over F such that $f(X)/g(X) = q(X) + r(X)/g(X)$, where $r(X) = 0$ or $\deg r(X) < \deg g(X)$. Call the unique element a_0 the *head* of f/g.

Lemma 3.2. Let F be a differential field with derivation D and C its subfield of constants. Let t be transcendental over F such that $D(t)/t$ is an Ei element over F. Assume that the subfield of constants of $F(t)$ is C. If $\gamma \in F$ is an Ei element over $F(t)$, then γ is also an Ei element over F.

Proof. Since γ is an Ei element over $F(t)$, then there exist

(i) $a_i \in C$, v_0 algebraic over F(t) and nonzero elements v_i algebraic over $F(t)$ for all $i \in I$,

(ii) $b_i \in C$, nonzero elements w_i, and x_i algebraic over $F(t)$ for all $i \in J$,

such that

$$\gamma = D(v_0) + \sum_{i \in I} a_i D(v_i)/v_i + \sum_{i \in J} b_i \frac{D(w_i)}{w_i} x_i ,$$

where I and J are all finite indexing sets, and $\frac{D(x_i)}{x_i} = D(w_i)$ for all $i \in J$. We may assume that F is algebraically closed, for if F is not algebraically closed, then let $\bar{F}$ be an algebraic closure of F. Note that $\bar{F}(t)$, $\bar{F}$, F have the same subfield of constants C. Since γ is an Ei element over $F(t)$ and $F(t) \subset \bar{F}(t)$, γ is also an Ei element over $\bar{F}(t)$. In this case, we could replace F by $\bar{F}$. It is easy to see that if γ is an Ei element over $\bar{F}$, then γ is also an Ei element over F. For each $i \in J$, we have $D(x_i) = D(w_i)x_i$, then by Lemma 2.1, we have that $w_i \in F$ and there exist rational numbers υ_i and p_i in F such that $x_i = p_i t^{\upsilon_i}$. Note that we can arrange so that υ_i are actually integers. To see this, let $\upsilon_i = g_i/m$ where g_i and m are integers. Let $\bar{t} = t^{1/m}$. Hence $\frac{D(\bar{t})}{\bar{t}} = \frac{1}{m}\frac{D(t)}{t}$ and $F \subset F(\bar{t})$.

If we replace t by $\bar{t}$, we still have fields of the appropriate form and furthermore, $x_i = p_i \bar{t}^{g_i}$ where g_i, are integers. We shall use the old notation but from now on assume that υ_i is integer. Let K be a finite Galois extension over $F(t)$ and let σ be an element of the Galois group of K over $F(t)$. Then

$$\gamma = \sigma(\gamma) = D(\sigma v_0) + \sum_{i \in I} a_i D(\sigma v_i)/(\sigma v_i) + \sum_{i \in J} b_i \frac{D(w_i)}{w_i} x_i .$$

Summing over all σ yields, for some M in $\mathbf{Z}$,

$$(1) \qquad M\gamma = D(Tv_0) + \sum_{i \in I} a_i D(Nv_i)/(Nv_i) + M \sum_{i \in J} b_i \frac{D(w_i)}{w_i} x_i ,$$

where T and N denote the trace and norm of K into $F(t)$ respectively.

Write $Tv_0 = \sum_{i=0}^{n} h_i t^i + \sum\sum (a_{ij}/(t-t_i)^j)$,

where h_i, a_{ij} and t_i are in F. Hence the head of $D(Tv_0)$ is $D(h_0)$.

For each $i \in I$ write $Nv_i = k_i \prod_{j=1}^{\alpha_i} (t - \mu_j)^{n_{ij}}$,

where the $\alpha_i \in \mathbf{Z}+$, the $k_i \in F\backslash\{0\}$, the $\mu_j \in F$ and the $n_{ij} \in \mathbf{Z}$.

Therefore the head of $\sum_{i\in I} a_i D(Nv_i)/(Nv_i)$ is $\sum_{i\in I} a_i D(k_i)/k_i + \sum_{i\in I}\sum_{j=1}^{\alpha_i} a_i n_{ij}\frac{D(t)}{t}$.

For each $i \in J$, recall $x_i = p_i \ t^{\upsilon_i}$.

Therefore the head of $\sum_{i\in J} b_i \frac{D(w_i)}{w_i} x_i$ is $\sum_{\substack{i\in J\\ \upsilon_i=0}} b_i \frac{D(w_i)}{w_i} p_i$.

