

Krawtchouk's Polynomial for Hypergeometric Distribution Approximation

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ABSTRACT

A formula for the expansion of the hypergeometric probability approximation in terms of Krawtchouk's polynomial was proposed—this formula is called a modified binomial probability—and its accuracy was investigated in terms of the total variation distance. In addition, an efficiency comparison with a binomial probability and Ord's probability was conducted using a simulation study for 288 situations. It was found that the total variation distance of a modified binomial probability was less than those of the binomial probability and Ord's probability for all situations and tended to zero for a small sampling fraction. Even for a large population size of 20,000, there seemed to be no difference in the efficiency of the modified binomial probability, Ord's probability and the binomial probability when the sampling fraction was greater than 0.1 at all levels of the proportion of the population that had the specified attribute.

Keywords: hypergeometric, binomial, sampling fraction, Krawtchouk's polynomials, accuracy

INTRODUCTION

A dichotomous finite population of size N can be classified as either having or not having a specified attribute. If a sample of size n is drawn at random with replacement, the process of observing whether or not each member selected has the specified attribute constitutes a Bernoulli trial. Each member in the population has an equal probability of being selected and members are sampled independently and sequentially with replacement. This process is known as simple random sampling with replacement. Hence, the number of members contained that have a specified attribute is a binomial random variable. Specifically, the number of members obtained that have the specified attribute has a binomial distribution with parameters n and p where n is the sample size and p is the proportion of the population that has the

specified attribute. However, if sampling is without replacement, the trials are not independent. Hence, the process of observing whether or not each member selected has the specified attribute does not constitute a Bernoulli trial. Each member in the population has an unequal probability of being selected for each trial. This process is known as simple random sampling without replacement. In particular, the number of members obtained that have the specified attribute has a hypergeometric distribution with parameters N , n and D where N is the population size, n is the sample size and D is the number of items that fall into a class of interest when $D \leq N$ (Ghahramani, 2000; Johnson *et al.*, 2005).

A family of hypergeometric random variables is closely related to a family of binomial random variables. Weiss (2006) mentioned that if a sample size n is small relative to a population size

N , the hypergeometric distribution with parameters N , n and D can be approximated by the binomial distribution. As a rule of thumb, the hypergeometric distribution can be adequately approximated by the binomial distribution, provided that the sample size does not exceed 5% of the population size (Weiss, 2006). Additionally, Montgomery (2009) and Evans *et al.* (2000) mentioned that if the sampling fraction $\left(f = \frac{n}{N}\right)$ is small—it is not greater than 0.1—then the binomial distribution with parameters $p = \frac{D}{N}$ and n is an efficient approximation to a hypergeometric distribution.

The hypergeometric distribution plays an important role in many areas of statistics, including sampling survey theory (Lahiri *et al.* 2007; Walpole *et al.* 2011) and statistical quality control where sometimes it is useful to approximate a hypergeometric distribution with a binomial distribution or an asymptotic binomial distribution (Ord, 1968). This is particularly helpful in situations where the original distribution is difficult to manipulate analytically. For example, this approximation is useful in the design of acceptance-sampling plans (Banks, 1989; Besterfield, 2004; Montgomery, 2009). However, if the sampling fraction is greater than 0.1, then the approximation to a hypergeometric distribution with a binomial distribution is no better. Further, a normal approximation to a hypergeometric probability is classical in the cases where the sampling fraction f and the proportion of the population that has the specified attribute p take values near the boundary values of 0 and 1 (Feller, 1968). However, the extreme cases where f or p take values near the boundary values 0 and 1 are very important in sample surveys and quality control problems.

Therefore, this study proposed a modified binomial approximation to the hypergeometric distribution using an expansion of hypergeometric probabilities in terms of Krawtchouk's polynomial (Blázquez and Miño, 2000). This is an efficient

approximation to a hypergeometric distribution whatever the sampling fraction and the proportion of the population that has the specified attribute.

MATERIALS AND METHODS

An accuracy comparison of the modified binomial approximations in terms of the total variation distance (Majsnerowska, 1998; Kennedy and Quine, 1989) with a binomial approximation and Ord's approximation was empirically performed using a simulation study.

Distributions of random variable

Consider a finite population of size N in which each member is classified as either having or not having a specified attribute. Let D be the number in the population having the specified attribute ($D \leq N$), then $N - D$ corresponds to the number in the population not having the specified attribute. These numbers are not known in advance. A random sample of size n ($n \leq N$) is taken without replacement from the population. The sample contains X elements that have the specified attribute. Then X is called a hypergeometric random variable and is said to have a hypergeometric distribution with parameters N , n and D . Additionally, the probability mass function is given by Equation 1 (Ghahramani, 2000; Johnson *et al.*, 2005):

$$h(x; n, D, N) = \frac{\binom{D}{x} \binom{N-D}{n-x}}{\binom{N}{n}} \quad (1)$$

where $\text{Max}\{0, n - (N - D)\} \leq x \leq \text{Min}\{n, D\}$, $n \in \mathbb{Z}^+$, $N \in \mathbb{Z}^+$ and $D \leq N$.

