

Numerical Schemes for Solving One-dimensional Transport Equation

Settapat Chinviriyasit* and Jirapa Khamta

ABSTRACT

The numerical manners based on finite element and finite difference methods for solving one-dimensional transport equation are presented. The results of both schemes are compared. The finite difference method is developed by replacing the space derivative with the backward difference to obtain a system of differential equation. Finite difference approximation is estimated via Pade' approximant. The finite difference method is unconditionally stable. Finite element method is applied to obtain system of ordinary differential equation and the system is solved by the forth-order Runge- Kutta method. The numerical approximation of both methods are demonstrated good agreement with the analytical solution.

Key words: finite element method, Runge-Kutta method, transport equation, finite difference method, Pade' approximations

INTRODUCTION

The transport equation arises in many applications in science and engineering such as reactor analysis propagation of photons in stellar and thermal pollutants. In this work the new finite difference schemes is developed and is compared to the numerical manner based on finite element method.

Our work is mainly stimulated by Arigul *et al.* (1996) who proposed the finite difference method via Pade' approximants for treating the first order hyperbolic PDEs. The methods based on lower and higher difference approximations were tested on the three linear problems. Kessler (1999) improved the accuracy of an adaptive transport scheme by dynamically creating and modifying a computational mesh specifically tailored to the properties of the model problem. Chen and Liu (2003) illustrated the iterative method of the energy transport model which was a new self-adjoint formulation of the carrier energy balance equation. Chan (2003) investigated the effects of the coarse grid correction process on the multigrid convergence for one and two dimensional hyperbolic problems.

The outline of the paper is as follow. In next section, the numerical schemes are discussed for solving one-dimensional transport equations. In the third section, the numerical experiments are presented. The last section, a conclusion on the problem is given.

MATERIALS AND METHODS

Numerical formulation for treating one-dimensional transport equations

Consider a one-dimensional transport equation which represents in the form:

$$\frac{\partial u(x,t)}{\partial t} + a \frac{\partial u(x,t)}{\partial x} = 0, \quad 0 \leq x \leq 1, \quad t > 0, \quad (1)$$

with initial conditions given by

$$u(x,0) = g(x), \quad 0 \leq x \leq 1, \quad (2)$$

and boundary conditions

$$u(0,t) = v(t), \quad u(1,t) = w(t), \quad t > 0 \quad (3)$$

where $a > 0$ is constant, $g(x)$, $v(t)$ and $w(t)$ are the function of x and t . It is clear from (2) and (3) that a discontinuity between the initial and boundary conditions exists at the point $(x,t) = (0,0)$ if $g(0) \neq v(0)$ and/or $g(1) \neq w(0)$.

The finite element in space and Runge-Kutta order 4 in time

Suppose that the boundary condition of equation (1) are assumed to be homogeneous and a is chosen to be 1. Applying the weighted residual via Galerkin method leads to

$$\tilde{\mathbf{M}}\dot{\mathbf{u}} + \tilde{\mathbf{S}}\mathbf{u} = \tilde{\mathbf{f}} \quad (4)$$

where

$$\tilde{\mathbf{M}} = [\tilde{M}_{ij}], \quad \tilde{M}_{ij} = \int_0^1 \varphi_i \varphi_j dx, \quad \tilde{\mathbf{S}} = [\tilde{S}_{ij}], \quad \tilde{S}_{ij} = \int \frac{d\varphi_i(x)}{dx} \varphi_j(x) dx, \quad (5)$$

$$\mathbf{u} = [u_i], \quad u_i = [u_1(t), u_2(t), \dots, u_n(t)]^T, \quad \tilde{\mathbf{f}} = [\tilde{f}_i], \quad \tilde{f}_i = \int f_i(x) \varphi_i(x) dx \quad (6)$$

The finite element method

Let $\tilde{\Omega}$ be a uniform partition of the domain Ω into the linear element $\tilde{\Omega}_e$. Applied the homogeneous boundary condition to equation (4), we obtain

$$\mathbf{M}\dot{\mathbf{u}} + \mathbf{S}\mathbf{u} = \mathbf{f} \quad (7)$$

where

$$\mathbf{M} = \frac{h}{6} \begin{bmatrix} 4 & 1 & & & 0 \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & & 1 & 4 \end{bmatrix}, \quad \mathbf{S} = \frac{1}{2} \begin{bmatrix} 0 & 1 & & & 0 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & & -1 & 0 \end{bmatrix} \quad (8)$$

