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Jordan Derivations on Rings

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ABSTRACT

An additive mapping $d : R \rightarrow R$ is called a Jordan derivation on a ring R if $d(a^2) = d(a)a + ad(a)$ for all $a \in R$. Two general forms of $d^n(aba)$ and $d^n(abc+cba)$, where $a,b,c \in R$ and n ∈ □, are established. It is also shown that if d is a Jordan derivation on a commutative ring R and P is a semiprime ideal or prime ideal of R where R/P is characteristic – free, then $d(P) \subseteq P$ if and only if $d^n(P) \subseteq P$ for all positive integers n ≥ 2.

Key words: derivation, Jordan derivation, ring

INTRODUCTION

An additive mapping $d : R \rightarrow R$ is called a derivation on R if $d(ab) = d(a)b + ad(b)$ for all pairs a, b ∈ R. An additive mapping $d : R \rightarrow R$ is called a Jordan derivation if $d(a^2) = d(a)a + ad(a)$ holds for all $a \in R$. Obviously, every derivation is a Jordan derivation.

Herstein(1957) showed that if d is a Jordan derivation on a ring R such that characteristic is not 2, then $d(aba) = d(a)ba + ad(b)a + abd(a)$ *and* $d(abc+cba) = d(a)bc + ad(b)c + abd(c) + d(c)ba + cd(b)a + cbd(a)$ for all $a,b,c \in R$.

In 1998, Creedon investigated a ring of characteristic – free, semiprime ideal on ring and proved that, if d is a derivation on a ring R and P is a semiprime ideal of R, such that R/P is characteristic – free and $d^k(P) \subseteq P$, for any fixed positive integer k, then $d(P) \subseteq P$.

In this paper we will consider d as a Jordan derivation on a ring R and establish general forms of $d^n(aba)$ and $d^n(abc+cba)$ for all positive integer n.

We also show that if d is a Jordan derivation on a commutative ring R and P is a semiprime ideal or prime ideal of R where R/P is characteristic – free, then we have $d(P) \subseteq P$ if and only if $d^n(P) \subseteq P$ for all positive integer $n \geq 2$.

MATERIALS AND METHODS

For the proof results we need the following lemmas.

Lemma 1. [Herstein, Lemma 3.1] *Let R be a ring such that characteristic is not 2 and let d be a Jordan derivation on R. Then for all a, b, c ∈ R the following statements hold :*

(*i*) $d(ab + ba) = d(a)b + ad(b) + d(b)a + bd(a)$,
(*ii*) $d(aba) = d(a)ba + ad(b)a + abd(a)$,
(*iii*) $d(abc + cba) = d(a)bc + ad(b)c + abd(c) + d(c)ba + cd(b)a + cbd(a)$.

Lemma 2. [Creedon, Lemma 4] *Suppose P is a semiprime ideal of a ring R and L is a left ideal satisfying $L^n \subseteq P$, for some positive integer n. Then $L \subseteq P$.*

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RESULTS AND DISCUSSION

Theorem 1. *Let R be a ring such that characteristic is not 2. If d is a Jordan derivation on R, then for all a, b ∈ R and for each positive integer n,* $d^n(aba)$

$$= \sum_{\substack{i+j+k=n \\ i,j,k\in\{0,1,2,\ldots,n\}}} \binom{n}{i\ j\ k} d^i(a)d^j(b)d^k(a)$$

PROOF. The proof will be finished by induction on n.

If $n = 1$, the result is just by Lemma 1 (ii) in materials and methods, so we assume $n > 1$ and assume that $d^n(aba)$

$$= \sum_{\substack{i+j+k=n \\ i,j,k\in\{0,1,2,\ldots,n\}}} \binom{n}{i\ j\ k} d^i(a)d^j(b)d^k(a).$$

Since $d^{n+1}(aba) = d(d^n(aba))$, $d^{n+1}(aba) =$

$$d\left[\sum_{\substack{i+j+k=n \\ i,j,k\in\{0,1,2,\ldots,n\}}} \binom{n}{i\ j\ k} d^i(a)d^j(b)d^k(a)\right]$$

$$= \sum_{\substack{i+j+k=n+1 \\ i,j,k\in\{0,1,2,\ldots,n,n+1\}}} \binom{n+1}{i\ j\ k} d^i(a)d^j(b)d^k(a).$$

the proof is complete. #

Theorem 2. *Let R be a ring such that characteristic is* not 2. *If d is a Jordan derivation on R, then for all a,b,c ∈ R and for each positive integer n,*

