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Some Basic Properties of Almost-Prime Left Ideals in Γ-Semirings

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ABSTRACT

The purpose of this paper was to introduce the notion of almost-prime left ideals in Γ-semirings by studying prime and weakly almost-prime left ideals in Γ-semirings. Some characterizations of almost-prime and weakly almost-prime left ideals were obtained. Moreover, the relationships between almost-prime and weakly almost-prime left ideals in Γ –semirings were investigated.
Keywords: Γ -semiring, ideal, quasi-ideal, almost-prime left ideal, weakly almost-prime left ideal

INTRODUCTION

The notion of Γ-semirings, was introduced in 1995 (Murali Krishna Rao, 1995) that is, a semigroup S is called a Γ -semiring if Γ is a semigroup and there exists a mapping

$$S\times\Gamma\times S\to S$$

satisfying some certain properties. (Dutta and Sardar, 2002a), gave the meaning of left and right operator semirings for a given Γ -semiring. Moreover, some relationships between Γ -semirings and their left and right operator semirings have been illustrated (Dutta and Sardar, 2002b).

It is natural to extend the concept of quasi-ideals in Γ-semirings and this was done by Chinram (2008) as a generalization of quasi-ideals in Γ -semigroups. The Γ -semirings introduced by Dutta and Sardar (2002c) and Chinram (2008) are different.

In Γ -semirings, the properties of their ideals, prime ideals, semiprime ideals and their generalizations play an important role in the theory of their structure; however, the properties of an ideal in semirings and Γ-semirings are somewhat different from the properties of the usual ring ideals (Dutta and Sardar, 2000).

BASIC RESULTS

This section refers to Sardar and Dasgupta (2004), Jagatap and Pawar (2009), and Sardar and Mukhopadhyay (2010) for some elementary aspects and quotes a few definitions and examples which are essential to step through this study. More detail can be found in the papers in the references.

Definition 1. (Sardar and Mukhopadhyay, 2010) Let $(S, +)$ and $(\Gamma, +)$ be commutative semigroups. Then, we call S a Γ -semiring if there exists a map $S\times\Gamma\times S\to S$ written (x,γ,y) by $x\gamma y$, such that it satisfies the following axioms (Equations 1–3) for all $x, y, z \in S$ and $\gamma,\alpha\in\Gamma$:

$$x\gamma(y + z) = x\gamma y + x\gamma z \text{ and } (x+y)\gamma z = x\gamma z + y\gamma z \quad (1)$$

$$x(\gamma + \alpha)y = x\gamma y + x\alpha y \quad (2)$$

$$(x\gamma y)\alpha z = x\gamma(y\alpha z) \quad (3)$$

Clearly, every semiring S is a Γ-semiring but not conversely. For this, let us consider the following example.

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Example 2. (Jagatap and Pawar, 2009) Let Q be the set of rational numbers. $(S,+)$ be the commutative semigroup of all 2×3 matrices over Q and $(\Gamma, +)$ be the commutative semigroup of all 3× 2 matrices over Q Define $A\alpha B$ = usual matrix product of A, α and B for all $A, B \in S$ and for all $\alpha \in \Gamma$. Then S is a Γ-semiring but not a semiring.

Definition 3. (Jagatap and Pawar, 2009) A nonempty subset A of S is called a left (respectively right) ideal of S if A is a subsemigroup of $(S, +)$ and $r\gamma a \in A$ (respectively $a\gamma r \in A$) for all $a \in A$, $r \in S$ and for all $\gamma \in \Gamma$. If A is both left and right ideal of S, then A is known as an ideal.

Definition 4. (Sardar and Mukhopadhyay, 2010) A left (right, two-sided) ideal A of a Γ-semiring S is said to be a left (right, two-sided) k-ideal of S if a, $a + x \in A$, then $x \in A$ for any $x \in S$.