In the case of nonhomogeneous boundary condition is applied to (1), we transform to homogeneous by setting

$$z(x,t) = u(x,t) - \phi(x,t) \quad (9)$$

where

$$\phi(x,t) = v(t)\varphi_0(x) + w(t)\varphi_{n+1}(x) \quad (10)$$

Applying equations (9) and (10) to equation (1) and use the method of weighted residual. This yields

$$\mathbf{M}\dot{\mathbf{u}} + \mathbf{S}\mathbf{u} = \mathbf{g} \quad (11)$$

where

$$\mathbf{g} = \begin{bmatrix} \int_0^1 f_1 \varphi_1 dx - \frac{dv}{dt} \int_0^1 \varphi_0 \varphi_1 dx - \frac{dw}{dt} \int_0^1 \varphi_{n+1} \varphi_1 dx - v \int_0^1 \frac{d\varphi_0}{dx} \varphi_1 dx - w \int_0^1 \frac{d\varphi_{n+1}}{dx} \varphi_1 dx \\ \int_0^1 f_2 \varphi_2 dx - \frac{dv}{dt} \int_0^1 \varphi_0 \varphi_2 dx - \frac{dw}{dt} \int_0^1 \varphi_{n+1} \varphi_2 dx - v \int_0^1 \frac{d\varphi_0}{dx} \varphi_2 dx - w \int_0^1 \frac{d\varphi_{n+1}}{dx} \varphi_2 dx \\ \vdots \\ \int_0^1 f_{n-1} \varphi_{n-1} dx - \frac{dv}{dt} \int_0^1 \varphi_0 \varphi_{n-1} dx - \frac{dw}{dt} \int_0^1 \varphi_{n+1} \varphi_{n-1} dx - v \int_0^1 \frac{d\varphi_0}{dx} \varphi_{n-1} dx - w \int_0^1 \frac{d\varphi_{n+1}}{dx} \varphi_{n-1} dx \\ \int_0^1 f_n \varphi_n dx - \frac{dv}{dt} \int_0^1 \varphi_0 \varphi_n dx - \frac{dw}{dt} \int_0^1 \varphi_{n+1} \varphi_n dx - v \int_0^1 \frac{d\varphi_0}{dx} \varphi_n dx - w \int_0^1 \frac{d\varphi_{n+1}}{dx} \varphi_n dx \end{bmatrix}$$

$\mathbf{M}$ and $\mathbf{S}$ are the matrices of equation (8). Equation (11) can be determined by Runge-Kutta order 4.

Finite difference approximation via Pade' approximant

Development of the numerical method

Suppose that the solution $u(x, t)$ is sought in some arbitrary region $R = [0 < x \leq X, t > 0]$ of the first quarter plane. The interval $0 \leq x \leq X$ is divided into N equal parts each of width h so that $Nh = X$ and the interval $t \geq 0$ is discretized at the points $t_n = n\ell$ ($n=0, 1, 2, \dots$) where ℓ is time increment. The open interval $\Omega = [0 < x < X] \times [t > 0]$ and its boundary $\partial\Omega$ consisting of the line $x = 0$ and $t = 0$ have thus been covered by a rectangular mesh with coordinates $(x_m, t_n) = (mh, n\ell)$, where $m = 0, 1, 2, \dots, N$ and $n = 0, 1, 2, \dots$. The theoretical solution of the model problem (1)-(3) at point (x_m, t_n) is $u(mh, n\ell)$ and the solution of an approximating numerical method at the same point is denoted by U_m^n . The space derivative in (1) is approximated by backward difference replacement

$$\frac{\partial u}{\partial x} = \frac{u(x, t) - u(x-h, t)}{h} + O(h) \text{ as } h \rightarrow 0. \quad (12)$$