However, if a random sample of size n is taken with replacement from a finite population of size N where each element in the population has an equal and independent probability p of having a specified attribute, then X is called a binomial random variable and is said to have a binomial

distribution with parameters n and p . The probability mass function is given by Equation 2 (Ghahramani, 2000; Evans *et al.*, 2000; Johnson *et al.*, 2005):

$$b(x; n, p) = \binom{n}{x} p^x q^{n-x} \quad (2)$$

where $p+q = 1$, $p>0$, $q>0$, n is a positive integer and $x = 0, 1, 2, \dots, n$.

Modified binomial approximation to a hypergeometric distribution

This section proposes a modified binomial approximation to the hypergeometric distribution using an expansion of hypergeometric probabilities in term of Krawtchouk's polynomial. A procedure to derive the proposed distributions follows.

Let $\alpha(x)$ be a step function with the jump function, at the point x , of

$$j(x) = \binom{n}{x} p^x q^{n-x} \quad \text{where } p+q = 1, p>0, q>0, n \text{ is a positive integer and } x = 0, 1, 2, \dots, n.$$

Then Krawtchouk's polynomial is defined by Equation 3 (Szegő, 2003; Chihara, 2011):

$$k_m(x; n, p) = \sum_{j=0}^m \binom{x}{j} \binom{n-x}{m-j} (-p)^{m-j} q^j \quad (3)$$

for $m = 0, 1, \dots, n$.

For any integers x , N , D and n satisfying the four conditions as follows:

- 1) $\text{Max}\{0, n - (N - D)\} \leq x \leq \text{Min}\{n, D\}$
- 2) $1 \leq n \leq N$
- 3) $D \leq N$
- 4) $0 < p < 1, 0 < q < 1$ and $p+q = 1$,

an expansion of the hypergeometric probability in term of Krawtchouk's polynomial is given by Equation 4:

$$h(x; n, D, N) = b(x; n, p) \sum_{m=0}^n \binom{N}{m}^{-1} (pq)^{-m} k_m(x; n, p) k_m(D; N, p) \quad (4)$$

An approximation of the relationship between the hypergeometric and the binomial probabilities is given by Equation 5:

$$\begin{aligned} b_r(x; n, p) &= b(x; n, p) \sum_{m=0}^r \binom{N}{m}^{-1} (pq)^{-m} k_m(x; n, p) k_m(D; N, p) \\ &= b(x; n, p) \sum_{m=0}^r \binom{N}{m}^{-1} (pq)^{-m} k_m(D; N, p) k_m(x; n, p) \\ &= b(x; n, p) \sum_{m=0}^r \delta_m k_m(x; n, p) \end{aligned} \quad (5)$$

where $r = 0, 1, \dots, n$

$$\delta_m = \binom{N}{m}^{-1} (pq)^{-m} k_m(D; N, p) \quad \text{for } m = 0, 1, \dots, n$$

$$\delta_{m+1} = -\frac{m}{(N-m)pq} [(q-p)\delta_m + \delta_{m-1}] \quad \text{for } m = 1, \dots, n-1$$

$$p = \frac{D}{N}, \quad q = 1-p$$

$k_m(x;n,p)$ and $k_m(D;N,p)$ are Krawtchouk's polynomials.

For a large r ($r \leq n$) and any $\alpha < \frac{r}{2} + 1$, $\lim_{N \rightarrow \infty} N^\alpha \sum_{x=0}^n |h(x;n,D,N) - b_r(x;n,p)| = 0$.

However, an approximation to the hypergeometric distribution using an expansion of hypergeometric probabilities in terms of the Krawtchouk's polynomial is difficult to calculate for a large r . The sufficiency value to approximate the hypergeometric probabilities in term of Krawtchouk's polynomial occurs when $r = 4$ because Blázquez and Miño (2000) indicated that there is not much difference between the values of $h(x;n,D,N)$ and $b_r(x;n,p)$ for some example comparisons. Then, $k_r(x;n,p)$, $k_r(D;N,p)$ and δ_r for $r = 1, 2, 3, 4$ are as follows:

$$\begin{aligned}
 k_0(x;n,p) &= \sum_{j=0}^0 \binom{x}{j} \binom{n-x}{0-j} (-p)^{0-j} q^j = 1 \\
 k_1(x;n,p) &= \sum_{j=0}^1 \binom{x}{j} \binom{n-x}{1-j} (-p)^{1-j} q^j = x - np \\
 k_2(x;n,p) &= \sum_{j=0}^2 \binom{x}{j} \binom{n-x}{2-j} (-p)^{2-j} q^j \\
 &= \frac{1}{2} [x(x-1)q^2 - 2x(n-x)pq + (n-x)(n-x-1)p^2] \\
 k_3(x;n,p) &= \sum_{j=0}^3 \binom{x}{j} \binom{n-x}{3-j} (-p)^{3-j} q^j \\
 &= \frac{1}{6} \left[x(x-1)(x-2)q^3 - 3x(x-1)(n-x)pq^2 \right. \\
 &\quad \left. + 3x(n-x)(n-x-1)p^2q - (n-x)(n-x-1)(n-x-2)p^3 \right] \\
 k_4(x;n,p) &= \sum_{j=0}^4 \binom{x}{j} \binom{n-x}{4-j} (-p)^{4-j} q^j \\
 &= \frac{1}{24} \left[x(x-1)(x-2)(x-3)q^4 \right. \\
 &\quad - 4x(x-1)(x-2)(n-x)pq^3 \\
 &\quad + 6x(x-1)(n-x)(n-x-1)p^2q^2 \\
 &\quad - 4x(n-x)(n-x-1)(n-x-2)p^3q \\
 &\quad \left. + (n-x)(n-x-1)(n-x-2)(n-x-3)p^4 \right] \\
 k_0(D;N,p) &= \sum_{j=0}^0 \binom{D}{j} \binom{N-D}{0-j} (-p)^{0-j} q^j = 1 \\
 k_1(D;N,p) &= \sum_{j=0}^1 \binom{D}{j} \binom{N-D}{1-j} (-p)^{1-j} q^j = Dq - (N-D)p \\
 k_2(D;N,p) &= \sum_{j=0}^2 \binom{D}{j} \binom{N-D}{2-j} (-p)^{2-j} q^j \\
 &= \frac{1}{2} [D(D-1)q^2 - 2D(N-D)pq + (N-D)(N-D-1)p^2] \\
 k_3(D;N,p) &= \sum_{j=0}^3 \binom{D}{j} \binom{N-D}{3-j} (-p)^{3-j} q^j
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{6} \left[\begin{aligned} &D(D-1)(D-2)q^3 - 3D(D-1)(N-D)pq^2 \\ &+ 3D(N-D)(N-D-1)p^2q - (N-D)(N-D-1)(N-D-2)p^3 \end{aligned} \right] \\
k_4(D; N, p) &= \sum_{j=0}^4 \binom{D}{j} \binom{N-D}{4-j} (-p)^{4-j} q^j \\
&= \frac{1}{24} \left[\begin{aligned} &D(D-1)(D-2)(D-3)q^4 \\ &- 4D(D-1)(D-2)(N-D)pq^3 \\ &+ 6D(D-1)(N-D)(N-D-1)p^2q^2 \\ &- 4D(N-D)(N-D-1)(N-D-2)p^3q \\ &+ (N-D)(N-D-1)(N-D-2)(N-D-3)p^4 \end{aligned} \right]
\end{aligned}$$

$$\delta_0 = \binom{N}{0}^{-1} (pq)^{-0} k_0(D; N, p) = 1$$

$$\delta_1 = \binom{N}{1}^{-1} (pq)^{-1} k_1(D; N, p) = 0$$

$$\delta_2 = \delta_{1+1} = -\frac{1}{(N-1)pq} [(q-p)\delta_1 + \delta_{1-1}] = -\frac{1}{(N-1)pq}$$

$$\delta_3 = \delta_{2+1} = -\frac{2}{(N-2)pq} [(q-p)\delta_2 + \delta_{2-1}] = \frac{2(q-p)}{(N-2)(N-1)(pq)^2}$$

$$\begin{aligned}
\delta_4 = \delta_{3+1} &= -\frac{3}{(N-3)pq} [(q-p)\delta_3 + \delta_{3-1}] \\
&= \frac{3}{(N-3)(N-1)(pq)^2} \left[1 - \frac{2(q-p)^2}{(N-2)pq} \right]
\end{aligned}$$

Hence, a modified binomial distribution has a probability distribution in the form of Equation 6:

$$b_4(x; n, p) = \begin{cases} \eta_4 \times b(x; n, p) & \text{for } x = 0, 1, 2, \dots, n \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

where $b(x; n, p)$ is the probability mass function of a binomial random variable

$$\eta_4 = 1 - \frac{1}{2(N-1)pq} A + \frac{(q-p)}{3(N-2)(N-1)p^2q^2} B + \frac{1 - \frac{2(q-p)^2}{(N-2)pq}}{8(N-3)(N-1)(pq)^2} C$$

$$A = x(x-1)q^2 - 2xx'pq + x'(x'-1)p^2$$

$$B = x(x-1)(x-2)q^3 - 3x(x-1)x'pq^2 + 3xx'(x'-1)p^2q - x'(x'-1)(x'-2)p^3$$

$$C = x(x-1)(x-2)(x-3)q^4 - 4x(x-1)(x-2)x'pq^3 + 6x(x-1)x'(x'-1)p^2q^2 \\ - 4xx'(x'-1)(x'-2)p^3q + x'(x'-1)(x'-2)(x'-3)p^4$$

$$x' = n - x$$

$$p = \frac{D}{N} \text{ and } q = 1 - p.$$

Further, the accuracy and reliability of the modified binomial distribution approximation to the hypergeometric distribution can be investigated in terms of the total variation distance as follows.

