Kasetsart J. (Nat. Sci.) 49 : 983 - 989 (2015)

Optimal Control Problem of Food Intake of Swine During Post Weaning Period

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ABSTRACT

The amount of food intake clearly affects swine body weight. The most suitable weight of swine on the day of sale is the weight that fetches the best price. The objective of this research was to minimize the amount of food fed to swine in the post weaning period so that the weight reaches a desirable final value on a fixed day of sale. Optimal control problems were derived by assuming that the weight increase follows either the logistic or the Gompertz equations with parameters estimated from actual growth data. Numerical solutions of the optimal control problem were obtained and discussed.
Keywords: swine farm, optimal control, logistic model, Gompertz equation

INTRODUCTION

Swine have traditionally been an important part of the integrated farming system in Thailand and in particular, pork has become the second most important meat in Thai consumption, with average consumption in the late 1990s of about 4.7 kg per person per year (Food and Agriculture Organization Corporate Document Repository, 2002). The amount of food fed daily to the animals is an important factor for consideration in order to increase the production and profitability of a swine farm (Thatchai, 2011).

This research aimed to minimize the total amount of food that should be fed to swine in the post-weaning period from 30 to 170 d after birth so that the weight at the day of the sale was 100 kg, which was taken to be the most profitable weight. The growth rate of swine as a function of age and food intake has been modeled by the logistic equation (Frank *et al.*, 2002) and by the Gompertz equation (Zeide, 1993) and a Michaelis-Menten relationship (Murray, 2007). For the current study, the parameters in the models were estimated by fitting the equations to the data shown in Table 1 (Ministry of Agriculture and Cooperatives, 2014). The problem can then be formulated as an optimal control problem with the daily food intake as the control variable. The optimal system is derived and then solved numerically for specific parameter values.

OPTIMAL CONTROL PROBLEM

This research work aimed to minimize the amount of food that should be fed to swine in the 140 d of the post-weaning period, namely from 30 to 170 d after birth, so that the weight on the day of the sale was 100 kg, which was taken to be the most profitable weight. If the swine are heavier,

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Received date : 26/04/15 Accepted date : 03/11/15

they would have too much accumulated fat and if the weight is lower, then the price would also be lower. The objective function to be minimized is shown in Equation 1:

$$J\{u\} = \int_{t_0}^{t_f} u(t)\, dt \tag{1}$$

where $u(t)$ is the daily food intake at time t, and where the weight $x(t)$ is the solution of an ordinary differential equation (Equation 2):

$$\frac{dx}{dt} = f\left(x(t), u(t)\right) \tag{2}$$

where $f(x(t),u(t))$ is an increasing function of x and u that describes the growth rate of the swine.

Modeling growth rate of swine

The model of body weight was based on the data shown in Table 1 for measurements of weight and growth rate as a function of age and food for swine in Thailand. The data on age and weight in Table 1 was used to estimate the functions which relate the body weight to the age of the swine for four different functional forms: a) linear function, b) exponential function, c) logistic growth model and d) Gompertz growth model. Figure 1 shows the data (marked with *) and the best fit for the different functions. The solid line represents the linear function, $x(t) = x_0 + rt$, the dashed line represents the exponential function, $x(t) = x_0 + e^{rt}$, the dotted line represents the logistic growth model (Equation 3),

$$x(t) = \frac{x_0 K}{x_0 + (K - x_0)e^{-rt}} \tag{3}$$

and the dash-dot line describes the Gompertz growth model (Equation 4), where

$$x(t) = K e^{\ln\left(\frac{x_0}{K}\right) e^{-rt}}. \tag{4}$$

As shown in Table 2, the Gompertz function gave the best fit to the age and weight data in Table 1 with the coefficient of determination $R^2 = 0.9986$. The linear, exponential and logistic functions yielded R^2 values of 0.9643, 0.8484 and 0.9969, respectively. Table 2 also shows the parameter values of the growth rate factors r and the constants K in the above-mentioned functions.

