

Mathematical Model of Traveling Wave Front Solutions in a Neural Network with a Lateral Excitation Connection Function

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ABSTRACT

The shape of a traveling wave front solution for double continuous layers of nerve cells was considered, involving a mutually inhibitory layer of cells coupled via inhibitory connections to a second layer of cells that provides excitatory feedback connections to the first layer. Results are presented on the qualitative behavior of wave fronts that depends on the threshold parameter and the time scale parameter.

Keywords: traveling wave front solutions, mathematical modeling, neural network, neural fields, wave front shape

INTRODUCTION

The human body is made up of several systems of organs that work together as one unit. These systems contain a network of specialized cells called neurons. A neuron or a nerve cell is an electrically excitable cell that transmits information between different parts by electrical or chemical signals and coordinate the actions of the body. The nervous system can be found in only some kinds of animals especially in multi-cellular animals. However, the structure of a nervous system that responds to activation from the world outside to all other nervous cells can be found in vertebrates.

At the most basic level, the function of the nervous system is to send signals from one cell to other cells or from one part of the body to other parts. Neurons in the nervous system send signals to other cells as electrochemical waves traveling along thin fibers called axons, which cause chemicals called neurotransmitters to be

released at junctions called synapses. When a cell receives a synaptic signal, it may be excited, inhibited, or otherwise modulated. There are many types of neurons such as sensory neurons, motor neurons and central neurons. Sensory neurons are activated by physical stimuli impinging on them, and send signals that inform the central nervous system of the state of the body and the external environment. Motor neurons situated either in the central nervous system or in peripheral ganglia, connect the nervous system to muscles or other organs of the effector. Central neurons, which in vertebrates greatly outnumber the other types, make all of their input and output connections with other neurons. The interactions of all these types of neurons form neural circuits that generate an organism's perception of the world and determine its behavior.

Normally, the human brain has approximately one billion neurons and 1×10^{15} synapses. Each neuron is connected to form a

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complicated network system. Nervous cells are used efficiently to carry and receive messages or impulses as electrical or chemical signals quickly from one part to another part of the body (Nicholson and Kraig, 1981). These impulses may carry information from the outside world to the nervous system which allows the body to quickly respond to them.

Abnormal electrical discharges from brain cells result in a recurrent seizure disorder such as epilepsy and migraine (Ermentrout and Cowan, 1979; Chervin *et al.*, 1988; Connors and Amitai, 1993). Thus, progress in understanding the spatial structure in neural tissues and how activity patterns are generated is very important. An analysis of theoretical models for networks of nerve cells is necessary and new types of medication to treat some neurological diseases may be suggested.

Work on neural fields had its origins in the pioneering work of Amari (1977) and Takeuchi and Amari (1979). Ermentrout and McLeod (1993) studied the existence and uniqueness of monotonic traveling waves using integral differential equations. Moreover, Zhang (2003) studied the asymptotic stability of homoclinic orbit-traveling wave solutions to a singularly perturbed nonlinear system of integral-differential equation arising from neural networks. Ruktamatakul *et al.* (2006) discussed the shape of traveling wave-front solutions to a model for a single and double continuous layer of nerve cells with a lateral inhibition connection function. Ruktamatakul and Yimprayoon (2008) studied the shape of a traveling wave front solution for a single continuous layer of nerve cells with a lateral inhibition connection function.

MATHEMATICAL MODEL

In the current research, the case of a mutually inhibitory layer of cells coupled via inhibitory connections to a second layer of cells

that provides excitatory feedback connections to the first layer was investigated. Results were derived on the qualitative behavior of wave fronts conditional on parameters that represent the excitation threshold and time scale of the excitation process.

Ruktamatakul *et al.* (2006) introduced a double layer model of nerve cells, that is, if $u(x, t)$ represents the potential of a cell in the first layer located at x at time t , and $v(x, t)$ represents the potential of a cell in the second layer at location x at time t , then the neural network of interest for $(x, t) \in \mathfrak{R} \times \mathfrak{R}^+$ is given by Equations 1 and 2:

$$u_t(x, t) + u(x, t) = \alpha_1 \int_{\mathfrak{R}} K_1(x - y) H(u(y, t) - \theta_1) dy - \alpha_2 \int_{\mathfrak{R}} K_2(x - y) H(v(y, t) - \theta_2) dy \quad (1)$$

$$\tau v_t(x, t) + v(x, t) = \int_{\mathfrak{R}} K_3(x - y) H(u(y, t) - \theta_3) dy. \quad (2)$$

Equations 1 and 2 are considered non-dimensional, with $\tau > 0$ representing the ratio of time scales associated with u and v ; α_1 and α_2 are intrinsic properties of single cells; θ_1 , θ_2 and θ_3 are the fixed threshold values. A further assumption was made, motivated by Amari (1977), that the spread of excitatory influence from layer 1 to layer 2 is very narrow. This allows the idealization of the excitatory connection function to $K_3(x) = k_3 \delta(x)$, where k_3 is the positive constant and $\delta(x)$ is the Dirac delta function. Thus, Equation 2 becomes Equation 3:

$$\tau v_t(x, t) + v(x, t) = k_3 H(u(x, t) - \theta_3). \quad (3)$$

A further reduction was undertaken to let $\theta_1 = \theta_3 = \theta$. The aim was to look for solutions of the form $(u(x, t), v(x, t)) = (U(z), V(z))$, $z = x + vt$, for some wave speed $v > 0$.