We conclude that the head of the right hand side of (1) is

$$D(\bar{v}_0) + \sum \bar{a}_i \frac{D(\bar{v}_i)}{\bar{v}_i} + \sum \bar{b}_i \frac{D(\bar{w}_i)}{\bar{w}_i}\bar{x}_i ,$$

where v_0, $x_i \in F$, $\bar{v}_i, \bar{w}_i \in F\backslash\{0\}$, $\bar{a}_i$, $\bar{b}_i \in$ C and $\frac{D(\bar{x}_i)}{\bar{x}_i} = D(\bar{w}_i)$.

Then comparing the head of (1) and dividing by M, we get the correct sum of $'$.

Lemma 3.3. Let F be a differential field with derivation D and C its subfield of constants. Let t be transcendental over F such that D(t) is an Ei element over F. Assume that the subfield of constants of F(t) is C. If γ F is an Ei element over F(t), then γ is also an Ei element over F.

Proof. Since γ is an Ei element over F(t), then there exist

(i) ai $\in$ C, vo algebraic over F(t) and nonzero elements vi algebraic over F(t) for all i * I,

(ii) bi $\in$ C, nonzero elements wi, and xi algebraic over F(t) for all i $\in$ J,

such that

$$\gamma = D(v_0) + \sum_{i\in I} a_i D(v_i)/v_i + \sum_{i\in J} b_i \frac{D(w_i)}{w_i} x_i ,$$

where I and J are all finite indexing sets, and $\frac{D(x_i)}{x_i} = D(w_i) \quad i \in J$

Similar to Lemma 3.2, we may assume that F is algebraically closed.

For each $i \in J$, we have that $D(x_i) = D(w_i)x_i$, then by Lemma 2.1, we get $x_i \in F$ and there exist $\lambda_i \in C$, $p_i \in F$ such that $w_i = \lambda_i t + p_i$.

Let K be a finite Galois extension over $F(t)$ and let σ be an element of the Galois group of K over $F(t)$. Then

$$\gamma = \sigma(\gamma) = D(\sigma v_0) + \sum_{i\in I} a_i D(\sigma v_i)/(\sigma v_i) + \sum_{i\in J} b_i \frac{D(w_i)}{w_i} x_i .$$

Summing over all σ yields, for some M in **Z**,

$$\text{(2)} \qquad M\gamma = D(Tv_0) + \sum_{i\in I} a_i D(Nv_i)/(Nv_i) + M\sum_{i\in J} b_i \frac{D(w_i)}{w_i} x_i ,$$

where T and N denote the trace and norm of K into $F(t)$ respectively.

Consider $\sum a_i \dfrac{D(Nv_i)}{Nv_i}$.

Write $Nv_i = k_i \prod_{j=1}^{\alpha_i} (t-\mu_j)^{n_{ij}}$ where the $\alpha_i \in \mathbf{Z}^+$, the $k_i \in F\setminus\{0\}$, the $\mu_i \in F$ and the $n_{ij} \in \mathbf{Z}$.

So $\sum_{i\in I} a_i \dfrac{D(Nv_i)}{Nv_i} = \sum_{i\in I} a_i \dfrac{D(k_i)}{k_i}$ + an element in $F(t)\setminus F[t]$.

Next, consider $\sum b_i \dfrac{D(w_i)}{w_i} x_i$. Recall $w_i = \lambda_i t + p_i$ for all $i \in J$.

If $\lambda_i = 0$ then $w_i \in F$.

Assume that $\lambda_i \neq 0$. Therefore $\dfrac{D(w_i)}{w_i} = \dfrac{\lambda_i D(t) + D(p_i)}{\lambda_i t + p_i} \in F(t)\setminus F[t]$.

From (2) we conclude that

(3) $\quad M\gamma = D(Tv_0) + \sum_{i\in I} a_i \dfrac{D(k_i)}{k_i} + M \sum_{\lambda_i = 0} b_i \dfrac{D(w_i)}{w_i} x_i$ + an element in $F(t)\setminus F[t]$.

Now consider $D(Tv_0)$.

Write $Tv_0 = \sum_{j=0}^{n} \bar{v}_j t^j$ + an element in $F(t)\setminus F[t]$, where n is nonnegative integer and the $\bar{v}_j \in F$ and $\bar{v}_n \neq 0$. So

(4) $\quad D(Tv_0) = D(\bar{v}_n)t^n + \sum_{j=1}^{n} \left(j\,\bar{v}_j D(t) + D(\bar{v}_{j-1})\right) t^{j-1}$ + an element in $F(t)\setminus F[t]$.