In the optimal control problem, the food intake per day was the control variable $u(t)$ in kilograms per day and it was assumed that the growth rate of the swine was a function of weight and daily food intake and could be modeled by the differential equation in Equation 2. It was assumed that the growth rate function $f(x(t),u(t)) = F(x(t))$

Table 1 Data on swine body weight, age and food intake (Source: Ministry of Agriculture and Cooperatives, 2014).

Age (d)	Weight (kg)	Growth rate ($kg.d^{-1}$)	Daily feed (kg)
30	6.5	0.15	0.3
42	9.0	0.33	0.5
60	15.0	0.50	1.0
70	22.0	0.60	1.4
82	30.0	0.65	1.5
94	40.0	0.70	2.0
106	50.0	0.72	2.2
120	60.0	0.75	2.4
133	70.0	0.78	2.6
145	80.0	0.80	2.8
158	90.0	0.80	3.0
170	100.0	0.80	3.0

$G(u(t))$, where $F(x(t))$ is either the Gompertz or logistic growth-rate function with the parameters given in Table 2, and $G(u(t))$ is given by the Michaelis-Menten (Holling type-II) function (Equation 5);

$$G(u(t)) = \frac{\alpha u(t)}{1+\beta u(t)}, \quad (5)$$

where $\alpha, \beta > 0$ have been estimated from a least-squares fit to the growth rate and daily feed data in Table 1 to be $\alpha = 0.8497$ d.kg^{-1} and $\beta = 0.7129$ d.kg^{-1} .

Solution of optimal control problem with unbounded control

Logistic growth function

The optimal control was determined assuming that the food intake control variable $u(t)$ is unbounded and that the weight increase follows the modified logistic equation in the 140 d following the weaning period. The optimal control problem is shown in Equations 6 and 7:

$$\text{Minimize} \quad J\{u\} = \int_0^{140} u(t)\, dt \quad (6)$$

subject to the ordinary differential equation:

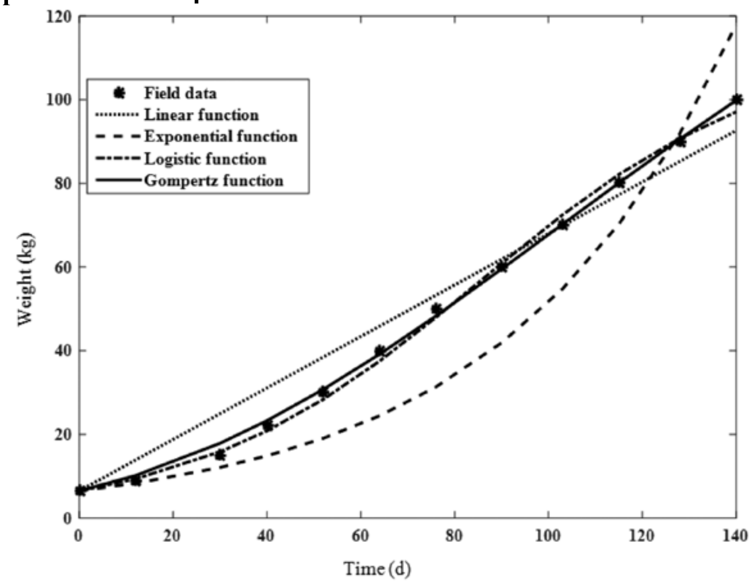

Figure 1 Graphs of functions relating weight to age for linear, exponential, logistic and Gompertz functions.

Table 2 Parameter values of *r*, *K* and coefficient of determination (R^2) for the best fits for the linear, exponential, logistic and Gompertz equations to the age-weight data.