Here, it is assumed that the following conditions hold:

(A1) $K_j, j=1,2$, are positive, even, smooth, single humped functions on \mathfrak{R} , with $\int_{\mathfrak{R}} K_j(x) dx = 1$, $\int_{\mathfrak{R}} |K'_j(x)| dx < \infty$, and decay exponentially for $|x|$

sufficiently large. Also, $K_1(x) < K_2(x)$ for $|x| < x_0$, and $K_1(x) > K_2(x)$ for $|x| > x_0$.

$$(A2) \quad U(0) = \theta.$$

$$(A3) \quad U(z) > \theta \text{ if and only if } z > 0.$$

Using the assumptions (A2) and (A3), produces

$$\int_{\Re} K_1(z-y)H(U(y)-\theta)dy = \int_0^\infty K_1(z-y)dy = \int_{-\infty}^z K_1(x)dx,$$

and

$$H(U(z) - \theta) = H(z).$$

For non-trivial solutions, Equations 1 and 3 thus become Equations 4 and 5, respectively:

$$vU' + U = \alpha_1 \int_{-\infty}^z K_1(x)dx - \alpha_2 \int_{\Re} K_2(z-y)H(V(y) - \theta_2)dy \quad (4)$$

and

$$v\tau V' + V = k_3 H(z). \quad (5)$$

The solution to Equation 5 is given by Equation 6:

$$V(z) = k_3 \left(1 - e^{-\frac{z}{v\tau}} \right) H(z) \quad (6)$$

with

$$V'(z) = \left(\frac{k_3}{v\tau} \right) e^{-\frac{z}{v\tau}} H(z) > 0.$$

Then, there exists a unique $z = z_1 > 0$ such that $V(z_2) = \theta_2$, so that

$$\theta_2 = k_3 \left(1 - e^{-\frac{z_1}{v\tau}} \right)$$

or

$$z_1 = \tau v \ln \left[\frac{k_3}{k_3 - \theta_2} \right]$$

provided $k_3 > \theta_2$.

Now,

$$\int_{\Re} K_2(z-y)H(V(y)-\theta_2)dy = \int_{z_1}^\infty K_2(z-y)dy = \int_{-\infty}^{z-z_1} K_2(x)dx.$$

For $v > 0$, Equation 4 thus becomes Equation 7:

$$vU' + U = \alpha_1 \int_{-\infty}^z K_1(x)dx - \alpha_2 \int_{-\infty}^{z-z_1} K_2(x)dx \quad (7)$$

and the solution to Equation 7 becomes Equation 8:

$$U(z) = \alpha_1 \int_{-\infty}^z K_1(x)dx - \alpha_1 \int_{-\infty}^z e^{\frac{(x-z)}{v}} K_1(x)dx - \alpha_2 \int_{-\infty}^{z-z_1} K_2(x)dx + \alpha_2 \int_{-\infty}^z e^{\frac{(x-z)}{v}} K_2(x-z_1)dx. \quad (8)$$

Since z_1 depends linearly on τ , as $\tau \rightarrow 0^+$, $z_1 \rightarrow 0$, with this limit, Equation 7 is reduced to Equation 9:

$$vU' + U = \int_{-\infty}^z (\alpha_1 K_1(x) - \alpha_2 K_2(x))dx = \alpha \int_{-\infty}^z K(x)dx. \quad (9)$$

If $\alpha_1 K_1(x)$ has smaller amplitude and wider spread than $\alpha_2 K_2(x)$, then $\alpha K(x) = \alpha_1 K_1(x) - \alpha_2 K_2(x)$ in Equation 9 is a lateral-excitation type kernel.

Existence and uniqueness of traveling wave front solutions

Theorem 1. Suppose that α_1 , α_1 , θ , θ_2 and k_3 are positive constants, such that $0 < 2\theta < \alpha_1 - \alpha_2$ and $0 < \theta_2 < k_3$. Besides the properties in (A1), assume that $K_1(x) \geq ce^{\mu x}$ for $x < 0$, $K_2(x) \geq de^{\rho x}$ for $x < 0$, and $\frac{\alpha_2 d \tau \xi e^{-\rho z_1}}{1 + \rho v} + \frac{\alpha_2 d e^{-\rho z_1}}{(1 + \rho v)^2} < \frac{\alpha_1 c}{(1 + \mu v)^2}$,

where $\tau > 0$, $\xi = \ln \left[\frac{k_3}{k_3 - \theta_2} \right] > 0$, $z_1 = \tau \xi v >$

