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Symbolic Integration for Line Integrals II

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ABSTRACT

The version of Liouville Theorem for line integrals is furthered from elementary to EL-extensions so as to include not only elementary but also special functions such as error functions and logarithmic integrals.

Key words : Liouville Theorem, symbolic integration, special functions, error functions

INTRODUCTION

The problem of line integration is sometimes referred to as the problem of multivariate integration. Roughly speaking : if $f_1(x_1,...,x_n),...,f_n(x_1,...,x_n)$ are functions of n variables over the field of complex numbers and if there exists a g such that

$$\nabla g = (\partial g/\partial x_1,....,\partial g/\partial x_n) = (f_1,.....,f_n),$$

then the line integral $\int(f_1dx_1+...+ f_ndx_n)$ is explicitly integrable into certain special forms.

A multivariate generalization for line integrals of the Weak Liouville Theorem (Caviness and Rothstein, 1975) states that:let G be a differential field that is regular elementary over the differential field F of characteristic zero. Let a be in F^n. If there exists b in G such that $\nabla(b,...,b) = a$, then there exist constants $c_1,...,c_m$ in F and elements $d_0, d_1,..., d_m$ in F such that

$$a = \nabla(d_0,...,d_0) + \sum_{i=1}^{m}(c_i,...,c_i)\nabla(d_i,...d_i)/(d_i,...d_i)$$

For the proof of this theorem, B.F. Caviness and M. Rothstein made use of a new derivation ∇ on R^n, where R is a commutative ring with identity, which is defined as follows: let $D_1,...,D_n$ be derivations on R; define

$$\nabla (a_1,...,a_n) = (D1a_1,...,D_na_n)$$

for all $(a_1,...,a_n)$ in R^n.

In Leerawat and Laohakosol (1992), a generalization of Liouville Theorem for line integrals, from elementary to generalized elementary extensions, was presented. Thereupon, it was remarked that another generalization, to the socalled EL-elementary extension (Singer *et al.*, 1985) that includes more special functions such as error functions and logarithmic integrals, seemed possible. Here, we affirmatively make complete this remark.

Section 2 contains preliminary definitions and some lemmas basic to the proof of the main result.

In section 3, we give the main results of the paper.

MATERIALS AND METHODS

A derivation of a commutative ring with identity R is a mapping D of R into itself such that

$D(x+y) = D(x) + D(y)$ and

$D(xy) = xD(y) + yD(x)$ for all x, y in R.

A differential ring is a commutative ring with identity and an indexed family $\{D_i/i\in I\}$ of derivations of the ring. A differential field is a differential ring that is a field. The constants of the differential ring are $\bigcap_{i\in I} \ker D_i$.

A differential extension field of a differential field F with a family of derivations $\{D_i/i\in I\}$ is an extension field K of F together with a family of derivations $\{D'_i/i\in I\}$ of K indexed by the same set such that the restriction of each D'_i to F is D_i.

Consider $R^n = \{(a_1,...,a_n)/a_i$ is in R for i =

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$1,...,n\}$. R^n is a commutative ring with identity when addition and multiplication are defined as follows:

$(a_1,...,a_n) + (b_1,...,b_n) = (a_1+b_1,...,a_n+b_n)$ and
$(a_1,...,a_n)(b_1,...,b_n) = (a_1b_1,...,a_nb_n)$
for all $(a_1,...,a_n)$, $(b_1,...,b_n)$ in R^n.

R^n contains a subring $\underline{R} = \{\underbrace{(a,...,a)}_{\text{n-tuples}}/a \text{ in } R\}$.

For brevity, write $\underline{a}$ for $(a,...,a)$.
Moreover, the mapping $a \to \underline{a}$ clearly defines an isomorphism of R onto $\underline{R}$.

Let F be a differential field with derivations $D_1,...,D_n$. Define $\nabla(a_1,...,a_n) = (D_1a_1,...,D_na_n)$ for all $(a_1,...,a_n)$ in F^n. In any differential extension field E of F, we also use the same symbols $D_1,...,D_n$ for derivations of E.