We now prove that $n \leq 1$. Suppose that $n > 1$. Replacing (4) in (3), we have that the right hand side of (3) would contain an expression of the form t^i with $i \geq 2$.

Comparing the terms of degree n and n-1 in (3), $D(\bar{v}_n) = 0$ and $(n\,\bar{v}_n D(t) + D(\bar{v}_{n-1})) = 0$. Since $D(\bar{v}_n) = 0$, $\bar{v}_n \in$ C.

Thus $D(n\,\bar{v}_n t + \bar{v}_{n-1}) = n\,\bar{v}_n D(t) + D(\bar{v}_{n-1}) = 0$.

So $n\,\bar{v}_n t + \bar{v}_{n-1} \in C$. Thus t is algebraic over F, a contradiction.

From (4), we get $D(Tv_0) = D(\bar{v}_1)t + (\bar{v}_1 D(t) + D(\bar{v}_0))$ + an element in $F(t)\setminus F[t]$.

Considering the degree of t in (3), we get $\bar{v}_1 \in C$.

Hence $D(Tv_0) = v_1 D(t) + D(\bar{v}_0)$ + an element in $F(t)\backslash F[t]$.
Replacing $D(Tv_0)$ in (3) and comparing the head, we get

$$(5) \qquad M\gamma = v_1 D(t) + D(v_0) + \sum_{i \in I} a_i D(k_i)/k_i + M \sum_{\lambda_i = 0} b_i \frac{D(w_i)}{w_i} x_i .$$

Dividing by M, we obtain the correct sum of $'$.

Proof of Theorem 3.1. Let m = tr.deg. E/F. The proof is by induction on m. If $m = 0$, then E is algebraic over F, and the theorem is trivially true. Assume that $m > 0$. Suppose that the theorem is true for any Ei extension L of a field F' such that tr.deg. $L/F' < m$. Since tr.deg. $E/F = m$, we can choose a transcendence basis $t_1,...,t_m$ of E over F such that $F = F_0 \subset F_{i-1} = F_1 \subset \cdots \subset F(t_1,...,t_m) = F_m \subset E$ and each t_i satisfies either $D(t_i)$ or $\frac{D(t_i)}{t_i}$ is an Ei element over $\bar{F}_{i-1} \cap E$.

($\bar{F}_{i-1}$ denote the algebraic closure of F_{i-1}).

Note that E is also a Ei extension of F_1 and tr.deg. $E/F_1 = m-1 < m$. So by the induction hypothesis, we get that γ is an Ei element over F_1. By Lemmas 3.2 and 3.3, we get the result of the theorem.

Definition. Let γ be an element of a differential field F with derivation D. We call γ is an *Ei integrable* over F if there exist an Ei extension E of F with the same subfield of constants as F such that $\int \gamma \in E$.

Theorem 3.4 Let C be a differential field of constants with a derivation D. Let x be transcendental over C with $D(x) = 1$. Let f and g be elements of $C(x)$. If $g\ exp(f)$ is Ei integrable over $C(x, \exp(f))$ then there exist $d_1, d_2,...,d_n$ in C ,nonzero elements $w_1, w_2,..., w_n$ in $C(x)$ and v in $C(x)$ such that

$$g = D(v) + vD(f) + \sum_{i=1}^{n} \frac{D(w_i)}{w_i},$$

where for each w_i there exists c_i in C such that $w_i = f + c_i$.

Proof. Let $t = exp(f)$.

By Theorem 3.1 , gt is an Ei element over $C(x,t)$.

So there exist

(i) $a_i \in C$, v_0 algebraic over $C(x,t)$ and nonzero elements v_i algebraic over $C(x,t)$ for all $i \in$ I ,

(ii) $b_i \in C$, nonzero elements w_i and x_i algebraic over $C(x,t)$ for all $i \in$ J,

such that

$$(6) \qquad gt = D(v_0) + \sum_{i \in I} a_i \frac{D(v_i)}{v_i} + \sum_{i \in J} b_i \frac{D(w_i)}{w_i} x_i ,$$

where I and J are all finite indexing sets and $\frac{D(x_i)}{x_i} = D(w_i)$ for all $i \in J$.