0 , $c > d > 0$, and $\mu, \rho > 0$. Then, there exists a unique wave front $(u(x, t), v(x, t)) = (U(z), V(z))$, where $z = x + v_0 t$, to the system of Equations 4 and 5, with unique wave speed $v_0 > 0$ such that

$$\alpha_1 \int_{-\infty}^0 e^{\frac{x}{v_0}} K_1(x)dx - \alpha_2 e^{\frac{z_1}{v_0}} \int_{-\infty}^{\frac{z_1}{v_0}} e^{\frac{x}{v_0}} K_2(x)dx = \frac{\alpha_1}{2} - \alpha_2 \int_{-\infty}^{-z_1} K_2(x)dx - \theta.$$

The wave front solution is given by $(U(z), V(z))$ as described in Equation 8 with Equation 6. Furthermore, the limits are:

$$\lim_{z \rightarrow -\infty} (U(z), V(z)) = (0, 0), \quad \lim_{z \rightarrow +\infty} (U(z), V(z)) = (\alpha_1 - \alpha_2, k_3).$$

Proof. From Equation 7, the traveling wave front is connecting $U_- \equiv 0$ at $z = -\infty$ to $U_+ \equiv \alpha_1 - \alpha_2$ at $z = +\infty$. Multiplying the equation by an integrating factor and integrating, produces Equation 8.

To find the wave speed, from Equation 8, an auxiliary function $A(v)$ can be defined using the assumption (A2), so that

$$\theta \equiv A(v) = \frac{\alpha_1}{2} - \alpha_1 \int_{-\infty}^0 e^{\frac{x}{v}} K_1(x) dx - \alpha_2 \int_{-\infty}^{-z_1} K_2(x) dx + \alpha_2 e^{\frac{z_1}{v}} \int_{-\infty}^{-z_1} e^{\frac{x}{v}} K_2(x) dx.$$

Then, from the hypotheses,

$$\lim_{v \rightarrow +\infty} A(v) = 0 < \theta < \frac{\alpha_1 - \alpha_2}{2} = \lim_{v \rightarrow 0^+} A(v).$$

Furthermore,

$$A'(v) = \frac{\alpha_1}{v^2} \int_{-\infty}^0 x e^{\frac{x}{v}} K_1(x) dx - \frac{\alpha_2}{v^2} e^{\frac{z_1}{v}} \int_{-\infty}^{-z_1} x e^{\frac{x}{v}} K_2(x) dx$$

from the hypotheses

$$\begin{aligned} v^2 A'(v) &= \alpha_1 \int_{-\infty}^0 x e^{\frac{x}{v}} K_1(x) dx - \alpha_2 e^{\frac{z_1}{v}} \int_{-\infty}^{-z_1} x e^{\frac{x}{v}} K_2(x) dx \\ &\leq \alpha_1 \int_{-\infty}^0 x e^{\frac{x}{v}} \{c e^{\mu x}\} dx - \alpha_2 e^{\frac{z_1}{v}} \int_{-\infty}^{-z_1} x e^{\frac{x}{v}} \{d e^{\rho x}\} dx \\ &= -\frac{\alpha_1 c v^2}{(1 + \mu v)^2} + \frac{\alpha_2 d v^2 \tau \xi e^{-\rho z_1}}{1 + \rho v} + \frac{\alpha_2 d v^2 e^{-\rho z_1}}{(1 + \rho v)^2} < 0. \end{aligned}$$

By continuity and monotonicity of $A(v)$, there is a

$$\begin{aligned} \text{unique } v = v_0 > 0 \text{ such that } \alpha_1 \int_{-\infty}^0 e^{\frac{x}{v_0}} K_1(x) dx - \alpha_2 e^{\frac{z_1}{v_0}} \int_{-\infty}^{-z_1} e^{\frac{x}{v_0}} K_2(x) dx \\ = \frac{\alpha_1}{2} - \alpha_2 \int_{-\infty}^{-z_1} K_2(x) dx - \theta. \quad \square \end{aligned}$$

Analysis of wave shape with respect to $z_1(\tau)$

It can be shown that the wave front is

a monotone function on $(-\infty, 0)$ (Lemma 1). In addition, the wave shape is monotone on $(0, \infty)$ when τ approaches 0 (Lemma 2). On the other hand, for τ sufficiently large, the wave has a local maximum point in (x_0, z_1) , that is, it is an increasing function on $(0, x_0)$. It then decreases to its positive asymptotic value as $z \rightarrow +\infty$ (Lemma 3).

Lemma 1 For $0 < 2\theta < \alpha_1 - \alpha_2$, $0 < 2\theta < k_3$ and for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever

$$\tau > \frac{\delta}{\alpha_2 \inf_{(-\infty, z)} K'_2(x) v \ln \left[\frac{k_3}{k_3 - \theta_2} \right]} \equiv \tau_M^* > \tau_m^*,$$

$z < 0$, $U'(z) > 0$ on $z < 0$.