Let C be the subfield of constants of F. Let A and B be finite indexing sets and let

$$E = \{G_\alpha (\exp R_\alpha (Y))/\alpha \in A\},$$

$$L = \{ H_\beta (\log S_\beta (Y))/\beta \in B\},$$

be sets of expressions where:

1. G_α, R_α, H_β are in C(Y) for all $\alpha \in A$, $\beta \in B$,
2. for all $\beta \in B$, if $H_\beta (Y) = P_\beta (Y)/Q_\beta (Y)$ with P_β, Q_β in C[Y] and $Q_\beta \neq 0$, the deg $P_\beta \leq$ deg Q_β,
3. for all $\beta \in B$, $S_\beta = \bar{S}_\beta^{1/m_\beta}$ with $m_\beta \in Z^+$, and $\bar{S}_\beta \in C(Y)$.

We say that a differential extension K of F is an EL - extension of F if there exist a tower of fields

$$F = F_0 \subset F_1 \subset ... \subset F_n = K$$

such that $F_i = F_{i-1}(\theta_i)$ where for each i = 1,...,n, one of the following holds:

(i) θ_i is algebraic over F_{i-1},

(ii) $\nabla(\underline{\theta}_i)/\underline{\theta}_i = \nabla(\underline{u})$ for some $u \in F_{i-1}$,

(iii) $\nabla(\underline{\theta}_i) = \nabla(\underline{u})/\underline{u}$ for some nonzero $u \in F_{i-1}$,

(iv) for some $\alpha \in A$, there are u and nonzero v in F_{i-1} such that $\nabla(\underline{\theta}_i) = \underline{G}_\alpha (\underline{v})\nabla(\underline{u})$ where $v = \exp R_\alpha (u)$,

(v) for some $\beta \in B$, there are u, v in F_{i-1} such that $\nabla(\underline{\theta}_i) = \underline{H}_\beta (\underline{V})\nabla(\underline{u})$ where $v = \log S_\beta (u)$ and $S_\beta (u) \neq 0$.

The following result will be our tool for the proof of the main theorem.

Lemma 2.1 (Singer *et al.*, 1985). Let k be a field containing the algebraic closure of the rationals and let X and Y be indeterminates. Let A(Y) and B(Y) be relatively prime elements of k[Y]. Furthermore, assume A/B is not an n^{th} power in k(Y) for any positive integer n. Then the polynomial $B(Y)X^m - A(Y)$ is irreducible in K(X)[Y] for all positive integer m.

Lemma 2.2 (Singer *et al.*, 1985). Let k be a field, X and Y indeterminates, and A(Y) and B(Y) relatively prime elements of k[Y]. If a and b are elements of k with $a \neq 0$, then A(Y) - (aX+b) B(Y) is irreducible in K(X)[Y].

Lemma 2.3 (Rosenlicht, 1976). Let k be a differential field, of characteristic zero, K a differential extension field of k with the same constants, with K algebraic over k(t) for some given $t \in K$. Suppose that $c_1,...,c_n$ are constants of k that are linearly independent over **Q**, that $u_1,...,u_n$, v are elements of K, with $u_1,...,u_n$ nonzero, and that for given derivation D of K we have

$$\sum_{i=1}^{n} c_i Du_i / u_i + Dv \in k$$

If for each given derivation D of K we have $Dt \in k$, then $u_1,...,u_n$ are algebraic over k and there exists a constant c of k such that v+ct is algebraic over k. If for each given derivation D of K we have $Dt/t \in k$, then v is algebraic over k and there are integers $m_0, m_1,...,m_i$, with $m_0 \neq 0$, such that each $u_i^{m_0}t^{m_i}$ is algebraic over k.

RESULTS

For notational convenience, throughout the remaining discussion, EL-extension and all associated symbols, namely, A, B, G_α, R_α, H_β, P_β, Q_β, S_β and $\bar{S}_\beta$ take the meaning as described in section 2 and will not be explicitly prescribed.