Applying (1) together with (12) to every one of the N discrete points x_m at time level t at point m on this level leads to the system of first-order ordinary differential equations

$$\frac{d\mathbf{U}(t)}{dt} = -\mathbf{a}\mathbf{C}\mathbf{U}(t) + \mathbf{a}\mathbf{c}_t \quad (13)$$

where $\mathbf{U}(t) = [U_1(t), U_2(t), \dots, U_N(t)]^T$, and T denoting the transpose of a matrix, is the vector of approximate solutions of (1) at time $t > 0$, $\mathbf{C}$ is the bidiagonal square matrix of order N given by

$$\mathbf{C} = h^{-1} \begin{bmatrix} 1 & & & 0 \\ -1 & 1 & & \\ & \ddots & \ddots & \\ 0 & & -1 & 1 \end{bmatrix} \quad (14)$$

and $\mathbf{c}_t$ is the vector with N components given by

The solution of (13) with initial conditions in (2) is

$$\mathbf{U}(t) = \mathbf{C}^{-1}\mathbf{c}_t + \exp(-at\mathbf{C})\{\mathbf{g} - \mathbf{C}^{-1}\mathbf{c}_t\}, \quad (15)$$

where $\mathbf{g} = [g(h), g(2h), \dots, g(Nh)]^T$ is the vector of initial conditions. It is easy to show that, for a time step l , equation (15) satisfies the recurrence relation

$$\mathbf{U}(t + \ell) = \mathbf{C}^{-1}\mathbf{c}_t + \exp(-a\ell\mathbf{C})\{\mathbf{U}(t) - \mathbf{C}^{-1}\mathbf{c}_t\}. \quad (16)$$

The Pade' approximation to exponential function

This section focuses on the approximation of the exponential term, $\exp(-a\ell\mathbf{C})$, in (16) using the Pade' approximation. The $[K/M]$ Pade' approximant ($M > K$) (Smith, 1995) defined for a scalar θ by

$$R_{M,N}(\theta) = \frac{P_K(\theta)}{Q_M(\theta)} + O(\theta)^{M+K+1}, \text{ where } P_K(\theta), Q_M(\theta) \text{ are polynomials of degrees } K \text{ and } M \text{ respectively.}$$

The $[2/3]$ Pade' approximants to e^θ and their leading error terms.

Table 1 Pade' approximants to e^θ

$[K/M]$ Pade'	Approximant	Order of error, $Cq\theta^{M+K+1}$
$[2/3]$	$\frac{1 + \frac{2}{5}\theta + \frac{1}{20}\theta^2}{1 - \frac{3}{5}\theta + \frac{3}{20}\theta^2 - \frac{1}{60}\theta^3}$	$\frac{-1}{7200}\theta^6$

The numerical method is developed by replacing the exponential matrix function in (16) with its $[2/3]$ Pade' approximant, which leads to

$$\mathbf{U}(t + \ell) = \mathbf{C}^{-1}\mathbf{c}_t + \frac{\mathbf{I} - \frac{2}{5}a\ell\mathbf{C} + \frac{1}{20}(a\ell\mathbf{C})^2}{\mathbf{I} + \frac{3}{5}a\ell\mathbf{C} + \frac{3}{20}(a\ell\mathbf{C})^2 + \frac{1}{60}(a\ell\mathbf{C})^3} \{\mathbf{U}(t) - \mathbf{C}^{-1}\mathbf{c}_t\} + O(\ell^6). \quad (17)$$

Pre-multiplying (17) by the matrix $\{\mathbf{I} + \frac{3}{5}a\ell\mathbf{C} + \frac{3}{20}(a\ell\mathbf{C})^2 + \frac{1}{60}(a\ell\mathbf{C})^3\}$, and writing the term

$\{\frac{2}{5}a\ell - \frac{1}{20}(a\ell)^2\mathbf{C}\}\mathbf{c}_t$ on the right-hand side of the resulting equation, to its left-hand side as

$$\{\frac{3}{5}a\ell + \frac{3}{20}(a\ell)^2\mathbf{C} + \frac{1}{60}(a\ell)^3\mathbf{C}^2\}\mathbf{c}_{t+\ell},$$

$$\{\mathbf{I} + \frac{3}{5}a\ell\mathbf{C} + \frac{3}{20}(a\ell\mathbf{C})^2 + \frac{1}{60}(a\ell\mathbf{C})^3\}\mathbf{U}(t + \ell) - \{\frac{3}{5}a\ell + \frac{3}{20}(a\ell)^2\mathbf{C} + \frac{1}{60}(a\ell)^3\mathbf{C}^2\}\mathbf{c}_{t+\ell}$$

$$= \{\mathbf{I} - \frac{2}{5}a\ell\mathbf{C} + \frac{1}{20}(a\ell\mathbf{C})^2\}\mathbf{U}(t) + \{\frac{2}{5}a\ell - \frac{1}{20}(a\ell)^2\mathbf{C}\}\mathbf{c}_t. \quad (18)$$