Proof. From Equation 8, we have Equation 10:

$$U'(z) = \frac{1}{v} \int_{-\infty}^z e^{\left(\frac{x-z}{v}\right)} \{\alpha_1 K_1(x) - \alpha_2 K_2(x - z_1)\} dx. \quad (10)$$

For $z \in (-\infty, 0)$, then, for $\varepsilon \in (0, -z)$, there is a $\delta > 0$ such that for $x \in (-\infty, z + \varepsilon)$, $\alpha_2 K_2(x) - \alpha_1 K_1(x) < \delta$.

Letting $\inf_{(-\infty, z)} K'_2(x) = \Delta$ (which is

$$\text{positive}), \text{ for } \tau > \frac{\delta}{\alpha_2 \inf_{(-\infty, z)} K'_2(x) v \ln \left[\frac{k_3}{k_3 - \theta_2} \right]},$$

this implies $z_1 > \frac{\delta}{\alpha_2 \Delta}$, and for some $\psi \in (x - z_1, x)$, producing

$$\begin{aligned} \alpha_1 K_1(x) - \alpha_2 K_2(x - z_1) &= \alpha_1 K_1(x) - \alpha_2 \{K_2(x) - z_1 K'_2(\psi)\} \\ &> \alpha_1 K_1(x) - \alpha_2 K_2(x) + \alpha_2 z_1 \Delta \\ &> \alpha_1 K_1(x) - \alpha_2 K_2(x) + \delta. \end{aligned}$$

$$\begin{aligned} \text{Thus, } \int_{-\infty}^z e^{\left(\frac{x-z}{v}\right)} \{\alpha_1 K_1(x) - \alpha_2 K_2(x - z_1)\} dx &> \\ \int_{-\infty}^z e^{\left(\frac{x-z}{v}\right)} \{\alpha_1 K_1(x) - \alpha_2 K_2(x)\} dx + \delta \int_{-\infty}^z e^{\left(\frac{x-z}{v}\right)} dx \\ &> -\delta \int_{-\infty}^z e^{\left(\frac{x-z}{v}\right)} dx + \delta \int_{-\infty}^z e^{\left(\frac{x-z}{v}\right)} dx = 0, \end{aligned}$$

giving $U'(z) > 0$ for $x \in (-\infty, z + \varepsilon)$. Since this holds for all $\varepsilon \in (0, z)$, then $U'(z) > 0$ for $x \in (-\infty,$

z). Thus, $U'(z) > 0$ on $z < 0$. \square

Lemma 2 For $0 < 2\theta < \alpha_1 - \alpha_2$, θ_2 sufficiently small, and for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever

$$0 < \tau < \frac{-\delta}{\alpha_2 \inf_{(0,z)} K'_2(x) \nu \ln \left[\frac{k_3}{k_3 - \theta_2} \right]} \equiv \tau_{m_1}^{**}, \quad z > 0,$$

$U'(z) > 0$ on $z > 0$.

Proof. From Equation 8, obtain

$$\begin{aligned} \nu U'(z) e^{\frac{z}{\nu}} &= \left\{ \frac{\alpha_1}{2} - \alpha_2 \int_{-\infty}^{-z_1} K_2(x) dx - \theta \right\} + \int_0^{\frac{z}{\nu}} e^{\frac{x}{\nu}} \\ &\quad \{ \alpha_1 K_1(x) - \alpha_2 K_2(x - z_1) \} dx \\ &> \int_0^{\frac{z}{\nu}} e^{\frac{x}{\nu}} \{ \alpha_1 K_1(x) - \alpha_2 K_2(x - z_1) \} dx. \end{aligned}$$

For $z \in (0, \infty)$, then for any $\varepsilon \in (0, z)$, there is a $\delta > 0$ such that for $x \in (0, z - \varepsilon)$, $\alpha_1 K_1(x) - \alpha_2 K_2(x) > \delta$.

Letting $\inf_{(0,z)} K'_2(x) = \Delta$ (which is negative), for $\tau < \frac{-\delta}{\alpha_2 \inf_{(0,z)} K'_2(x) \nu \ln \left[\frac{k_3}{k_3 - \theta_2} \right]}$,

this implies $z_1 < \frac{-\delta}{\alpha_2 \Delta}$, and for some $\psi \in (x - z_1, x)$, producing

$$\begin{aligned} \alpha_1 K_1(x) - \alpha_2 K_2(x - z_1) &= \alpha_1 K_1(x) - \alpha_2 \{ K_2(x) - z_1 K'_2(\psi) \} \\ &> \alpha_1 K_1(x) - \alpha_2 K_2(x) + \alpha_2 z_1 \Delta \\ &> \alpha_1 K_1(x) - \alpha_2 K_2(x) - \delta. \end{aligned}$$

Thus,

$$\begin{aligned} \int_0^{\frac{z}{\nu}} e^{\frac{x}{\nu}} \{ \alpha_1 K_1(x) - \alpha_2 K_2(x - z_1) \} dx &> \int_0^{\frac{z}{\nu}} e^{\frac{x}{\nu}} \{ \alpha_1 K_1(x) \\ &- \alpha_2 K_2(x) \} dx - \delta \int_0^{\frac{z}{\nu}} e^{\frac{x}{\nu}} dx \\ &> \delta \int_0^{\frac{z}{\nu}} e^{\frac{x}{\nu}} dx - \delta \int_0^{\frac{z}{\nu}} e^{\frac{x}{\nu}} dx = 0, \end{aligned}$$

giving $\nu U'(z) e^{\frac{z}{\nu}} > 0$ for $x \in (0, z - \varepsilon)$. Since this holds for all $\varepsilon \in (0, z)$, then $\nu U'(z) e^{\frac{z}{\nu}} > 0$ for $x \in (0, z)$. Thus, $U'(z) > 0$ on $z > 0$. \square

Lemma 3 For $0 < 2\theta < \alpha_1 - \alpha_2$, $0 < \theta_2 < k_3$, assume, besides (A1), that $K_1(x)$ and $K_2(x)$ satisfy

the following.