Function	$\frac{dx}{dt}$	$x(t)$	K	r	R^2
Linear	r	$x_0 + rt$	-	0.6154	0.9643
Exponential	$rx_0\, e^{rt}$	$x_0\, e^{rt}$	-	0.0207	0.8484
Logistic	$rx\left(1-\frac{x}{K}\right)$	$\frac{x_0 K}{x_0 + (K-x_0)e^{-rt}}$	113.4	0.03275	0.9969
Gompertz	$rx\ln\left(\frac{K}{x}\right)$	$K e^{\ln\left(\frac{x_0}{K}\right) e^{-rt}}$	188.6	0.01191	0.9986

$$\frac{dx}{dt} = F(x(t))G(u(t)) = rx(t)\left(1 - \frac{x(t)}{K}\right)\frac{\alpha u(t)}{1+\beta u(t)} \quad (7)$$

with boundary conditions: $x(0) = 6.5$ kg and $x(140) = 100$ kg. Here, $x(t)$ is the body weight in kilograms at time t, $u(t)$ is the daily food intake in kilograms per day at time t, $r > 0$ is the specific rate of increase for the logistic function, K is the carrying capacity in the logistic function and α, β are positive constants for the Michaelis-Menten function.

In combining the fitted growth-rate functions F for the logistic equation with the Michaelis-Menten function G, it is necessary to combine the constants $r\alpha$ into one constant R so that the average value of R corresponds to a daily food intake control variable u within the interval range of 0.3 to 3.1 of Table 1. From the numerical results, a suitable choice is $R = 3r\alpha$.

The principal technique for solving the type of optimal control problem in Equations 6 and 7 is to use the Pontryagin maximum principle (Pontryagin *et al.*, 1962) and set up the state and adjoint equations which are the necessary conditions for the optimal value of $u(t)$ (Clark, 1990). This paper minimized the objective function and defined the Hamiltonian so that the Hamiltonian must be minimized with respect to the control $u(t)$ (for example, Lenhart and Workman, 2007). For the optimal control problem in Equations 6 and 7, the Hamiltonian, H, is shown in Equation 8:

$$H(x,u,\lambda) = u + \lambda\frac{Ru}{1+\beta u}x\left(1 - \frac{x}{K}\right), \quad (8)$$

and the differential equation for the state variable $x(t)$ is given by Equation 7 and the differential equation for the adjoint variable $\lambda(t)$ is given by Equation 9 (for example, Lenhart and Workman, 2007).

$$\frac{d\lambda}{dt} = -\frac{\partial H}{\partial x} = -\lambda\frac{Ru(-2x+K)}{K(\beta u+1)}. \quad (9)$$

Since the control is unbounded, the minimum value of the Hamiltonian with respect to u will be given by the optimality condition $\frac{\partial^2 H}{\partial u^2} > 0$ as in Equation 10:

$$\frac{\partial H}{\partial u} = 1 + \lambda\frac{Rx(K-x)}{K(\beta u+1)^2} = 0 \quad (10)$$

whiich then gives the optimal control as shown in Equation 11:

$$u^*(t) = \frac{-K + \sqrt{\lambda(t)KRx(t)(x(t)-K)}}{K\beta}. \quad (11)$$

Since $x \le K$ and all parameters are positive, $u^*(t)$ can be real and non-negative only if $\lambda < 0$. To check that Equation 11 gives a minimum, the second derivative was calculated (Equation 12):

$$\frac{\partial^2 H}{\partial u^2} = -\frac{2\lambda Rx(K-x)}{K(\beta u+1)^3}. \quad (12)$$

The strict Legendre condition $\frac{\partial^2 H}{\partial u^2} > 0$ is satisfied on $[t_0, t_f]$, if $\lambda(t) < 0$ for $t \in [t_0, t_f]$. This is also a necessary condition for $u^*(t)$ to be real and non-negative.

The solution of the optimal control problem then involves the solution of the differential equation in Equation 7, the adjoint equation in Equation 9, with $u(t) = u^*(t)$ from Equation 11, subject to the boundary conditions $x(0) = 6.5$ and $x(140) = 100$. This boundary value problem was solved numerically using the *bvp5c* function in Matlab.