Theorem. Let F be an algebrically closed differential field of characteristic zero with derivations $D_1,...,D_n$. Let C be an algebraically closed subfield of constants of F. Let E be an EL-extension of F having the same subfield of constants. Let $\gamma \in F^n$. If there exists b in E such that $\nabla\underline{b} = \gamma$, then there exist

b_i, $c_{i\alpha}$, $d_{i\alpha}$ in C, v_i in F and $w_{i\alpha}$, $x_{i\alpha}$, $y_{i\beta}$, $z_{i\beta}$ in F, such that

$$\gamma = \nabla(\underline{v}_0) + \sum_{i\in J} \underline{b}_i \nabla(\underline{v}_i)/\underline{v}_i + \sum_{\alpha\in A}\sum_{i\in I_\alpha} \underline{c}_{i\alpha}\nabla(\underline{w}_{i\alpha})\underline{G}_\alpha(\underline{x}_{i\alpha}) + \sum_{\beta\in B}\sum_{i\in J_\beta} \underline{d}_{i\beta}\nabla(\underline{y}_{i\beta}) H_\beta(\underline{z}_{i\beta}),$$

where A, B, J, I_α and J_β are all finite indexing sets, $x_{i\alpha} = \exp R_\alpha(w_{i\alpha})$ and $z_{i\beta} = \log S_\beta(y_{i\beta})$ and $S_\beta(y_{i\beta}) \neq 0$ for all α, β and i.

The proof of Theorem is by induction on the transcendence degree of E over F and can be seen to follow immediately from the following Lemmas.

Lemma 3.1. Let F be an algebrically closed differential field of characteristic 0 with derivations $D_i,...,D_n$ and C being its algebraically closed subfield of constants. Assume that $\overline{S}_\beta \notin C(Y)^n$ for all positive integer $n \geq 2$

Let θ be transcendental over F and satisfy

(*)-------$\nabla(\underline{\theta}) = \underline{u\theta}$ for some $u \in F$.

Let E be a finite algebraic differential extension of $F(\theta)$ equipped with extended derivations $D_1,...D_n$. Assume that the field of constants of E is C. Let $\gamma \in F^n$. Assume that there exist

1. $b_i \in C$, $v_0 \in E$, $v_i \in E\setminus\{0\}$ $\forall i \in J$,
2. $c_{i\alpha} \in C$, $w_{i\alpha}, x_{i\alpha} \in E\setminus\{0\}$ $\forall i \in I_\alpha, \alpha \in A$,
3. $d_{i\beta} \in C$, $y_{i\beta}, z_{i\beta} \in E\setminus\{0\}$ $\forall i \in J_\beta, \beta \in B$,

such that

(1)-------$$\gamma = \nabla(\underline{v}_0) + \sum_{i\in J} \underline{b}_i\nabla(\underline{v}_i)/\underline{v}_i + \sum_{\alpha\in A}\sum_{i\in I_\alpha} \underline{c}_{i\alpha}\nabla(\underline{w}_{i\alpha})\ \underline{G}_\alpha(\underline{x}_{i\alpha}) + \sum_{\beta\in B}\sum_{i\in J_\beta} \underline{d}_{i\beta}\nabla(\underline{y}_{i\beta})\ \underline{H}_\beta(\underline{z}_{i\beta})$$

where J, I_α and J_β are finite indexing sets, $x_{i\alpha} = \exp R_\alpha(w_{i\alpha})$ $\forall i \in I_\alpha, \alpha \in A$ and $z_{i\beta} = \log S_\beta(y_{i\beta})$ and $S_\beta(y_{i\beta}) \neq 0$ $\forall i \in J_\beta, \beta \in B$.