- 1) There exist $m_1, m_2, \rho_1, \rho_2 > 0$, such that $K_1(x) < m_1 e^{-\rho_1 x}$, $K_2(x) < m_2 e^{-\rho_2 x}$ for $x > 0$.
- 2) $\nu \neq \frac{1}{\rho_1}$ and $\nu \neq \frac{1}{\rho_2}$.
- 3) $\inf_{[0,\infty)} K'_2(x) > -\infty$.

Then, for all $\varepsilon > 0$, there exists a δ such that

$$\text{whenever } \tau < \frac{-\delta}{\alpha_2 \inf_{(0,x_0)} K'_2(x) \nu \ln \left[\frac{k_3}{k_3 - \theta_2} \right]} \equiv \tau_M^{**},$$

there is a unique $z = z_2 \in (x_0, z_1)$ which is a local maximum point for $U(z)$ so that $U'(z) > 0$ on $[0, z_2)$, and $U'(z) < 0$ on (z_2, ∞) .

Proof. For $z \in (0, x_0)$, from Equation 8, obtain

$$\begin{aligned} \nu U'(z) e^{\frac{z}{\nu}} &= \left\{ \frac{\alpha_1}{2} - \alpha_2 \int_{-\infty}^{-z_1} K_2(x) dx - \theta \right\} + \int_0^{\frac{z}{\nu}} e^{\frac{x}{\nu}} \\ &\quad \{ \alpha_1 K_1(x) - \alpha_2 K_2(x - z_1) \} dx \\ &> \int_0^{\frac{z}{\nu}} e^{\frac{x}{\nu}} \{ \alpha_1 K_1(x) - \alpha_2 K_2(x - z_1) \} dx. \end{aligned}$$

Then, for $\varepsilon \in (0, x_0)$, there is a $\delta > 0$ such that for $x \in (0, x_0 - \varepsilon)$, $\alpha_1 K_1(x) - \alpha_2 K_2(x) > \delta$.

Letting $\inf_{(0,x_0)} K'_2(x) = \Delta < 0$, for

$$\tau < \frac{-\delta}{\alpha_2 \inf_{(0,x_0)} K'_2(x) \nu \ln \left[\frac{k_3}{k_3 - \theta_2} \right]}, \text{ this implies}$$

$$\begin{aligned} z_1 &< \frac{-\delta}{\alpha_2 \Delta}, \text{ and for some } \psi \in (x - z_1, x), \text{ producing} \\ \alpha_1 K_1(x) - \alpha_2 K_2(x - z_1) &= \alpha_1 K_1(x) - \alpha_2 \{ K_2(x) - z_1 K'_2(\psi) \} \\ &> \alpha_1 K_1(x) - \alpha_2 K_2(x) + \alpha_2 z_1 \Delta \\ &> \alpha_1 K_1(x) - \alpha_2 K_2(x) - \delta. \end{aligned}$$

Thus,

$$\begin{aligned} \int_0^{\frac{z}{\nu}} e^{\frac{x}{\nu}} \{ \alpha_1 K_1(x) - \alpha_2 K_2(x - z_1) \} dx &> \int_0^{\frac{z}{\nu}} e^{\frac{x}{\nu}} \{ \alpha_1 K_1(x) \\ &- \alpha_2 K_2(x) \} dx - \delta \int_0^{\frac{z}{\nu}} e^{\frac{x}{\nu}} dx \\ &> \delta \int_0^{\frac{z}{\nu}} e^{\frac{x}{\nu}} dx - \delta \int_0^{\frac{z}{\nu}} e^{\frac{x}{\nu}} dx = 0, \end{aligned}$$

giving $U'(z) > 0$ on $z \in (0, x_0 - \varepsilon)$. Since this holds for all $\varepsilon \in (0, x_0)$, then $U'(z) > 0$ on $z \in (0, x_0)$. Thus, $U(z)$ is monotonic increasing on $(0, x_0)$.