Applying (18) to the mesh point $(h, n\ell)$ gives

$$\begin{aligned} & \{1 + \frac{3}{5}ar + \frac{3}{20}(ar)^2 + \frac{1}{60}(ar)^3\}U_1^{n+1} - \{\frac{3}{5}a\ell + \frac{3}{20}\frac{(a\ell)^2}{h} + \frac{1}{60}\frac{(a\ell)^3}{h^2}\}v_{t+\ell} \\ &= \{1 - \frac{2}{5}ar + \frac{1}{20}(ar)^2\}U_1^n + \{\frac{2}{5}a\ell - \frac{1}{20}\frac{(a\ell)^2}{h}\}v_t; \quad n=0. \end{aligned} \quad (19)$$

Applying (18) to the mesh point $(2h, n\ell)$ gives

$$\begin{aligned} & \{-\frac{3}{5}ar - \frac{3}{10}(ar)^2 - \frac{1}{20}(ar)^3\}U_1^{n+1} + \{1 + \frac{3}{5}ar + \frac{3}{20}(ar)^2 + \frac{1}{60}(ar)^3\}U_2^{n+1} \\ & + \{\frac{3}{20}\frac{(a\ell)^2}{h} + \frac{1}{30}\frac{(a\ell)^3}{h^2}\}v_{t+\ell} = \{\frac{2}{5}ar - \frac{1}{10}(ar)^2\}U_1^n + \{1 - \frac{2}{5}ar + \frac{1}{20}(ar)^2\}U_2^n - \frac{1}{20}\frac{(a\ell)^2}{h}v_t. \end{aligned} \quad (20)$$

Applying (18) to the mesh point $(3h, n\ell)$ gives

$$\begin{aligned} & \{\frac{3}{20}(ar)^2 + \frac{1}{20}(ar)^3\}U_1^{n+1} + \{-\frac{3}{5}ar - \frac{3}{10}(ar)^2 - \frac{1}{20}(ar)^3\}U_2^{n+1} + \{\frac{3}{5}ar + \frac{3}{20}(ar)^2 \\ & + \frac{1}{60}(ar)^3\}U_3^{n+1} - \frac{1}{60}\frac{(a\ell)^3}{h^2}v_{t+\ell} = \frac{1}{20}(ar)^2U_1^n + \{\frac{2}{5}ar - \frac{1}{10}(ar)^2\}U_2^n \\ & + \{1 - \frac{2}{5}ar + \frac{1}{20}(ar)^2\}U_3^n, \end{aligned} \quad (21)$$

and to the mesh points $(mh, n\ell)$, $m = 4, \dots, N$ gives

$$\begin{aligned} & -\frac{1}{60}(ar)^3U_{m-3}^{n+1} + \{\frac{3}{20}(ar)^2 + \frac{1}{20}(ar)^3\}U_{m-2}^{n+1} + \{-\frac{3}{5}ar - \frac{3}{10}(ar)^2 - \frac{1}{20}(ar)^3\}U_{m-1}^{n+1} \\ & + \{1 + \frac{3}{5}ar + \frac{3}{20}(ar)^2 + \frac{1}{60}(ar)^3\}U_m^{n+1} = \frac{1}{20}(ar)^2U_{m-2}^n + \{\frac{2}{5}ar - \frac{1}{10}(ar)^2\}U_{m-1}^n \\ & + \{1 - \frac{2}{5}ar + \frac{1}{20}(ar)^2\}U_m^n \end{aligned} \quad (22)$$