Then there exist

1. $\bar{a}$, $\bar{b}_i \in C$, $\bar{v}_0 \in F$, $\bar{v}_i \in F/\{0\}$ $\forall i \in \bar{J}$,
2. $\bar{c}_{i\alpha} \in C$, $\bar{w}_{i\alpha}, \bar{x}_{i\alpha} \in F\setminus\{0\}$ $\forall i \in \bar{I}_\alpha, \alpha \in \bar{A}$,
3. $\bar{d}_{i\beta} \in C$, $\bar{y}_{i\beta}, \bar{x}_{i\beta} \in F\setminus\{0\}$ $\forall i \in \bar{J}_\beta, \beta \in \bar{B}$,

such that

(2)---- $$\gamma = \nabla(\underline{\bar{v}}_0) + \underline{\bar{a}}\nabla(\underline{\theta})/\underline{\theta} + \sum_{i\in \bar{J}} \underline{\bar{b}}_i\nabla(\underline{\bar{v}}_i/\underline{\bar{v}}_i) + \sum_{a\in \bar{A}}\sum_{i\in \bar{I}_a} \underline{\bar{c}}_{ia}\nabla(\underline{\bar{w}}_{ia})\ \underline{G}_a(\underline{\bar{x}}_{ia}) + \sum_{b\in \bar{B}}\sum_{i\in \bar{J}_b} \underline{\bar{d}}_{ib}\nabla(\underline{\bar{y}}_{ib})\ \underline{H}_b(\underline{\bar{z}}_{ib})$$

where $\bar{A}, \bar{B}, \bar{J}, \bar{I}_\alpha$ and $\bar{J}_\beta$ are all finite indexing sets,

$\bar{x}_{i\alpha} = \exp R_\alpha(\bar{w}_{i\alpha})$ $\forall i \in \bar{I}_\alpha$, $\alpha \in \bar{A}$ and

$\bar{z}_{i\beta} = \log S_\beta(\bar{y}_{i\beta})$ and $S_\beta(\bar{y}_{i\beta}) \neq 0$ $\forall i \in \bar{J}_\beta$, $\beta \in \bar{B}$.

Proof. We may assume that for all α in A, $R \notin C$, because if $R_{\alpha_0} \in C$ for some $\alpha_0 \in A$, then for each $i \in I_{\alpha_0}$, $x_{i\alpha_0} \in C$. Hence $G_{\alpha_0}(x_{i\alpha_0}) \in C$. Thus $\sum_{i\in I_{\alpha_0}} \underline{c}_{i\alpha_0}\nabla(\underline{w}_{i\alpha_0})\underline{G}_{\alpha_0}(\underline{x}_{i\alpha_0})$ is of form $\nabla(\underline{v})$ where $v \in E$.

By Lemma 2.3, $x_{i\alpha} = p_{i\alpha}\theta^{\gamma_{i\alpha}}$, $S_\beta(y_{i\beta}) = q_{i\beta}\theta^{S_{i\beta}}$, for some rational integers $r_{i\alpha}$, $s_{i\beta}$ and elements $p_{i\alpha}$ $q_{i\beta}$ in F and that the $w_{i\alpha}$ and the $z_{i\beta}$ are in F. Note that we can arrange so that $r_{i\alpha}$ and $s_{i\beta}$ are actually integers. To see this, let $r_{i\alpha} = t_{i\alpha}/m$ and $s_{i\beta} = l_{i\beta}/m$, where $t_{i\alpha}$, $l_{i\beta}$ and m are integers. Let $\bar{\theta} = \theta^{i/m}$ Hence $\nabla(\underline{\bar{\theta}}) = \underline{(u/m)/\bar{\theta}}$ and $F \subset F(\bar{\theta}) \subset E(\bar{\theta})$. If we replace E by $E(\bar{\theta})$ and θ by $\bar{\theta}$, we still have fields of the appropriate form and furthermore, $x_{i\alpha} = p_{i\alpha}\bar{\theta}^{t_{i\alpha}}$, and $S_\beta(y_{i\beta}) = q_{i\beta}\bar{\theta}^{l_{i\beta}}$, where $t_{i\alpha}$ and $l_{i\beta}$ are integers.