If τ is sufficiently large, then

$$z_1 = \tau v \ln \left[\frac{k_3}{k_3 - \theta_2} \right] \gg 1 \text{ and } z_1(\tau) > x_0. \text{ When}$$

$z \in (z_1, \infty)$,

$$\begin{aligned} vU'(z)e^{\frac{z}{v}} &= \alpha_1 \int_{-\infty}^{\frac{z}{v}} e^{\frac{x}{v}} K_1(x) dx + \alpha_2 e^{\frac{z_1}{v}} \int_{-\infty}^{\frac{z}{v} - \frac{z_1}{v}} e^{\frac{x}{v}} K_2(x) dx \\ &< \alpha_1 \int_{-\infty}^{\frac{z}{v}} e^{\frac{x}{v}} \left\{ m_1 e^{-\rho_1 x} \right\} dx + \alpha_2 e^{\frac{z_1}{v}} \int_{-\infty}^{\frac{z}{v} - \frac{z_1}{v}} e^{\frac{x}{v}} \left\{ m_2 e^{-\rho_2 x} \right\} dx \\ &= \frac{\alpha_1 m_1 v}{1 - \rho_1 v} \left\{ e^{\left(\frac{1}{v} - \rho_1\right)z} - 1 \right\} + \frac{\alpha_2 m_2 e^{\frac{z_1}{v}} v}{1 - \rho_2 v} \left\{ e^{\left(\frac{1}{v} - \rho_2\right)(z - z_1)} - 1 \right\}. \end{aligned}$$

That is,

$$\begin{aligned} vU'(z) &< \frac{\alpha_1 m_1 v}{1 - \rho_1 v} \left\{ e^{-\rho_1 z} - e^{-\frac{z}{v}} \right\} + \frac{\alpha_2 m_2 e^{\frac{z_1}{v}} v}{1 - \rho_2 v} \\ &\left\{ e^{-\rho_2(z - z_1) - \frac{z_1}{v}} - e^{-\frac{z}{v}} \right\}. \end{aligned}$$

If $z > z_1$ and $z \rightarrow +\infty$, then

$$e^{-\rho_2(z - z_1) - \frac{z_1}{v}} - e^{-\frac{z}{v}} \rightarrow 0$$

for $z \in (z_1, \infty)$. Hence $U'(z) < 0$ on (z_1, ∞) , given conditions 1)-3). So, there exists a $z = z_2$, $z_2 \in (x_0, z_1)$, such that $U'(z_2) = 0$ and $U(z_2)$ is a local maximum.

Next, is proven that z_2 is unique. Suppose it is not, then there are at least two different z' 's, say z_2 and z'_2 such that $U'(z_2) = 0$, $U'(z'_2) = 0$ and $z_2, z'_2 \in (x_0, z_1)$. One is larger, say $x_0 < z_2 < z'_2 < z_1$. So,

$$U'(z_2) = \frac{\alpha_1}{v} \int_{-\infty}^{z_2} e^{\left(\frac{x - z_2}{v}\right)} K_1(x) dx - \frac{\alpha_2}{v} \int_{-\infty}^{z_2} e^{\left(\frac{x - z_2}{v}\right)}$$

$$K_2(x - z_1) dx = 0$$

or

$$\alpha_1 \int_{-\infty}^{z_2} e^{\frac{x}{v}} K_1(x) dx - \alpha_2 e^{\frac{z_1}{v}} \int_{-\infty}^{\frac{z_2}{v} - \frac{z_1}{v}} e^{\frac{x}{v}} K_2(x) dx = 0, \quad (11)$$

and

$$U'(z'_2) = \frac{\alpha_1}{v} \int_{-\infty}^{z'_2} e^{\left(\frac{x - z'_2}{v}\right)} K_1(x) dx - \frac{\alpha_2}{v} \int_{-\infty}^{z'_2} e^{\left(\frac{x - z'_2}{v}\right)}$$

$$K_2(x - z_1) dx = 0$$

or

$$\alpha_1 \int_{-\infty}^{z'_2} e^{\frac{x}{v}} K_1(x) dx - \alpha_2 e^{\frac{z_1}{v}} \int_{-\infty}^{\frac{z'_2}{v} - \frac{z_1}{v}} e^{\frac{x}{v}} K_2(x) dx = 0.$$

Therefore,

$$\alpha_1 \int_{-\infty}^{z_2} e^{\frac{x}{v}} K_1(x) dx + \alpha_1 \int_{z_2}^{z'_2} e^{\frac{x}{v}} K_1(x) dx - \alpha_2 e^{\frac{z_1}{v}} \int_{-\infty}^{\frac{z_2}{v} - \frac{z_1}{v}} e^{\frac{x}{v}}$$

$$K_2(x) dx - \alpha_2 e^{\frac{z_1}{v}} \int_{\frac{z_2}{v} - \frac{z_1}{v}}^{\frac{z'_2}{v} - \frac{z_1}{v}} e^{\frac{x}{v}} K_2(x) dx = 0. \quad (12)$$

Subtracting Equation 11 from Equation 12, produces

$$\alpha_1 \int_{z_2}^{z'_2} e^{\frac{x}{v}} K_1(x) dx - \alpha_2 e^{\frac{z_1}{v}} \int_{\frac{z_2}{v} - \frac{z_1}{v}}^{\frac{z'_2}{v} - \frac{z_1}{v}} e^{\frac{x}{v}} K_2(x) dx = 0$$

or,

$$\alpha_1 \int_{z_2}^{z'_2} e^{\frac{x}{v}} K_1(x) dx - \alpha_2 \int_{z_2}^{z'_2} e^{\frac{x}{v}} K_2(x - z_1) dx = 0$$