Suppose P and Q are two probability measures. Then the total variation distance between P and Q on a sigma algebra $\mathcal{K}$ of subsets of the sample space Ω is defined via Equation 7:

$$d(P, Q) = \sup_{A \in \mathcal{K}} |P(A) - Q(A)| \quad (7)$$

SIMULATION RESULTS

A simulation study was conducted to empirically evaluate the accuracy and reliability of the three distributions—the binomial distribution, Ord's distribution and the modified binomial distribution—as approximations to the hypergeometric distribution. In the study, finite populations of size $N = 100, 200, 500$ and $20,000$ were generated in the form of hypergeometric distributions with the sampling fractions $\left(f = \frac{n}{N}\right)$ at $0.01, 0.02, 0.04, 0.06, 0.09, 0.1, 0.2$ and 0.4 . Moreover, proportions having a specified attribute $\left(p = \frac{D}{N}\right)$ were studied at $0.02, 0.04, 0.06, 0.08, 0.1, 0.2, 0.4, 0.6$ and 0.8 . Thus, a total of 288 situations were created for the simulation study. Then, the total variation distances were compared empirically among the three distributions.

Let $d(b, h)$ be a total variation distance between the binomial and the hypergeometric distribution,

$d(\text{Ord}, h)$ be a total variation distance between the Ord's distribution and the hypergeometric distribution,

$d(\text{mod_b}, h)$ be a total variation distance between the modified binomial distribution and the hypergeometric distribution.

Let mod_b , Ord , b and h be the abbreviation of a modified binomial distribution, the Ord's distribution, a binomial distribution and a hypergeometric distribution, respectively.

The simulation results in Figures 1–8 reveal the total variation distances of the binomial, the Ord and the modified binomial approximations to the hypergeometric distribution. Figure 1a indicates that when the population size and the sampling fraction are 100 and 0.01, respectively, the total variation distances of the three estimations approximate zero at all levels of p . Almost all situations in this study, except the case of $N = 100$ and $f = 0.01$, indicated that the values of $d(\text{Ord}, h)$ and $d(\text{mod_b}, h)$ seemed to be less than the value of $d(b, h)$ at all levels of p . However, for a large population size of $N = 20,000$ and all levels of f , the total variation distances of the three estimations were likely to be slightly different, whatever the levels of p . Additionally, Figures 1–8 show the total variation distances of the three estimations tended to increase whenever the sampling fraction increased, whatever the levels of p .

When the population sizes were not greater than 500 and the sampling fraction was $f = 0.4$, as shown in Figures 8a–8c, the value of $d(\text{mod_b}, h)$ seemed to be less than the values of $d(\text{Ord}, h)$ and $d(b, h)$, for all levels of p . When the population sizes were not greater than 500, the values of $d(\text{Ord}, h)$ and $d(\text{mod_b}, h)$ did not seem to be different and they were less than the value of $d(b, h)$, for all levels of p and f .

In addition, the value of $d(b, h)$ was not greater than 0.025 when the sampling fraction was not greater than 0.1, for all the values of p