Take the trace, T, over an appropriate Galois extension of $F(\theta)$ to $F(\theta)$ on both sides of (1) to get

$$(3)\text{-----}\quad M\gamma = \nabla(T\underline{v}_0) + \sum b_i \nabla(\underline{Nv_i}) / N\underline{v}_i + M\sum\sum \underline{c}_{i\alpha}\nabla(\underline{w}_{i\alpha})\underline{G}_\alpha(\underline{x}_{i\alpha}) + \sum\sum \underline{d}_{i\beta}\nabla(\underline{Ty}_{i\beta})\,\underline{H}_\beta(\underline{z}_{i\beta})$$

where N denote the corresponding norm and $M \in Z$. We now consider the coefficient of θ^0 in each component on the right hand side of (3).

Since $Tv_0 \in F(0)$, the expression of $\nabla(\underline{Tv_0})$ which is in F^n is $\nabla(\underline{\tilde{v}}_0)$ for some $\tilde{v}_0$ in F.

Since $Nv_i \in F(\theta)$, the expression of $\sum \underline{b}_i\nabla(\underline{Nv}_i)/\underline{Nv}_i$ which is in F^n is $\sum \underline{b}_i\nabla(\underline{k}_i)/\underline{k}_i + \sum\sum \underline{b}_i\underline{n}_j\underline{u}$, where the $k_i \in F\setminus\{0\}$ and $n_{ij} \in Z$. Next, write

$$\sum\sum \underline{c}_{i\alpha}\nabla(\underline{w}_{i\alpha})G_\alpha(\underline{x}_{i\alpha}) = \sum_{r_{i\alpha}=0}\sum + \sum_{r_{i\alpha}\neq 0}\sum$$

If $r_{i\alpha} = 0$, then $G_\alpha(x_{i\alpha})$ are in F. Assume that $r_{i\alpha} \neq 0$. The expression of $\sum_{r_{i\alpha}\neq 0}\sum \underline{c}_{i\alpha}\nabla(\underline{w}_{i\alpha})G_\alpha(\underline{x}_{i\alpha})$

which is in F^n is $\sum_{r_{i\alpha}\neq 0}\sum \underline{c}'_{i\alpha}\nabla(\underline{w}_{i\alpha})$ where the

$c'_{i\alpha}$ are constants. Therefore the expression of $\sum\sum \underline{c}_{i\alpha}\nabla(\underline{w}_{i\alpha})\underline{G}_\alpha(\underline{x}_{i\alpha})$ which is in F^n is

$$\sum_{r_{i\alpha}=0}\sum \underline{c}_{i\alpha}\nabla(\underline{w}_{i\alpha})G_\alpha(\underline{x}_{i\alpha}) + \sum_{r_{i\alpha}\neq 0}\sum \underline{c}_{i\alpha}\nabla(\underline{w}_{i\alpha})$$

Finally, we consider $\sum_\beta\sum_i \underline{d}_i\nabla(\underline{Ty}_i)\underline{H}_\beta(\underline{z}_{i\beta})$.

Write $\sum_\beta\sum_i = \sum_{s_{i\beta}=0}\sum + \sum_{s_{i\beta}\neq 0}\sum$.

If $s_{i\beta} = 0$, then $y_{i\beta}$ are in F, and hence $\sum_{s_{i\beta}=0}\sum \in F^n$.

Assume that $s_{i\beta} \neq 0$. Write $\overline{S}_\beta(Y) = A_\beta(Y)/B_\beta(Y)$ where A_β, B_β in C[Y], $B_\beta \neq 0$ and A_β and B_β are relatively prime . Each $y_{i\beta}$ satisfies

$$q_{i\beta}^{m_{i\beta}}\theta^{m_{i\beta}s_{i\beta}}B_\beta(Y) - A_\beta(Y) = 0.$$

By Lemma 2.1, $q_{i\beta}^{m_{i\beta}}\theta^{m_{i\beta}s_{i\beta}}B_\beta(Y) - A_\beta(Y)$ is irreducible over $F(\theta)$.