Definition 5. (Sardar and Dasgupta, 2004) Let S be a Γ-semiring. A proper ideal P of S is said to be prime if for any two ideals H and K of S, $H\Gamma K \subseteq P$ implies that either $H \subseteq P$ or $K \subseteq P$.

Definition 6. (Sardar and Dasgupta, 2004) A subsemigroup A of $(S, +)$ is a quasi-ideal of S if $(S\Gamma A) \cap (A\Gamma S) \subseteq A$.

IDEALS IN Γ-SEMIRINGS

The results of the following lemmas seem to play an important role in the study of Γ-semiring; these facts will be used so frequently that normally we shall make no reference to this lemma.

Lemma 6. If S is a Γ-semiring with identity, then $a\gamma b = a\alpha b$ for all $a, b \in S$ and $\gamma, \alpha \in \Gamma$.

Proof. Let S be a Γ-semiring and e be the identity of S, and let $a, b \in S$ and $\gamma, \alpha \in \Gamma$. therefore we have

$$\begin{aligned} a\gamma b &= a\gamma(e\alpha b) \\ &= (a\gamma e)\alpha b \\ &= a\alpha b. \end{aligned}$$

Hence $a\gamma b = a\alpha b$.

Lemma 7. Let S be a Γ-semiring with identity and let $a \in S$. If A is a left ideal of S, then $A\gamma a$ is a left ideal in S, where $\gamma \in \Gamma$.

Proof. Let S be a Γ-semiring with left identity and let $a \in S$. Now consider

$$\begin{aligned} s\gamma a + r\gamma a &= (s + r)\gamma a \\ &\in A\gamma a \end{aligned}$$

and

$$\begin{aligned} S\Gamma(A\gamma a) &\subseteq (S\Gamma A)\gamma a \\ &\subseteq A\gamma a \end{aligned}$$

for all $r, s \in A$ and $\gamma, \alpha \in \Gamma$. Therefore $A\gamma a$ is a left ideal in S.

Corollary 8. Let S be a Γ-semiring with identity and let $a \in S$. If A is a right ideal of S, then $a\gamma A$ is a right ideal in S, where $\gamma \in \Gamma$.

Proof. It is similar to the proof of Lemma 7.

Lemma 9. Let S be a Γ-semiring with identity, and let A, B be a left ideal of S. Then $(A{:}\Gamma{:}\,B)$ is a left ideal in S, for each left ideal B of S and $(A{:}\,\Gamma{:}\,B) = \{a \in S{:}\ a\Gamma B \subseteq A\}$.

Proof. Suppose that S is a Γ-semiring with left identity. Let $s \in S$ and let $a, b \in (A{:}\,\Gamma{:}\,B)$. Then $a\Gamma B \subseteq A$ and $b\Gamma B \subseteq A$ so that

$$\begin{aligned} (a + b)\Gamma B &= a\Gamma B + b\Gamma B \\ &\subseteq A + A \\ &= A \end{aligned}$$

and

$$\begin{aligned} (s\gamma a)\Gamma B &= s\gamma(a\Gamma B) \\ &\subseteq s\gamma A \\ &\subseteq A \end{aligned}$$

for all $\gamma \in \Gamma$. Therefore $a + b \in (A{:}\,\Gamma{:}\,B)$ and $S\Gamma(A{:}\,\Gamma{:}\,B) \subseteq (A{:}\,\Gamma{:}\,B)$. Hence $(A{:}\,\Gamma{:}\,B)$ is a left ideal in S.

Corollary 10. Let S be a Γ-semiring with identity, and let A be a left ideal of S. Then $(A{:}\gamma{:}\,r)$ is a left ideal in S, where $(A{:}\gamma{:}\,r) = \{ a \in S{:}\ a\gamma r \in \mathrm{A}\}$.

Proof. This follows from Lemma 9.

Remark Let S be a Γ -semiring and let A be left ideals of S. It is easy to verify that $(A{:}\Gamma{:}C) \subseteq (A{:}\Gamma{:}B)$, where $B \subseteq C$.