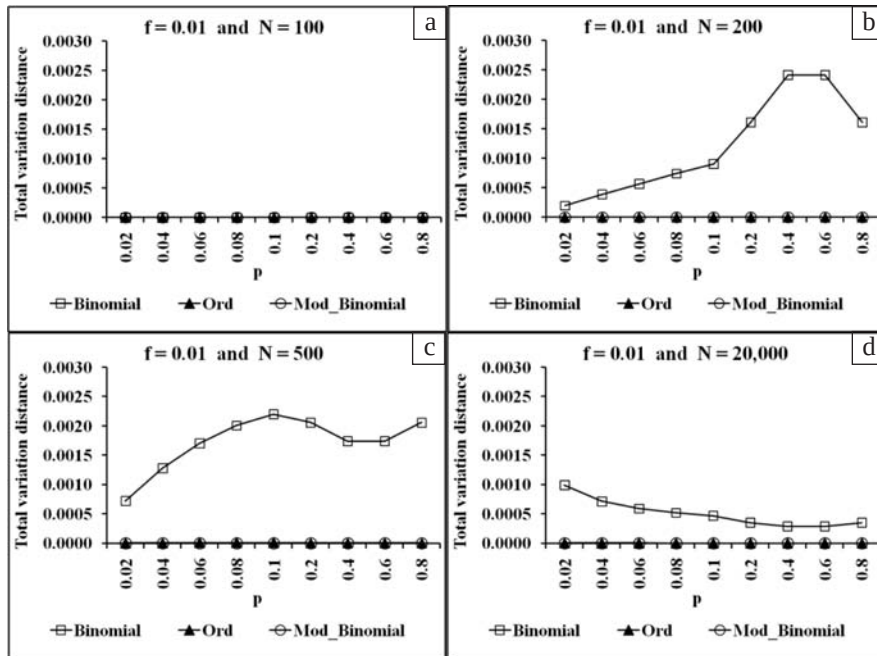


Figure 1 Total variation distance for the binomial, Ord and modified binomial probabilities from the hypergeometric probability when $f = 0.01$, with: (a) $N = 100$; (b) $N = 200$; (c) $N = 500$; (d) $N = 20,000$.

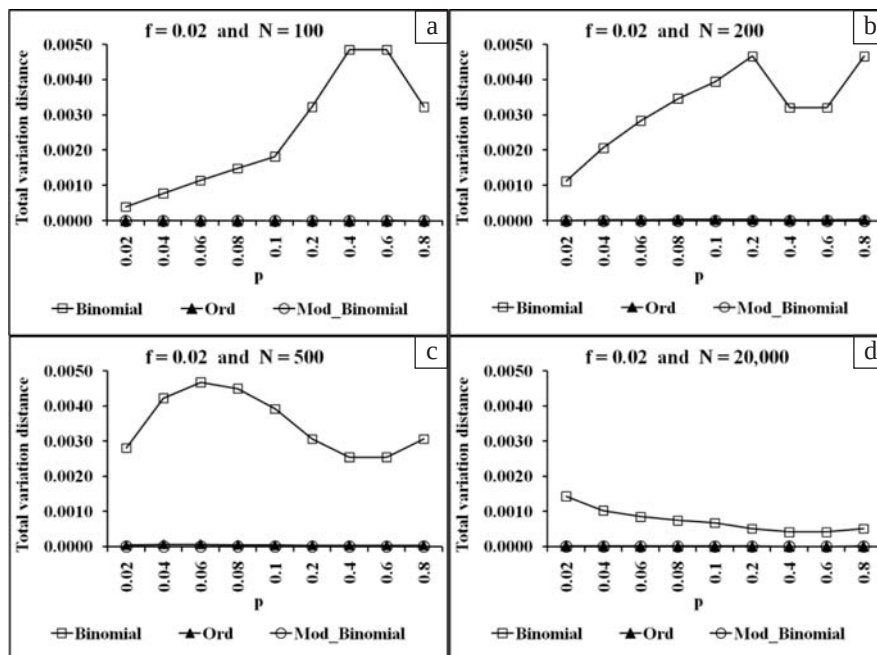


Figure 2 Total variation distance for the binomial, Ord and modified binomial probabilities from the hypergeometric probability when $f = 0.02$, with: (a) $N = 100$; (b) $N = 200$; (c) $N = 500$; (d) $N = 20,000$.

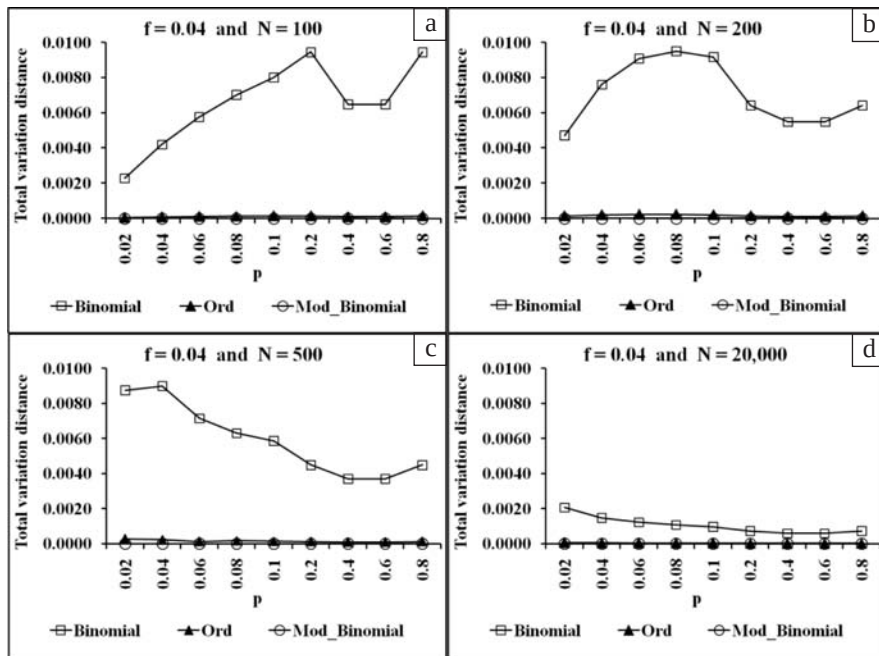


Figure 3 Total variation distance for the binomial, Ord and modified binomial probabilities from the hypergeometric probability when $f = 0.04$, with: (a) $N = 100$; (b) $N = 200$; (c) $N = 500$; (d) $N = 20,000$.