Gompertz growth model

The method of solution was similar to that given for the logistic growth model except that the function $F(x(t))$ in Equation 7 was replaced by the growth rate for the Gompertz model. The growth-rate equation is shown in Equation 13:

$$\frac{dx}{dt} = F(x(t)(G(u(t)) = r\ln\left(\frac{K}{x(t)}\right)x(t)\frac{\alpha u(t)}{1+\beta u(t)} \quad (13)$$

with boundary conditions: $x(0) = 6.5$ kg and $x(140) = 100$ kg, where $x(t)$ is the body weight in kg at time t, $u(t)$ is the daily food intake in kg/day at time t, $r > 0$ is the specific rate of increase in the Gompertz function, K is the carrying capacity in the Gompertz function and α, β are positive constants for the Michaelis-Menten function. As with the logistic growth model, it is necessary when combining the fitted growth-rate functions F for the Gompertz equation with the Michaelis-Menten function G to combine the constants $r\alpha$ into one constant R so that the average value of R corresponds to a daily food intake control variable u within the interval range of 0.3 to 3.1 $kg.d^{-1}$ of Table 1. From the numerical results, a suitable choice is $R = 3r\alpha$.

Following the same method of solution as in the logistic growth model, the Hamiltonian, H is defined as shown in Equation 14:

$$H(x,u,\lambda) = u + \lambda \frac{Ru}{1+\beta u} x \ln\left(\frac{K}{x}\right). \quad (14)$$

with the adjoint equation as shown in Equation 15:

$$\frac{d\lambda}{dt} = -\frac{\partial H}{\partial x} = -\lambda \frac{Ru\left[\ln\left(\frac{K}{x}\right) - 1\right]}{\beta u + 1}. \quad (15)$$

For the unbounded control, the optimality condition for the Hamiltonian with respect to u is given by Equation 16:

$$\frac{\partial H}{\partial u} = 1 + \frac{\lambda R x \ln\left(\frac{K}{x}\right)}{(\beta u + 1)^2} = 0. \quad (16)$$

Then solving Equation 16 for the optimal control $u^*(t)$ we obtain Equation 17:

$$u^*(t) = \frac{-1 + \sqrt{-\lambda R x \ln\left(\frac{K}{x}\right)}}{\beta}. \quad (17)$$

Since $x \le K$, $u^*(t)$ can be real and non-negative only if $r > 0$. To check that Equation 17 gives a minimum, the second derivative was calculated as shown in Equation 18:

$$\frac{\partial^2 H}{\partial u^2} = -\frac{2\lambda R\beta \ln\left(\frac{K}{x}\right)}{(\beta u + 1)^3}. \quad (18)$$

The strict Legendre condition $\frac{\partial^2 H}{\partial u^2} > 0$ is satisfied on $\left[t_0, t_f\right]$, if $\lambda(t) < 0$ for $t \in \left[t_0, t_f\right]$. This is also a necessary condition that $u^*(t)$ is real and non-negative.

The solution of the optimal control problem then involves the solution of the differential equation in Equation 13, the adjoint equation in Equation 15, with $u(t) = u^*(t)$ from Equation 17, subject to the boundary conditions $x(0) = 6.5$ and $x(140) = 100$. This boundary value problem was solved numerically using the *bvp5c* function in Matlab.

NUMERICAL RESULTS AND DISCUSSION

The two boundary value problems developed in the previous section for minimizing the total food intake were numerically solved using the *bvp5c* command in the Matlab software package.

The results of the optimal daily food intake control variable $u^*(t)$ are shown in Figures 2 and 3 for the unbounded logistic control and the unbounded Gompertz control, respectively. The food intake was 0.5824 $kg.d^{-1}$ according to the logistic model, and 0.5454 $kg.d^{-1}$ according to the Gompertz model. Both of these results are in the range of food intake shown in Table 1. The coefficient of determination using the logistic equation was 0.9887 and for the Gompertz equation was 0.9986.