So we see that $Ty_{i\beta} = m_{i\beta}\dfrac{\delta_{i\beta}q_{i\beta}^{m_{i\beta}}\theta^{m_{i\beta}s_{i\beta}} + \varepsilon_{i\beta}}{\mu_{i\beta}q_{i\beta}^{m_{i\beta}}\theta^{m_{i\beta}s_{i\beta}} + \nu_{i\beta}}$,

where $m_{i\beta} \in Z^+$, $\delta_{i\beta}, \varepsilon_{i\beta}, \mu_{i\beta}, \nu_{i\beta} \in C$.

It is straightforward to see that the expression of $\nabla\underline{Ty}_{i\beta}$ which is in F^n is 0. Hence the term of

$\sum\sum \underline{d}_{i\beta}\nabla(\underline{Ty}_{i\beta})\underline{H}_\beta(\underline{z}_{i\beta})$ which is in F^n is $M\sum_{s_{i\beta}=0}\sum \underline{d}_{i\beta}\nabla(\underline{y}_{i\beta})\underline{H}_\beta(\underline{z}_{i\beta})$.

Equating terms in each component of (3) with respect to θ^0, we obtain the correct sum of γ .

Lemma 3.2. Assume all hypothesis of Lemma 3.1, except for equation (*). Instead of (*), θ satisfies $\nabla(\underline{\theta}) \in F^n$ and assume that for all $\beta, \in B$, if $H_\beta(Y) = P_\beta(Y)/Q_\beta(Y)$ with P_β, Q_β in C[Y], and $Q_\beta \neq 0$, then $\deg P_\beta \leq \deg Q_\beta$. Then there exist

1. $\bar{a}, \bar{b}_i \in C, \bar{v}_0 \in F, \bar{v}_i \in F\setminus\{0\}\ \ \forall\, i \in \bar{J}$,
2. $\bar{c}_{i\alpha} \in C, \overline{w}_{i\alpha}, \bar{x}_{i\alpha} \in F\setminus\{0\}\ \ \forall\, i \in \bar{I}_\alpha, \alpha \in \overline{A}$
3. $\bar{d}_{i\alpha} \in C, \bar{y}_{i\beta}, \bar{z}_{i\beta} \in F\setminus\{0\}\ \ \forall\, i \in \bar{J}_\beta, \beta \in \overline{B}$

such that

$$\gamma = \nabla(\underline{\bar{v}}_0) + \underline{\bar{a}}\,\nabla(\underline{\theta}) + \sum_{i\in\bar{J}}\underline{\bar{b}}_i\nabla(\underline{\bar{v}}_i)/\underline{\bar{v}}_i + \sum_{\alpha\in\bar{A}}\sum_{i\in\bar{J}_\alpha}\underline{c}_{i\alpha}\nabla(\underline{w}_{i\alpha})\underline{G}_\alpha(\underline{x}_{i\alpha}) + \sum_{\beta\in\bar{B}}\sum_{i\in\bar{J}_\beta}\underline{d}_{i\beta}\nabla(\underline{y}_{i\beta})\underline{H}_\beta(\underline{z}_{i\beta})$$

where $\bar{A}$, $\bar{B}$, $\bar{J}$, $\bar{I}_\alpha$ and $\bar{J}_\beta$ are all finite indexing sets,

$\bar{x}_{i\alpha} = \exp R_\alpha(\overline{w}_{i\alpha})\ \ \forall\, i \in \bar{I}_\alpha, \alpha \in \overline{A}$ and

$\bar{z}_{i\beta} = \log S_\beta(\bar{y}_{i\beta})$ and $S_\beta(\bar{y}_{i\beta}) \neq 0\ \ \forall\, i \in \bar{J}_\beta, \beta \in \overline{B}$.