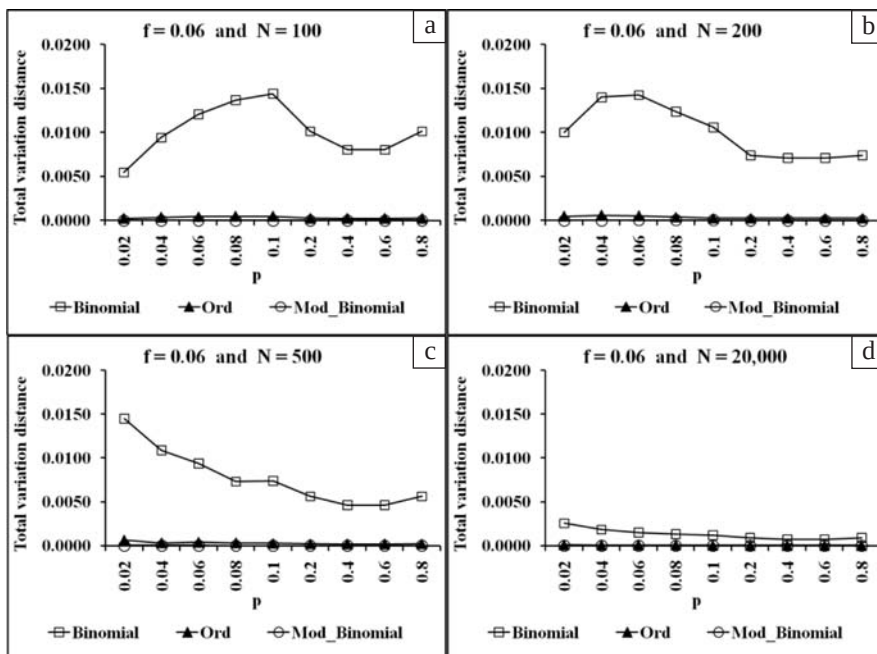


Figure 4 Total variation distance for the binomial, Ord and modified binomial probabilities from the hypergeometric probability when $f = 0.06$, with: (a) $N = 100$; (b) $N = 200$; (c) $N = 500$; (d) $N = 20,000$.

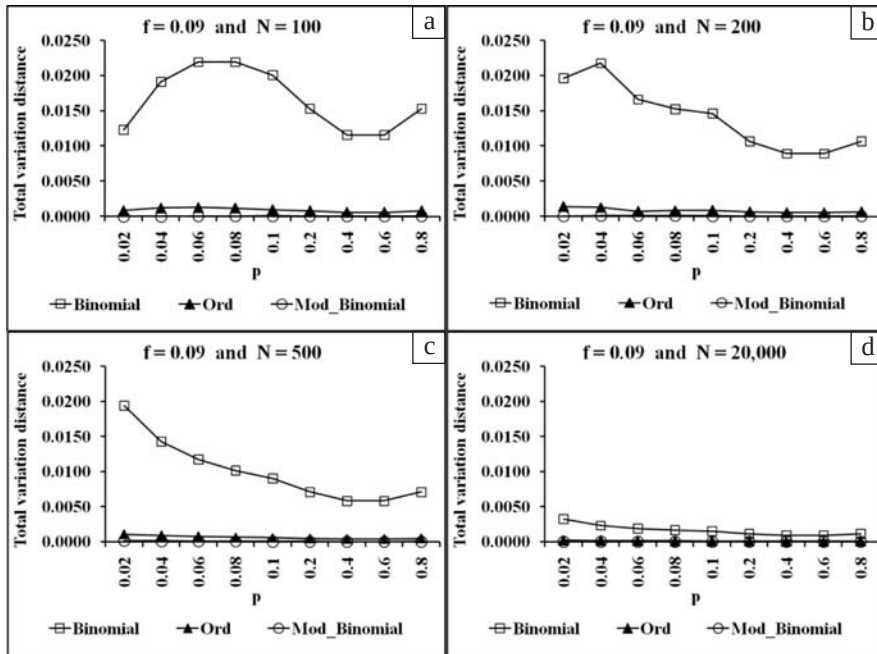


Figure 5 Total variation distance for the binomial, Ord and modified binomial probabilities from the hypergeometric probability when $f = 0.09$, with: (a) $N = 100$; (b) $N = 200$; (c) $N = 500$; (d) $N = 20,000$.

