

Global Stability of an SIQ Epidemic Model

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ABSTRACT

In this paper, an SIQ epidemic model is studied and analyzed. In the model the susceptibles and the infectious have constant immigration. The model exhibits an unique endemic state if $p > 0$, a disease-free state and an endemic state if $p = 0$. A basic reproduction number R_0 is derived. If $R_0 > 1$, the global stability of the endemic equilibrium is proved when $p \geq 0$.

Key words: epidemic model, global stability

INTRODUCTION

One intervention procedure to control the spread of infectious diseases is to isolate some infectives, in order to reduce transmissions of the infection to susceptibles. Isolation may have been the first infection control method, since biblical passages refer to the ostracism of lepers, and plague sufferers were often isolated (Hethcote *et al.*, 2002). Takeuchi *et al.* (2000) and Ma *et al.* (2002) studied the SIR infectious disease model in which an infectious disease is transmitted by a vector after an incubation time. Their model assumes that the birth rate and the death rate are all constant, so the dynamics of the total population is very simple. In order to investigate disease dynamics for the model with more demographic effects, it would be necessary to isolate the fraction of arriving infectives. We would like to mention some relative investigation in epidemic models. Beretta and Takeuchi (1995) investigated SIR models with constant and varying population size, respectively. Cooke *et al.* (1999), Brauer and Diessche (2001) investigated a model for disease transmission with a general contact rate and references therein.

The aim of this paper is to study global stability of a model for the transmission dynamics of infectious diseases that include a new class Q of quarantined individuals, who have been removed and isolated either voluntarily or coercively from the infectious class. The resulting model is of SIQ type. This paper is organized as follows: First, we formulate the model and discuss the existence of equilibrium; then, we consider the global stability of the endemic equilibrium when $p \geq 0$ and conclusion of this paper is given in the final section.

MATERIAL AND METHODS

The established model of infectious diseases focuses on the transmitting features of diseases with constant immigration into the susceptibles and the infected. According to different features of the transmitted diseases, the host population is partitioned into three compartments, the susceptible, infectious and recovered, with sizes denoted by S, I, Q, respectively. The host total population $N = S + I + Q$. The SIQ model with constant immigration is

described by the following system of differential equations

$$\begin{aligned} S' &= (1-p)A - \mu S - \beta SI, \\ I' &= pA + \beta SI - (\mu + \delta + \alpha)I, \\ Q' &= \alpha I - (\mu + \delta)Q, \end{aligned} \quad (1)$$

where μ is the rate of natural death, δ is non-negative constant and denote the rate of disease caused death. The constant α is the rate at which the infections individuals leaving the infective compartment for the quarantine compartment. The parameter β is the rates of the efficient contract in the infected period, $(1-p)A$, pA are constant recruitment of susceptibles, infectious, respectively.

Thus, the total population size N can be determined by $N(t) = S(t) + I(t) + Q(t)$ or from the differential equation

$$\frac{dN}{dt} = A - \mu N - \delta I - \delta Q$$

It is convenient to use I , Q and N as variables and replaced S by $N - I - Q$. This gives the model

$$\begin{aligned} I' &= pA + \beta I(N - I - Q) - (\mu + \delta + \alpha)I \\ Q' &= \alpha I - (\mu + \delta)Q \\ N' &= A - \mu N - \delta I - \delta Q \end{aligned} \quad (2)$$

The system (2) is equivalent to the system (1). This allows us to attack (1) by studying the system (2). From biological considerations, we study the system (2) in the closed set $T =$

$\left\{ (I, Q, N) \in \mathbb{R}_+^3 : 0 \leq I + Q \leq N \leq \frac{A}{\mu} \right\}$. It can be verified that T is positively invariant with respect to the system (2).