Proof. Proceeding as in the proof of Lemma 3.1, we get the $x_{i\alpha}, y_{i\beta}$ are in F, $R_\alpha(w_{i\alpha}) = \lambda_{i\alpha}\theta + p_{i\alpha}$, $z_{i\beta} = \overline{\lambda}_{i\beta}\theta + q_{i\beta}$ where $\lambda_{i\alpha}$, $\overline{\lambda}_{i\beta} \in C$ and $p_{i\alpha}$, $q_{i\beta} \in$

F and

$$(1)\text{-----}\quad M\gamma = \nabla(T\underline{v}_0) + \sum_{i\in J} \underline{b}_i \nabla(\underline{Nv}_i)/N\underline{v}_i + M\sum_{\alpha\in A}\sum_{i\in J_\alpha} \underline{c}_{i\alpha}\nabla(\underline{Tw}_{i\alpha})\underline{G}_\alpha(\underline{x}_{i\alpha}) + M\sum_{\beta\in B}\sum_{i\in J_\beta} \underline{d}_{i\beta}\nabla(\underline{Ty}_{i\beta})\,\underline{H}_\beta(\underline{z}_{i\beta})$$

where $M \in Z$.

We now consider each component in the right hand side of (1). For each derivation D,

$$(2)\text{--}\quad D(Tv_0)+\Sigma\, b_i\, D(Nv_i)/Nv_i+\Sigma\Sigma c_{i\alpha}\, D(Tw_{i\alpha})G(x_{i\alpha}) + M\Sigma\Sigma d_{i\beta}\, D(y_{i\beta})H(z_{i\beta}) \in F.$$

Since $Nv_i \in F(\theta)$, $\Sigma\, b_i\, D(Nv_i)/Nv_i = \Sigma\, b_i\, Dk_i/k_i$ + an expression belonging to $F(\theta)\backslash F[\theta]$, where the k_i are in F.

Next, consider $\sum_{\alpha\in A}\sum_{i\in I_\alpha} c_{i\alpha}\, D(Tw_i)G(x_i)$.

Write

$$\sum_{\alpha\in A}\sum_{i\in I_\alpha} c_{i\alpha}\, D(Tw_i)\, G(x_{i\alpha}) = \sum_{\lambda_{i\alpha}=0}\sum + \sum_{\lambda_{i\alpha}\neq 0}\sum .$$

If $\lambda_{i\alpha} = 0$, then $Tw_{i\alpha} = Mw_{i\alpha}$.

Assume that $\lambda_{i\alpha} \neq 0$. Write $R_\alpha(Y) = A_\alpha(Y)/B_\alpha(Y)$ where A_α and B_α are relatively prime in C[Y] and $B_\alpha \neq 0$. Each $w_{i\alpha}$ satisfies $A_\alpha(Y) - (\lambda_{i\alpha}\theta + p_{i\alpha})B_\alpha(Y)=0$. By Lemma2.2, $A_\alpha(Y)-(\lambda_{i\alpha}\theta + p_{i\alpha})B_\alpha(Y)$ is irreducible over $F(\theta)$. So

$$Tw_{i\alpha} = m_{i\alpha}\left(\frac{\delta_{i\alpha}(\lambda_{i\alpha}\theta+p_{i\alpha})+\varepsilon_{i\alpha}}{\mu_{i\alpha}(\lambda_{i\alpha}\theta+p_{i\alpha})+\nu_{i\alpha}}\right),$$

where $\delta_{i\alpha}, \varepsilon_{i\alpha}, \mu_{i\alpha}, \nu_{i\alpha} \in C$ and $m_{i\alpha} \in Z^+$.

Therefore, we conclude that

$$\sum\sum c_i\, D(Tw_i)\, G(x_i) = M\sum_{\lambda_{i\alpha}=0}\sum c_{i\alpha}\, D(w_{i\alpha})\, G(x_{i\alpha}) + D(\tilde{w}_0)+\sum_{i\in\bar{J}}\tilde{c}_i D(\tilde{w}_1)/(\tilde{w}_1) +$$

+ an expression belonging to $F(\theta)\backslash F[\theta]$,

where the $\tilde{c}_i \in C$, the $\tilde{w}_i \in F$ and $\bar{J}$ is a finite indexing set.

Finally, consider $\sum_{\beta\in B}\sum_{i\in J_\beta} d_{i\beta}\, D(y_{i\beta})\, H_\beta(z_{i\beta})$.