Theorem 11. Let S be a Γ -semiring with identity, and let A be a quasi-ideal of S. Then $(A{:}\ \Gamma{:}\ B)$ is a quasi-ideal in S.

Proof. Assume that A is a quasi-ideal of S. By Lemma 9, we have $(A{:}\ \Gamma{:}\ B)$ is a left ideal in S. Then

$$\begin{aligned}(S\Gamma(A{:}\ \Gamma{:}\ B)) \cap ((A{:}\ \Gamma{:}\ B)\Gamma S) &\subseteq (A{:}\ \Gamma{:}\ B) \cap (A{:}\ \Gamma{:}\ B)\\ &\subseteq (A{:}\ \Gamma{:}\ B).\end{aligned}$$

Hence $(A{:}\ \Gamma{:}\ B)$ is a quasi-ideal in S.

Theorem 12. Let S be a Γ-semiring with identity, and let A be a left k-ideal of S. Then $(A{:}\Gamma{:}B)$ is a left k-ideal in S.

Proof. Assume that A is a left k-ideal of S. By Lemma 9, we have $(A{:}\ \Gamma{:}\ B)$ is a left ideal in S. Let a, $a + x \in (A{:}\ \Gamma{:}\ B)$. So that $a\Gamma B \subseteq A$ and $(a + x)\Gamma B \subseteq A$ that is

$$a\Gamma B \subseteq A$$

and

$$a\Gamma B + x\Gamma B \subseteq A.$$

Then, we get $x\Gamma B \subseteq A$. Hence $(A{:}\ \Gamma{:}\ B)$ is a left k-ideal in S.

ALMOST-PRIME IDEALS IN Γ -SEMIRINS

We start with the following theorem that gives a relation between an almost-prime and a weakly almost-prime left ideal in Γ -semirings. Our starting point is the following definition:

Definition 13. Let A be a left ideal and B be a right ideal of S. A left ideal P is called almost- prime if $A\Gamma B \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$.

Remark It is easy to see that every almost-prime left ideal is prime.

Definition 14. Let A be a left ideal and B be a right ideal of S. A left ideal P is called weakly almost-prime if $\{0\} \neq A\Gamma B \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$.

Remark It is easy to see that every almost-prime left ideal is weakly almost-prime.

Lemma 15. Let S be a Γ -semiring with identity and P be an ideal of S. Then P is a almost-prime left ideal of S if $a\Gamma(S\Gamma b) \subseteq P$, then $a \in P$ or $b \in P$.

Proof. Let P be an almost-prime left ideal of a Γ -semiring S with identity. Now suppose that $a\Gamma(S\Gamma b) \subseteq P$. Then by hypothesis, we get

$$\begin{aligned}(S\Gamma a)\Gamma(b\Gamma S) &\subseteq (S\Gamma a)\Gamma S\Gamma(b\Gamma S)\\ &= (S\Gamma a)\Gamma(S\Gamma b)\Gamma S\\ &= S\Gamma(a\Gamma(S\Gamma b))\Gamma S\\ &\subseteq (S\Gamma P)\Gamma S\\ &\subseteq P\Gamma S\\ &\subseteq P\end{aligned}$$

that is $(S\Gamma a)\Gamma(b\Gamma S) \subseteq P$. Then $a = e\gamma a \in S\Gamma a \subseteq P$ or $b = b\gamma e \in b\Gamma S \subseteq P$. Hence $a \in P$ or $b \in P$.

Corollary 16. Let S be a Γ -semiring with identity and P be an ideal of S. Then P is a weakly almost-prime left ideal of S if $\{0\} \neq a\Gamma(S\Gamma b) \subseteq P$, then $a \in P$ or $b \in P$.

Proof. This follows from Lemma 15.