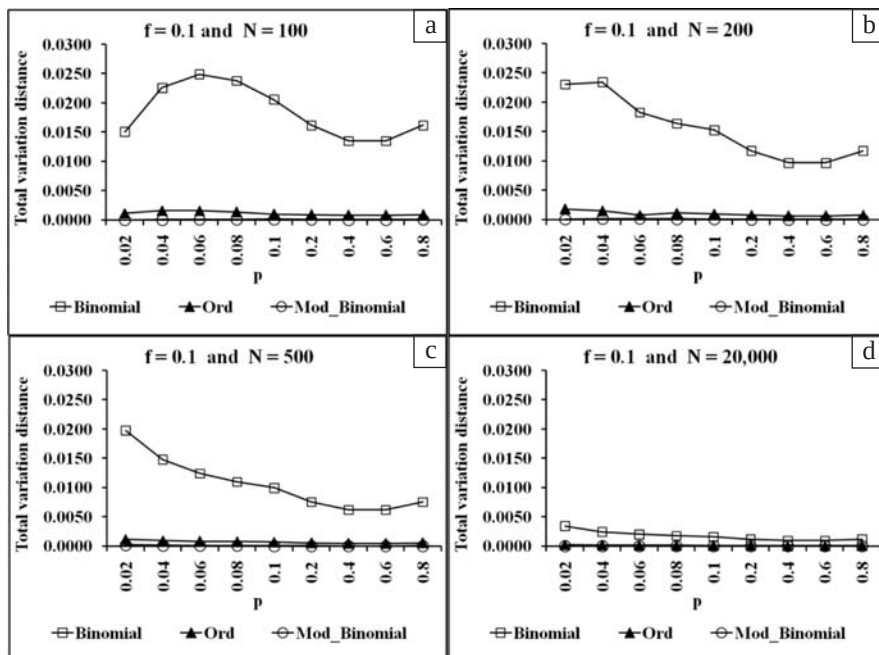


Figure 6 Total variation distance for the binomial, Ord and modified binomial probabilities from the hypergeometric probability when $f = 0.1$, with: (a) $N = 100$; (b) $N = 200$; (c) $N = 500$; (d) $N = 20,000$.

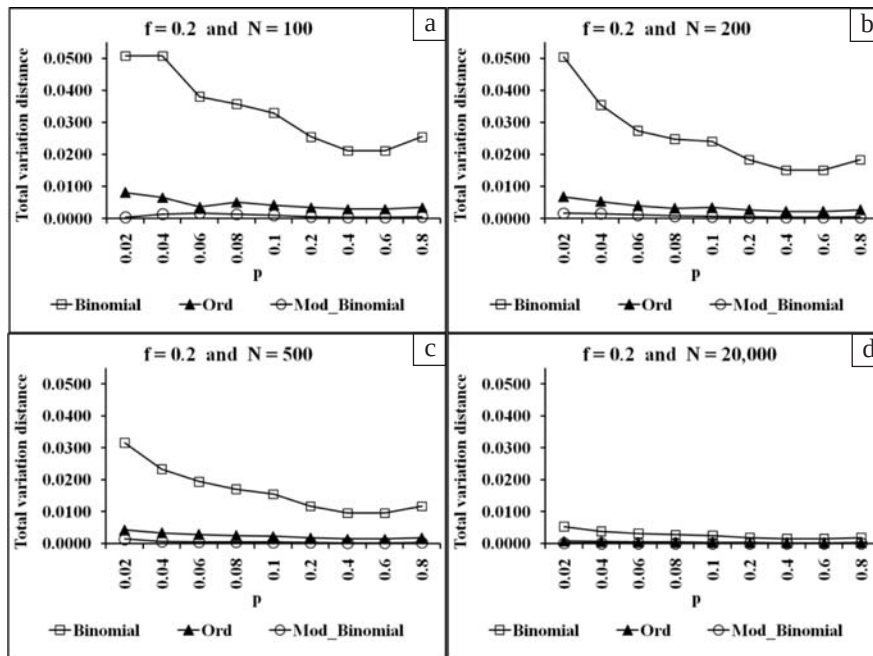


Figure 7 Total variation distance for the binomial, Ord and modified binomial probabilities from the hypergeometric probability when $f = 0.2$, with: (a) $N = 100$; (b) $N = 200$; (c) $N = 500$; (d) $N = 20,000$.