That is,

$$\int_{z_2}^{z'_2} e^{\frac{x}{v}} \{ \alpha_1 K_1(x) - \alpha_2 K_2(x - z_1) \} dx = 0. \quad (13)$$

For τ sufficiently large ($\tau \gg 1$), z_1 is consequently large enough so that

$$\alpha_1 K_1(x) < \alpha_2 K_2(x) < \alpha_2 K_2(x - z_1)$$

for $x \in (z_2, z'_2)$. Then,

$$\int_{z_2}^{z'_2} e^{\frac{x}{v}} \{ \alpha_1 K_1(x) - \alpha_2 K_2(x - z_1) \} dx < 0$$

which contradicts Equation 13. Hence $z_2 = z'_2$, that is, z_2 is unique. \square

Example Consider $K_1(x) = a_1 e^{-b_1|x|}$ and $K_2(x) = a_2 e^{-b_2|x|}$ where $0 < a_1 < a_2$, $0 <$

$b_1 < b_2$, $a_1 = \frac{b_1}{2}$ and $a_2 = \frac{b_2}{2}$. Now, from Equation 8, for a numerical example, let $a_1 = 1$, $b_1 = 2$, $a_2 = 2$, $b_2 = 4$, $\alpha_1 = 3$ and $\alpha_2 = 1$. Then, $K_1(x) = e^{-2|x|}$ and $K_2(x) = 2e^{-4|x|}$, whose graphs are shown in Figure 1. Now, $e^{-2x_0} - 2e^{-4x_0} = 0$, yields $x_0 = \frac{\ln 2}{2} \approx 0.347 > 0$.

In order to satisfy the conditions in Lemma 1 and Lemma 2, let $\tau = 0.01$, $\theta_2 = 0.1$, $\theta = 0.245$ and $k_3 = 0.9$. Thus,

$$z_1 = \tau v \ln \left[\frac{k_3}{k_3 - \theta_2} \right] = 0.01v \ln(1.125).$$

From $\theta \equiv A(v) = \frac{\alpha_1}{2} - \alpha_1 \int_{-\infty}^0 e^v K_1(x) dx - \alpha_2 \int_{-\infty}^{-z_1} K_2(x) dx + \alpha_2 e^v \int_{-\infty}^{-z_1} e^v K_2(x) dx$, is yielded $v = 2$.

The function $U(z)$ is a monotonic increasing function as shown in Figure 2. Moreover, if we let $\tau = 0.01$, $\theta_2 = 0.85$, $\theta = 0.256$ and $k_3 = 0.9$ which satisfy the same conditions in Lemma 1 and Lemma 2, Then, $v = 2$ and $z_1 = 0.0578$. The

function $U(z)$ is a monotonic increasing function as shown in Figure 3.

In order to satisfy the conditions in Lemma 1 and Lemma 3, let $\tau = 100$, $k_3 = 0.9$, $\theta = 0.3$, $\rho_1 = \rho_2 = 1$, $1 > m_1 > e^{-x}$ and $1 > m_2 > 2e^{-3x}$ for $x = 0$. Then, $K_1(x) = e^{-2x} < m_1 e^{-\rho_1 x}$ and $K_2(x) = 2e^{-4x} < m_2 e^{-\rho_2 x}$ on $x \in (x_0, \infty)$. Also, $v \neq \frac{1}{\rho_1}, \frac{1}{\rho_2}$, that is, $v \neq 1$. If $\theta_2 = 0.1$ and $\theta_2 = 0.85$, then $U(z)$ is monotonic for $z < 0$ but non-monotonic for $z > 0$ as shown in Figure 4 and Figure 5, respectively.

DISCUSSION AND CONCLUSION

This research investigated the shape of traveling wave front solutions for double continuous layers of nerve cells with the lateral excitation connection function. A characterization was given of the wave front shape that depends on the size of the firing threshold parameter θ_2 and the time scale parameter τ . The results on the shape of the wave front solutions show that the exhibited shape does not depend on the size of the firing threshold potential but depends only on the time scale parameter in that for small time scales, the wave shape is monotonic but for large time scales the wave shape becomes non-monotonic.

The results of this study may help to understand the spatially structured activity seen in neural tissues, such as how the activity patterns are generated and suggest new types of medication to treat some neurological diseases in the near future.