By Lemma 2.2 there exist $r_i \in \mathbf{Q}$, $c_i \in C$ such that $w_i = c_i + r_i f$.
Similar to Lemma 3.2, we may assume that r_i are integers.

Since $D(t^{r_i}) = t^{r_i} D(w_i)$ and $D(w_i) = \frac{D(x_i)}{x_i}$, $D(x_i t^{-r_i}) = 0$. Thus $x_i = d_i t^{r_i}$ for some $d_i \in C$. From (6), we have

$$gt = D(v_o) + \sum a_i \frac{D(v_i)}{v_i} + \sum b_i d_i \frac{D(w_i)}{w_i} t^{r_i}.$$

Let F be the algebraic closure of $C(x)$. Let K be a finite Galois extension over $F(t)$ and let σ be an element of the Galois group of K over $F(t)$. Then

$$gt = D(\sigma v_o) + \sum a_i \frac{D(\sigma v_i)}{\sigma v_i} + \sum b_i d_i \frac{D(w_i)}{w_i} t^{r_i}.$$

Summing over all σ yields, for some M in $\mathbf{Z}$,

$$\text{(7)} \qquad M\, gt = D(\mathrm{T} v_0) + \sum a_i \frac{D(N v_i)}{N v_i} + M \sum b_i d_i \frac{D(w_i)}{w_i} t^{r_i}.$$

Using a partial fraction decomposition, comparing the terms of t, and dividing by M, we get

$$g = D(h) + hD(f) + \sum d_i \frac{D(w_i)}{w_i}.$$

Now let K_l be a finite Galois extension over $C(x)$ containing h, and let σ be an element of the Galois group of K_l over $C(x)$. Then

$$g = \sigma(g) = D(\sigma h) + (\sigma h) D(f) + \sum d_i \frac{D(w_i)}{w_i}.$$

Summing over all σ yields, for some M_1 in $\mathbf{Z}$,

$$\text{(8)} \qquad M_1\, g = D(Th) + (Th)\, D(f) + M_1 \sum d_i \frac{D(w_i)}{w_i}$$

Since Th is in $C(x)$, dividing equalion (8) by M_1, we get the result. The proof is complete.
Example. Let $\mathbf{C}$ be the set of complex numbers and let $F = \mathbf{C}(x)$ be the set of rational functions with coefficients in $\mathbf{C}$. Then F is a differential field under the usual derivative: $D = \frac{d}{dx}$. We claim that $\int e^{x^2}$ is not an Ei integrable over F.

To see this, suppose that $\int e^{x^2}$ is an Ei integrable over F. Then there exist $d_1, d_2, \ldots d_n$ in **C** nonzero element $w_1, w_2, \ldots, w_n$ in F and v in F such that

$$(9) \qquad 1 = D(v) + vD\left(x^2\right) + \sum_{i=1}^{n} d_i \frac{D(w_i)}{w_i},$$

where for each w_i there exist c_i in **C** such that $w_i = x^2 + c_i$.

Write $$v = a_0 + a_1 x + \mathrm{L} + a_m x^m + \sum_{i=1}^{l} \sum_{j=1}^{\alpha_i} \frac{d_{ij}}{(x - \beta_i)^j}.$$

So $$D(v) = a_1 + 2a_2 x + \mathrm{L} + m a_m x^{m-1} + \sum_{i=1}^{l} \sum_{j=1}^{\alpha_i} \frac{-j d_{ij}}{(x - \beta_i)^{j+1}}$$

Since $w_i = x^2 + c_i$, $D(w_i) = 2x$.
From (9), we have

$$(10) \qquad 1 = a_1 + 2a_2 x + \mathrm{L} + m a_m x^{m-1} + \sum_{i=1}^{l} \sum_{j=1}^{\alpha_i} \frac{-j d_{ij}}{(x - \beta_i)^{j+1}} +$$

$$2x\left(a_1 + 2a_2 x + \mathrm{L} + m a_m x^{m-1} + \sum_{i=1}^{l} \sum_{j=1}^{\alpha_i} \frac{-j d_{ij}}{(x - \beta_i)^{j+1}}\right) + \sum_{i=1}^{n} \frac{2 d_i x}{x^2 + c_i}.$$

If the d_{ij} 's and the d_i 's are all zero, then we get x is algebraic over **C**, otherwise the expression on the right hand side of (10) would have a pole of order ≥ 1 which is impossible because the left-hand side of (10), being 1, can have no poles. Therefore $\int e^{x^2}$ is not an Ei integrable over F.

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