Theorem 17. Let S be a Γ -semiring with identity and let $a, b \in S$, $\gamma \in \Gamma$. Then a left ideal P of S is almost-prime if and only if $a\gamma b \in P$ implies that $a \in P$ or $b \in P$.

Proof. Let P be a left ideal of a Γ -semiring with identity. Now suppose that $a\gamma b \in P$, where $a, b \in S$ and $\gamma \in \Gamma$. Then by hypothesis, we get

$$\begin{aligned}(S\Gamma a)\gamma(b\Gamma S) &\subseteq S\Gamma((a\gamma b)\Gamma S)\\ &\subseteq S\Gamma(P\Gamma S)\\ &\subseteq S\Gamma P\\ &\subseteq P\end{aligned}$$

So by the definition of almost-prime, we have $a \in P$ or $b \in P$. Conversely, assume that if $a\gamma b \in P$ implies that $a \in P$ or $b \in P$. Let A be a left ideal of S. Suppose that $A\Gamma B \subseteq P$, where B is a right ideal of S such that $B \subseteq S - P$. Then there exists $b \in B$ such that $b \notin P$. Now we get $a\gamma b \in P$. So by hypothesis, $a \in P$, for all $a \in A$ implies that $A \subseteq P$. Hence P is almost-prime left ideal in S.

Corollary 18. Let S be a Γ-semiring with identity and let $a, b \in S$, $\gamma \in \Gamma$. Then a left ideal P of S is weakly almost-prime if and only if $0 \neq a\gamma b \in P$ implies that $a \in P$ or $b \in P$.
Proof. This follows from Theorem 17.

Theorem 19. Let S be a Γ -semiring with left identity and let A be an ideal of S. If A is a almost-prime left ideal of S, then $(A{:}\Gamma{:}B)$ is a almost-prime left ideal in S, where $B \subseteq S\text{-}A$.
Proof. Assume that A is a almost-prime left ideal of S. By Lemma 9, we have $(A{:}\ \Gamma{:}\ B)$ is a left ideal in S. Let $a\gamma b \in (A{:}\ \Gamma{:}\ B)$, where $a, b \in S$ and $\gamma \in \Gamma$. Suppose that $b \notin (A{:}\ \Gamma{:}\ B)$. Since $a\gamma b \in (A{:}\ \Gamma{:}\ B)$, we have $(a\gamma b)\Gamma B \subseteq A$. So by hypothesis,

$$\begin{aligned}(S\Gamma a)\gamma(b\Gamma B) &= S\Gamma((a\gamma b)\Gamma B)\\ &\subseteq S\Gamma A\\ &\subseteq A\end{aligned}$$

By the definition of almost-prime, we have $a = e\gamma a \in S\Gamma a \subseteq A$ or $b\Gamma B \subseteq A$ implies that $a\Gamma S \subseteq A\Gamma S \subseteq A$. Hence $(A{:}\ \Gamma{:}\ B)$ is a almost-prime left ideal in S.

Corollary 20. Let S be a Γ -semiring with left identity and let A be an ideal of S. If A is a weakly almost-prime left ideal of S, then $(A{:}\ \Gamma{:}\ B)$, is a weakly almost-prime left ideal in S, where $B \subseteq S - A$.
Proof. This follows from Theorem 19.

Corollary 21. Let S be a Γ -semiring with left identity and let A be an ideal of S. If A is an almost-prime left ideal of S, then $(A{:}\gamma{:}\ s)$, is an almost-prime left ideal in S, where $s \in S\text{-}A$ and $\gamma \in \Gamma$.
Proof. This follows from Theorem 19.

Corollary 22. Let S be a Γ -semiring with left identity and let A be an ideal of S. If A is a weakly almost-prime left ideal of S, then $(A{:}\ \gamma{:}\ s)$, is a weakly almost-prime left ideal in S, where $s \in S - A$ and $\gamma \in \Gamma$.
Proof. This follows from Theorem 18.

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