$d^n(abc+cba)$

$$= \sum_{\substack{i+j+k=n \\ i,j,k\in\{0,1,2,\ldots,n\}}} \binom{n}{i\ j\ k} [d^i(a)d^j(b)d^k(c) + d^i(c)d^j(b)d^k(a)]$$

PROOF. We linearize the result of Theorem 1 by replacing a by $a+c$ to obtain

$d^n((a+c)b(a+c))$

$$= \sum_{\substack{i+j+k=n \\ i,j,k\in\{0,1,2,\ldots,n\}}} \binom{n}{i\ j\ k} d^i(a+c)d^j(b)d^k(a+c)$$

$$= \sum_{\substack{i+j+k=n \\ i,j,k\in\{0,1,2,\ldots,n\}}} \binom{n}{i\ j\ k} d^i(a)d^j(b)d^k(a)$$

$$+ \sum_{\substack{i+j+k=n \\ i,j,k\in\{0,1,2,\ldots,n\}}} \binom{n}{i\ j\ k} d^i(a)d^j(b)d^k(c)$$

$$+ \sum_{\substack{i+j+k=n \\ i,j,k\in\{0,1,2,\ldots,n\}}} \binom{n}{i\ j\ k} d^i(c)d^j(b)d^k(a)$$

$$+ \sum_{\substack{i+j+k=n \\ i,j,k\in\{0,1,2,\ldots,n\}}} \binom{n}{i\ j\ k} d^i(c)d^j(b)d^k(c)$$

$$= d^n(abc)$$

$$+ \sum_{\substack{i+j+k=n \\ i,j,k\in\{0,1,2,\ldots,n\}}} \binom{n}{i\ j\ k} d^i(a)d^j(b)d^k(c)$$

$$+ \sum_{\substack{i+j+k=n \\ i,j,k\in\{0,1,2,\ldots,n\}}} \binom{n}{i\ j\ k} d^i(c)d^j(b)d^k(a)$$

$$+ d^n(cbc).$$

On the other hand, $d^n((a+c)b(a+c))$
$= d^n(aba + cbc + cba + abc)$
$= d^n(aba) + d^n(cbc) + d^n(cba+abc)$
Comparing the two equations, we get the result. #

For the next results, we need the following lemmas.

Lemma 3. *Let P be an ideal of a ring R such that characteristic is not 2. If d is a Jordan derivation on R, then for all a ∈ P and for all positive integers r we have* $d^r(a^s) \in P$ *for all* $s = r+1, r+2, \ldots$

PROOF. We will proof by induction on r. Obviously, for r = 1 $d(a^2) \in P$. If $s \geq 2$, $d(a^{s+1}) = d(aa^{s-1}a)$. Using Lemma1(ii) in Materials and methods, with $b = a^{s-1}$ we obtain $d(aa^{s-1}a) = d(a)a^{s-1}a + ad(a^{s-1})a + aa^{s-1}d(a)$. Since $a \in P$, $d(aa^{s-1}a) \in P$. Hence $d(a^s) \in P$ for all $s \geq 2$. Next, we assume that $r \geq 2$ and $d^r(a^s) \in P$ for all $r=2,...,n$ and $s = r+1, r+2,...$ Since $d^{n+1}(a^{n+2}) = d^{n+1}(aa^n a)$, by theorem 1 we have $d^{n+1}(aa^n a)$

$$= \sum_{\substack{i+j+k=n+1 \\ i,j,k \in \{0,1,2,\ldots,n,n+1\}}} \binom{n+1}{i \;\; j \;\; k} d^i(a) d^j(a^n) d^k(a).$$

Therefore $d^{n+1}(aa^n a) \in P$ because $a \in P$. Hence $d^{n+1}(a^{n+2}) \in P$. Let $n+2 = t$, $u \in$ □ and suppose that $d^{n+1}(a^s) \in P$ for all $s = t+1, t+2, ..., t+u$. Since $d^{n+1}(a^{t+u+1})$

$$= \sum_{\substack{i+j+k=n+1 \\ i,j,k \in \{0,1,2,\ldots,n,n+1\}}} \binom{n+1}{i \;\; j \;\; k} d^i(a) d^j(a^{n+u+1}) d^k(a)$$

and $a \in P$, then $d^{n+1}(a^{t+u+1}) \in P$. Hence $d^{n+1}(a^s) \in P$ *for* $s > n+1$. Thus we conclude that $d^r(a^s) \in P$ for all $s = r+1, r+2,...$ #