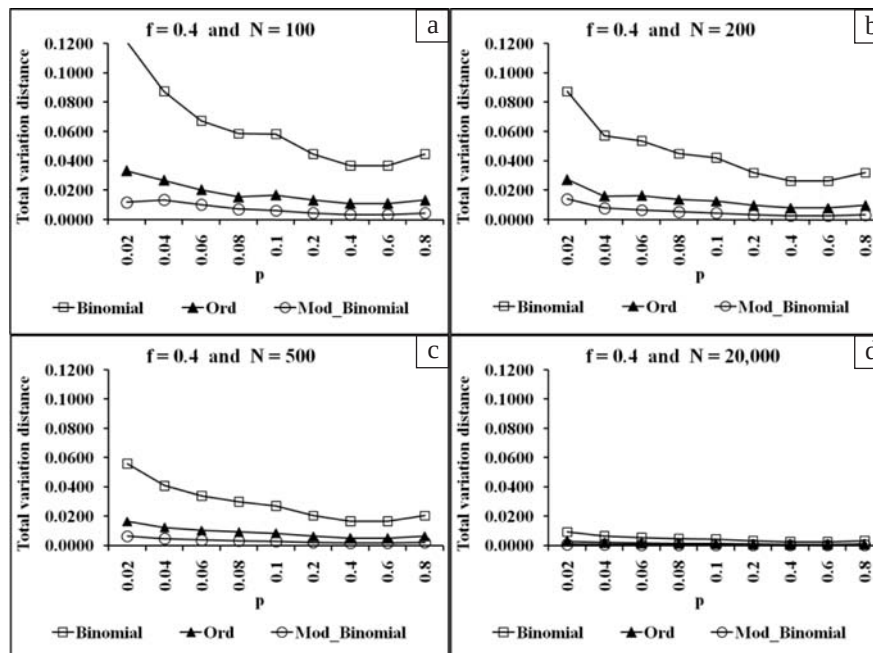


Figure 8 Total variation distance for the binomial, Ord and modified binomial probabilities from the hypergeometric probability when $f = 0.4$, with: (a) $N = 100$; (b) $N = 200$; (c) $N = 500$; (d) $N = 20,000$.

and N . However, when the sampling fraction was greater than 0.1, the value of (b, h) seemed to be large, especially for a sampling fraction of $f = 0.4$, $p = 0.02$ and a population size of 100, where the value of $d(b, h)$ approximated 0.12 and tended to decrease when p increased. The values of $d(\text{mod_}b, h)$ and $d(\text{Ord}, h)$ were not greater than 0.005 when the sampling fraction was not greater than 0.1, for all values of p and N . However, when the sampling fraction was greater than 0.1, the values of $d(\text{mod_}b, h)$ and $d(\text{Ord}, h)$ were not greater than 0.04 for population sizes of $N = 100, 200, 500$ and $20,000$.

DISCUSSION

The total variation distance of the binomial approximation to the hypergeometric distribution was not greater than 0.025 when the sampling fraction was not greater than 0.1 for all levels of the population size in this study, whatever the values of p . However, when the sampling fraction was greater than 0.1, the total variation distance of the binomial approximation to the hypergeometric distribution seemed to be large for all levels of the population size and a small value of p . Hence, the binomial distribution with parameters $p = \frac{D}{N}$ and n is a suitable approximation to the hypergeometric distribution for a small sampling fraction (sampling fraction not greater than 0.1) as mentioned by Weiss (2006), Montgomery (2009) and Evans *et al.* (2000).

The values of $d(\text{mod_}b, h)$ and $d(\text{Ord}, h)$ were not greater than 0.005 when the sampling fraction was not greater than 0.1 for all the values of p and N . However, when the sampling fraction was greater than 0.1, the values of $d(\text{mod_}b, h)$ and $d(\text{Ord}, h)$ were not greater than 0.04 for population sizes of $N = 100, 200, 500$ and $20,000$.

Even the total variation distances of the modified binomial and the Ord approximations to the hypergeometric distribution were not greater

than 0.005 when the sampling fraction was not greater than 0.1 for all levels of the population size in this study, whatever the values of p . When the sampling fraction was greater than 0.1, the total variation distances of the modified binomial and the Ord approximations to the hypergeometric distribution were not greater than 0.04 for all levels of the population size in this study. Therefore, the modified binomial approximation to the hypergeometric distribution using an expansion of the hypergeometric probability in terms of Krawtchouk's polynomial is an efficient approximation to the hypergeometric distribution whatever the sampling fraction and the population size.

CONCLUSION

This paper proposed a modified binomial approximation to the hypergeometric distribution using an expansion of hypergeometric probabilities in term of Krawtchouk's polynomial. The results of the simulation study indicated that the total variation distance of the modified binomial approximation to the hypergeometric distribution was less than those of the binomial and the Ord approximations for all situations of this study and tended to zero for a small sampling fraction. Even for a large population size of $N = 20,000$, the total variation distances of the three estimations seemed to be only slightly different at all levels of the sampling fraction and p .

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