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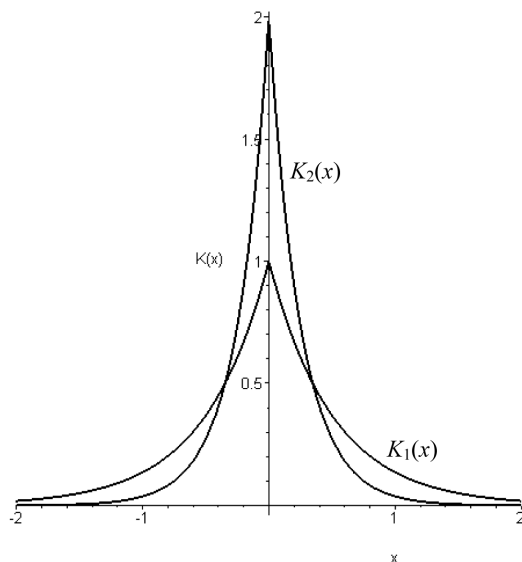


Figure 1 Graph of $K_1(x) = e^{-2|x|}$ and $K_2(x) = 2e^{-4|x|}$.

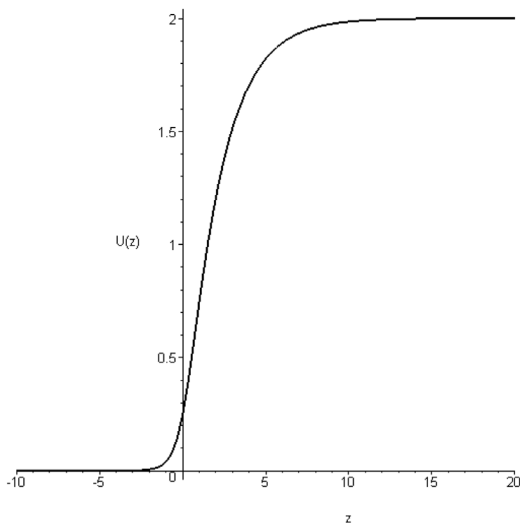


Figure 2 Graph of $U(z)$ with $\theta_2 = 0.1$, $\tau = 0.01$, $x_0 = 0.347$, $z_1 = 0.00236$ and $v = 2$.

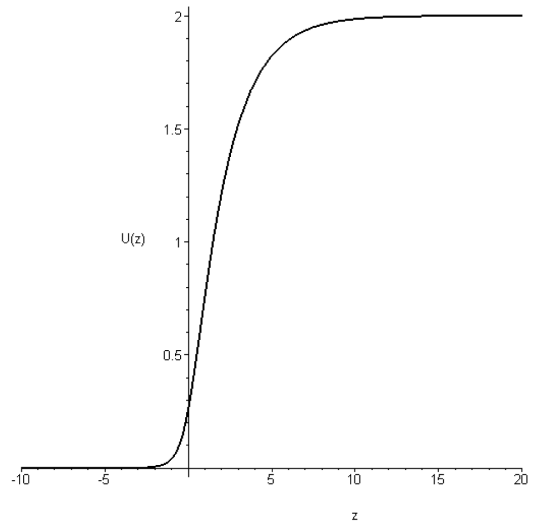


Figure 3 Graph of $U(z)$ with $\theta_2 = 0.85$, $\tau = 0.01$, $x_0 = 0.347$, $z_1 = 0.0578$ and $v = 2$.

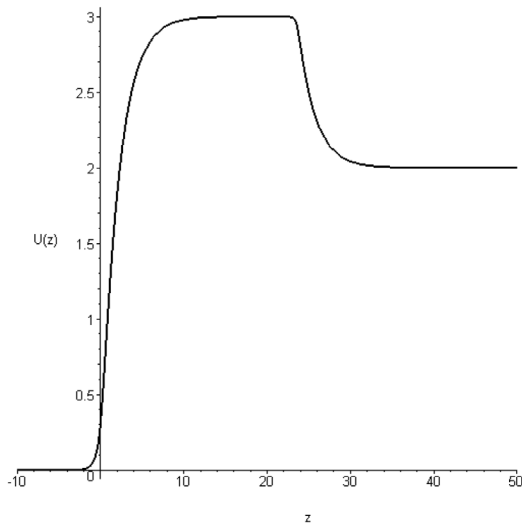


Figure 4 Graph of $U(z)$ with $\theta_2 = 0.1$, $\tau = 100$, $x_0 = 0.347$, $z_1 = 23.5566$ and $v = 2$.

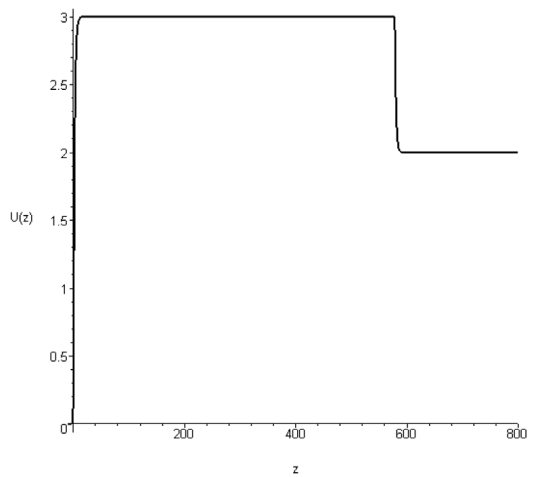


Figure 5 Graph of $U(z)$ with $\theta_2 = 0.85$, $\tau = 100$, $x_0 = 0.347$, $z_1 = 578.07$ and $v = 2$.

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