Equilibria of system (2) are given by

$$\begin{aligned} pA + \beta I(N - I - Q) - (\mu + \delta + \alpha)I &= 0, \\ \alpha I - (\mu + \delta)Q &= 0, \\ \mu N + \delta I + \delta Q &= A. \end{aligned}$$

We solve for N and Q in terms of I and then substitute N and Q into the first equation of the system (2). This gives the quadratic equation

$$\beta(\mu + \delta + \alpha)I^2 - [\beta A - \mu(\mu + \delta + \alpha)]I - \mu p A = 0. \quad (3)$$

If $p = 0$, one root is $I = 0$, and there is a second root

$$I = \frac{\beta A - \mu(\mu + \delta + \alpha)}{\beta(\mu + \delta + \alpha)}$$

which is positive if and only if

$$\sigma = \beta A - \mu(\mu + \delta + \alpha) > 0.$$

If $p > 0$, the quadratic equation (3) has one positive and one negative root (Brauer and Diessche, 2001). The disease-free equilibrium, $I = 0$, that occurs when $p = 0$ now become negative (not biologically feasible). The positive root is

$$I^* = \frac{\sigma + \sqrt{\sigma^2 + 4\beta A \mu(\mu + \delta + \alpha)}}{2\beta(\mu + \delta + \alpha)}, \quad (4)$$

with $\lim_{p \rightarrow 0} I^* = \begin{cases} 0 & (\sigma < 0), \\ \frac{\sigma}{\beta(\mu + \delta + \alpha)} & (\sigma > 0). \end{cases}$

Thus $\sigma = 0$ is a threshold, there is a basic reproduction number

$$R_0 = \frac{\beta A}{\mu(\mu + \delta + \alpha)}.$$

Thus, for $p > 0$, there is an unique equilibrium $P^* = (I^*, Q^*, N^*)$ which is given by

$$Q^* = \frac{\alpha}{\mu + \delta} I^*, \quad N^* = \frac{A}{\mu} - \frac{\delta}{\mu}(I^* + Q^*)$$

and I^* as in (4).

RESULTS

Theorem 1 For the system (2) if $R_0 > 1$, the endemic equilibrium P^* when $p \geq 0$ is local asymptotically stable.

Proof. The Jacobian matrix of the system (2) at a point $P^* = (I^*, Q^*, N^*)$ is

$$J(P^*) = \begin{bmatrix} -\beta I^* + \beta(N^* - I^* - Q^*) - (\mu + \delta + \alpha) & -\beta I^* & \beta I^* \\ \alpha & -(\mu + \delta) & 0 \\ -\delta & -\delta & -\mu \end{bmatrix}.$$

Its characteristic equation is $\det(J(P^*) - \lambda I) = 0$, where I is the unit matrix and

$$n = N^* - I^* - Q^* = \frac{A}{\mu} - \left[\frac{\delta}{\mu} + \frac{\alpha}{\mu + \delta} \frac{(\mu + \delta)}{\mu} + 1 \right] I^*.$$

So the characteristic equation become $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$, where

$$a_1 = \alpha + 2\delta + 3\mu + \beta I^* - \beta n,$$

$$a_2 = (\alpha + \delta + 2\mu)\beta I^* + (\delta + 2\mu)(\delta + \alpha + \mu) + \mu(\mu + \delta) - \beta n(\delta + 2\mu),$$

$$a_3 = \mu(\mu + \delta)(\mu + \delta + \alpha) + (\delta(\alpha + \mu + \delta) + \mu(\mu + \delta) - \mu\alpha)\beta I^* - \beta\mu(\mu + \delta)n.$$

Thus by Routh-Hurwitz criterion, the endemic equilibrium P^* is local asymptotically stable as it can be seen for $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 > 0$.