Lemma 4. *Let P be an ideal of R such that characteristic is not 2 and let $x \in P$. Suppose that d is a Jordan derivation on R. Then $d^n(x^n) - n!(dx)^n \in P$ for all positive integers n.*

PROOF. By definition of prime ideal, the result holds for $n = 1$. If $n \geq 2$, suppose $d^n(x^n) - n!\,(dx)^n \in P$. Since $d^{n+1}(x^{n+1}) = d^{n+1}(xx^{n-1}x)$, by Theorem 1 we have $d^{n+1}(xx^{n-1}x)$

$$= \sum_{\substack{i+j+k=n+1 \\ i,j,k \in \{0,1,2,\ldots,n,n+1\}}} \binom{n+1}{i \;\; j \;\; k} d^i(x) d^j(x^{n-1}) d^k(x).$$

Since $x \in P$,

$$\sum_{\substack{i+j+k=n+1 \\ i,j,k \in \{0,1,2,\ldots,n,n+1\}}} \binom{n+1}{i \;\; j \;\; k} d^i(x) d^j(x^{n-1}) d^k(x)$$

$$= \binom{n+1}{1 \;\; n-1 \;\; 1} d(x) d^{n-1}(x^{n-1}) d(x) + P.$$

Thus $d^{n+1}(x^{n+1}) \in (n+1)n\,[d(x)\, d^{n-1}(x^{n-1}) d(x)] + P$, and so $d^{n+1}(x^{n+1}) \in (n+1)!\,(dx)^{n-1} + P$. Hence $d^{n+1}(x^{n+1}) - (n+1)!(dx)^{n+1} \in P$. Inductively we have $d^n(x^n) - n!(dx)^n \in P$. #

Theorem 5. *If d is a Jordan derivation on a commutative ring R and P is a semiprime ideal of R for which R/P has characteristic-free, then $d(P) \subseteq P$ if and only if $d^n(P) \subseteq P$ for all positive integer $n \geq 2$.*

PROOF. Suppose that $d^n(P) \subseteq P$ for all positive integer $n \geq 2$. Lemma 4 says that $d^n(x^n) - n!(dx)^n \in P$ for all $x \in P$. Since $d^n(x^n) \in P$, we see that $n!\,[d(x)]^n \in P$. Hence, $[d(x)]^n \in P$. Consider the ideal $\langle d(x)\rangle + P$ of R, we have $(\langle d(x)\rangle + P)^n \subseteq P$. Therefore, by Lemma 2 in Materials and Methods, we see that $(\langle d(x)\rangle + P) \subseteq P$, and hence $d(P) \subseteq P$. The reverse implication is obvious. #

The following theorem can be proven in a similar way.

Theorem 6. *If d is a Jordan derivation on a commutative ring R and P is a prime ideal of R such that $P \neq R$ and R/P has characteristic-free, then $d(P) \subseteq P$ if and only if $d^n(P) \subseteq P$ for all positive integer $n \geq 2$.*

CONCLUSION

Results of the studying show that:

1. If d^n is a Jordan derivation on a ring R such that characteristic is not 2, then

1.1 $d^n(aba)$

$$= \sum_{\substack{i+j+k=n \\ i,j,k \in \{0,1,2,\ldots,n\}}} \binom{n}{i \;\; j \;\; k} d^i(a) d^j(b) d^k(a)$$

1.2 $(abc+cba)$

$$= \sum_{\substack{i+j+k=n \\ i,j,k \in \{0,1,2,\ldots,n\}}} \binom{n}{i \;\; j \;\; k} \left[d^i(a) d^j(b) d^k(c) \right.$$

$$\left. + \; d^i(c) d^j(b) d^k(a) \right]$$

for all $a,b,c \in R$ and for any positive integer n.

2. If d is a Jordan derivation on a commutative ring R and P is a semiprime ideal of R (or prime ideal *of R such that* $P \neq R$) for which R/P has characteristic-free, then $d(P) \subseteq P$ if and only if $d^n(P) \subseteq P$ for all positive integer $n \geq 2$.

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