Analysis of the developed method

Truncation errors and consistency

The principal part of the local truncation error may be written down for every finite difference scheme yielded by (16). The local truncation error associated with (19)-(22) to the mesh point $(mh, n\ell)$, is thus

$$\left(-\frac{1}{7200}\ell^5 \frac{\partial^2 u}{\partial t^2} - \frac{1}{2}ah \frac{\partial^2 u}{\partial x^2} \right)_m^n \quad (23)$$

From which it is seen that the finite scheme arising from the $[2/3]$ Pade' approximant is seen to be consistent with transport equation (1). In equation (23) the component $-\frac{1}{2}ah \frac{\partial^2 u}{\partial x^2}$ is due to the finite difference replacement (12) and will be included in the local truncation error of each finite difference

method derived from (16). The component $-\frac{1}{7200} \ell^5 \frac{\partial^2 u}{\partial t^2}$ is due to the used of the [2/3] Pade' approximant in (16), then constant $-\frac{1}{7200}$ in this component is given in Table 1.

Stability analysis

The concept of stability is concerned with the boundedness of the solution of the finite difference equations and this examined by finding conditions under which

$$Z_m^n = U_m^n - \tilde{U}_m^n \quad (24)$$

remains bounded as n increases, for fixed h and ℓ . In (24), $\tilde{U}_m^n$ denotes the numerical solution actually obtained and U_m^n is the solution of (17). The matrix method is successful in analyzing the method (17). This method shows that a perturbation $Z^0 = U^0 - \tilde{U}^0$ of the initial condition satisfies

$$Z^{n+1} = WZ^n, n = 0, 1, 2, \dots \quad (25)$$

will not grow in magnitude as time increase provided

$$\|W\| < 1 \quad (26)$$

where $W = \{I - \frac{2}{5} a\ell C + \frac{1}{20} (a\ell C)^2\} \{I + \frac{3}{5} a\ell C + \frac{3}{20} (a\ell C)^2 + \frac{1}{60} (a\ell C)^3\}^{-1}$ is called the amplification matrix. Since the eigenvalues of C in (14) are all $\frac{1}{h}$, then each eigenvalues of the

amplification matrix W are $\frac{1 - \frac{2}{5} ar + \frac{1}{20} (ar)^2}{1 + \frac{3}{5} ar + \frac{3}{20} (ar)^2 + \frac{1}{60} (ar)^3}$. It is clear that all eigenvalues of W are less

than unity in modulus for non-negative values of ar so that the perturbation errors Z^{n+1} , ($n = 0, 1, 2, \dots$) in (25) do not grow as $n \rightarrow \infty$. This indicates that the finite-difference scheme (19)-(22) is unconditionally stable. Therefore we can verify that the scheme is also convergent for any value of $ar > 0$

The error norms

The boundedness of the numerical solution are defined by

$$\|U^n\|_2^2 = h \sum_{m=0}^N |U_m^n|^2, \|Z\|_2^2 = h \sum_{m=0}^N |Z_m^n|^2, \|Z^n\|_\infty = \max_m |Z_m^n|. \quad (27)$$

RESULTS AND DISCUSSION

Problem 1

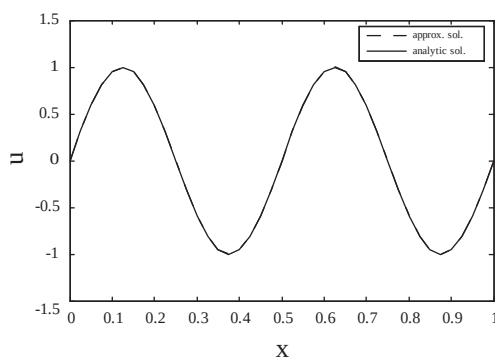
The initial and boundary condition are chosen to be

$$u(x, 0) = \sin(4\pi k), 0 \leq x \leq 1; \quad u(0, t) = \sin(-4\pi t), t > 0; \quad u(1, t) = \sin 4\pi(1 - t), t > 0$$

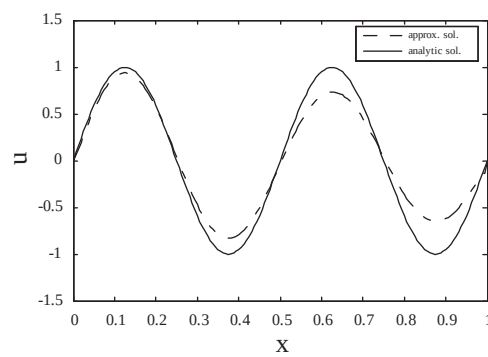
The error norm in Table 2 illustrates that the finite element solution can be evaluated when $r = (\ell/h) \leq 1$. The finite difference method is unconditionally stable. Figure 1 shows that the finite element solutions are more accuracy than finite difference solutions.