Write

$$\sum\sum d_{i\beta}\, D(y_{i\beta})\, H_\beta(z_{i\beta}) = \sum_{\bar{\lambda}_{i\beta}=0}\sum + \sum_{\bar{\lambda}_{i\beta}\neq 0}\sum .$$

Clearly, $\sum_{\bar{\lambda}_{i\beta}=0}\sum \in F$. To deal with the sum corresponding to $\bar{\lambda}_{i\beta} \neq 0$, recall that

deg (numerator H_β) $\leq$ deg (denominatior H_β).

So $H_\beta(Y) = \sum\sum a_{ij}/(Y-\alpha_i)^j + q_\beta$, where $a_{ij}, \alpha_i, q_\beta \in C$.

Hence

$$\sum_{\bar{\lambda}_{i\beta}\neq 0}\sum d_{i\beta}\, D(y_{i\beta})\, H_\beta(z_{i\beta}) = \sum_{\bar{\lambda}_{i\beta}\neq 0}\sum d_{i\beta}\, D(y_{i\beta})\, q_\beta +$$

+ an expression belonging to $F(\theta)\backslash F[\theta]$.

From (2), we can conclude that

$$(3)\text{----}\quad D(Tv_0)+\sum b_i D(k_i)/K_i + M\sum_{\lambda_{i\alpha}=0}\sum c_{i\alpha} D(w_{i\alpha})G_\alpha(x_{i\alpha}) + D(\tilde{w}_0)+\sum \tilde{c}_i D(\tilde{w}_i)/\tilde{w}_i + M\sum_{\bar{\lambda}_{i\beta}=0}\sum d_{i\beta} D(y_{i\beta})H_\beta(z_{i\beta}) \; M\sum_{\bar{\lambda}_{i\beta}\neq 0}\sum d_{i\beta} D(y_{i\beta})q_\beta +$$

+ an expression belonging to $F(\theta)\backslash F[\theta] \in F$.

Since $Tv_0 \in F(\theta)$,

$$D(Tv_0) = \tilde{v}_i\theta + \tilde{v}_i\, D(\theta) + D(\tilde{v}_0) +$$

+ an expression belonging to $F(\theta)\backslash F[\theta]$,

where the $\tilde{v}_i \in C$ and $\tilde{v}_0 \in F$.

Substituting $D(Tv_0)$ into (3) and then by (1), since $M\gamma \in F^n$, only coefficients of θ^0 can survive leading to the result of Lemma.

Example. Let F be the algebraic closure of **C**(x, y, log x), (**C** denote the field of complex numbers). Take $D_1 = \frac{\partial}{\partial x}$ and $D_2 = \frac{\partial}{\partial y}$.

Recall that the logarithmic integral is defined by $li(u) = \int \frac{D(u)}{\log u}$, where D is a derivation on a field containing u.

Let $z = li(x^2 + y)$, $E = F(z)$ and

$$\gamma = (2x/\log(x^2+y), 1/\log(x^2+y)) \in F^2$$

Note that
(i) E is an EL-extension over F with $E = \varnothing$ and $L = \{1/\log Y\}$,
(ii) the subfield of constants of E and F with respect to D_1 and D_2 is **C**,
(iii) z satisfies $\nabla \underline{z} = \gamma$.

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LITERATURE CITED

Caviness, B.F., and M. Rothstein. 1975. A liouville theorem on integration in finite terms for line integrals. Communications in algebra 3:781-795.

Leerawat, U. and V. Laohakosol. 1992. Symbolic integration for line integrals. Paper presented at the 2nd Mathematics Conference. Chiang Mai Univ.,Chiang Mai.7p.

Rosenlicht, M.1976 On Liouville's theorey of elementary functions. Pacific J. Math. 65: 485-492.

Singer,M.F.,B.D. Saunders, and B.F. Caviness. 1985. An extension of Liouville's theorem on integration in finite terms. SIAM J. of Computing 14:966-990.