CONCLUSION

A generic approach was used to calculate the daily amount of fed food that achieved a desirable final weight of swine in the post weaning period while minimizing the total food intake. The Gompertz equation as the growth model produced a more reliable result and made the most sense from a physical consideration.

The results of this paper should be regarded as a preliminary investigation into the application of optimal control to minimize the food intake of swine in Thailand. The method can provide useful results that could potentially increase the quality of livestock and result in a better economic output for the food production industry in Thailand. However, it is clear that reliable results would require the analysis of considerably more data to obtain a better model for the growth rate of swine as a function of age and daily food intake. Optimal control could be a useful method for obtaining optimal feeding schedules for other animals in Thailand.

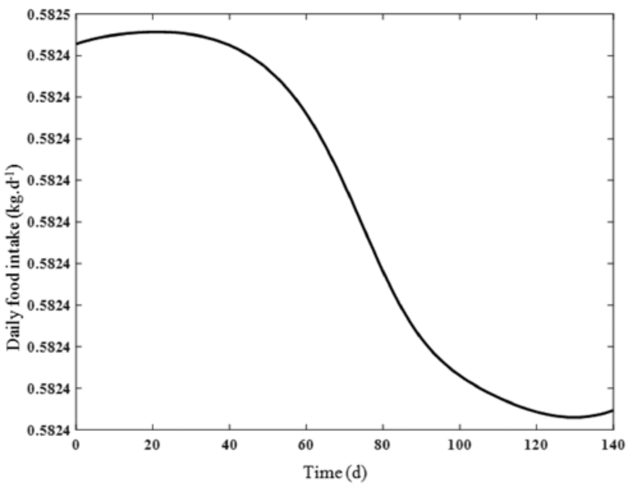

Figure 2 Amount of fed food with unbounded control. The weight increase is modeled using the logistic equation with parameter values: $R = 3r\alpha = 0.08348$ kg^{-1}, $K = 113.4$ kg, $\beta = 0.7129$ d.kg^{-1}, $x(0) = 6.5$ kg, $x(140) = 100$ kg.

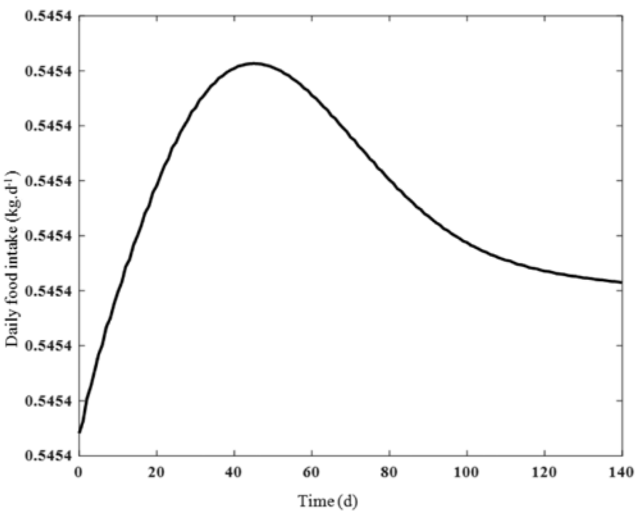

Figure 3 Amount of fed food with unbounded control. The weight increase is modeled using the Gompertz equation with parameter values: $R = 3r\alpha = 0.03036$ kg^{-1}, $K = 188.6$ kg, $\beta = 0.7129$ d.kg^{-1}, $x(0) = 6.5$ kg, $x(140) = 100$ kg.

ACKNOWLEDGEMENTS

This work was supported by the Centre of Excellence in Mathematics, Commission on Higher Education, Thailand.

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