Table 2 The error in L_2 norm at time $t = 8$.

Finite element in space and RK4 in time					Finite difference			
$h \backslash \ell$	0.05	0.025	0.0125	0.00625	0.05	0.025	0.0125	0.00625
0.05	1.93E-02	6.97E-03	6.25E-03	6.20E-03	5.87E-01	5.64E-01	5.55E-01	5.51E-01
0.025		3.08E-04	9.34E-05	8.18E-05		4.44E-01	4.34E-01	4.30E-01
0.0125			1.21E-05	1.67E-06			2.90E-01	2.86E-01
0.00625	unstable			9.21E-07				1.69E-01



(a)



(b)

Figure 1 The numerical results at time $t = 8$ for $h = 0.025$, $\ell = 0.025$: (a) finite element method (b) finite difference method.**Problem 2**

In this problem, the initial and boundary condition are given by

$$u(x, 0) = e^x, 0 \leq x \leq 1; \quad u(0, t) = e^{-t}, t > 0; \quad u(1, t) = e^{1-t}, t > 0$$

The numerical solution in Problem 2 has the same behavior as that in Problem 1, as shown in Table 3 and Figure 2.

Table 3 The error in L_2 norm at time $t = 2$.

Finite element in space and RK4 in time					Finite difference			
$h \backslash \ell$	0.05	0.025	0.0125	0.00625	0.05	0.025	0.0125	0.00625
0.05	6.99E-06	7.29E-07	7.44E-08	5.74E-08	4.12E-03	4.04E-03	3.87E-03	3.53E-03
0.025		9.93E-07	1.18E-07	8.62E-09		2.03E-03	1.95E-03	1.78E-03
0.0125			1.52E-07	1.99E-08			9.78E-04	8.93E-04
0.00625	unstable			2.24E-08				4.47E-04

CONCLUSION

The numerical schemes based on finite element and finite difference method are developed for treating one-dimensional transport equation. The numerical experiment of both problems in section 3

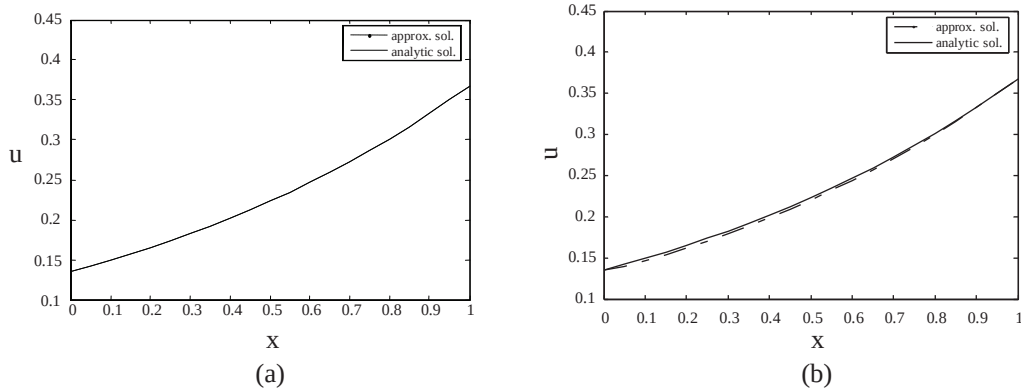


Figure 2 The numerical experimental at time $t = 2$ for $h = 0.05$, $\ell = 0.05$: (a) finite element method (b) finite difference method

are good agreement with the theoretical solutions. The numerical result based on finite element method is conditionally stable i.e. $r < 1$. The finite difference method numerical presents unconditionally stable. The test problems show that the numerical manner based on finite element method gives more acceptable solutions than the finite difference method.

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