To prove the global stability behavior of this equilibrium, we defined the new variables

$$x = \frac{\mu}{A}S, y = \frac{\mu}{A}I, z = \frac{\mu}{A}Q \text{ and parameters } \tau = \mu t, c = \frac{A}{\mu}, \tilde{\beta} = \frac{\beta}{\mu}, \tilde{\delta} = \frac{\delta}{\mu}, \tilde{\alpha} = \frac{\alpha}{\mu}.$$

Using these

change of variables, the system (2) becomes

$$\begin{aligned} \frac{dx}{d\tau} &= (1-p) - x - c\tilde{\beta}xy \\ \frac{dy}{d\tau} &= p + \tilde{\beta}cxy - (1 + \tilde{\delta} + \tilde{\alpha})y \\ \frac{dz}{d\tau} &= \tilde{\alpha}y - (1 + \tilde{\delta})z \end{aligned} \quad (5)$$

with $\tilde{N}(\tau) = x(\tau) + y(\tau) + z(\tau)$. The equation for the total population \tilde{N} is

$$\frac{d\tilde{N}}{d\tau} = 1 - \tilde{N} - \tilde{\delta}I - \tilde{\delta}Q. \quad (6)$$

Therefore, we study the stability of the model (5) in the region

$$\Gamma = \{(x, y, z) \in \mathbb{R}_+^3 : x, y, z \geq 0, x + y + z = \tilde{N} \leq 1\}$$

Consider the subset Γ^* of Γ given by

$$\Gamma^* = \{(x, y, z) \in \mathbb{R}_+^3 : x, y, z \geq 0, x + (1 + \tilde{\delta})y + (1 + \tilde{\delta})z = 1\}$$

From equation (6), it is obvious that $d\tilde{N}/d\tau = 0$ in Γ^* . If

$$\Gamma^* = \{(x, y, z) \in \mathbb{R}_+^3 : x, y, z \geq 0, x + (1 + \tilde{\delta})y + (1 + \tilde{\delta})z > 1\}$$

then $d\tilde{N}/d\tau < 0$ and if

$$\Gamma^* = \{(x, y, z) \in \mathbb{R}_+^3 : x, y, z \geq 0, x + (1 + \tilde{\delta})y + (1 + \tilde{\delta})z < 1\}$$

then $d\tilde{N}/d\tau > 0$. It follows that Γ^* is a positively

invariant set in Γ . Thus the ω -limit set of each solution of the system (5) is contained in Γ^* .

Theorem 2 The system (5) has no periodic orbits, homoclinic orbits or polygons in Γ^* .

Proof. Let f_1, f_2, f_3 denote the right-hand side in system (5), respectively. Let $\mathbf{f} = (f_1, f_2, f_3)$, $\mathbf{r} = (x, y, z)$, $\mathbf{g}(\mathbf{r}) = \frac{\mathbf{r} \times \mathbf{f}}{xyz}$. We can get $\mathbf{g} = (g_1, g_2, g_3)$ where

$$\begin{aligned} g_1 &= \frac{\tilde{\alpha}y}{xz} - \frac{(1 + \tilde{\delta})}{x} - \frac{p}{xy} - c\tilde{\beta} + \frac{(1 + \tilde{\delta} + \tilde{\alpha})}{x}, \\ g_2 &= -\frac{\tilde{\alpha}}{z} + \frac{(1 + \tilde{\delta})}{y} + \frac{1 - p}{xy} - \frac{1}{y} - c\tilde{\beta} \\ g_3 &= \frac{p}{yz} + \frac{c\tilde{\beta}x}{z} - \frac{(1 + \tilde{\delta} + \tilde{\alpha})}{z} - \frac{(1 - p)}{xz} + \frac{1}{z} + \frac{c\tilde{\beta}x}{z}, \end{aligned}$$

then $\mathbf{g} \cdot \mathbf{f} = 0$. Using the normal vector $\mathbf{n} = (1, 1 + \tilde{\delta}, 1 + \tilde{\delta})$ to Γ^* , it can be shown that

$$\begin{aligned} (\text{Curl } \mathbf{g}) \cdot (1, 1 + \tilde{\delta}, 1 + \tilde{\delta}) &= -\frac{p}{y^2 z} - \frac{c\tilde{\beta}\tilde{\delta}}{z} - \frac{\tilde{\alpha}}{z^2} \\ &\quad - (1 + \tilde{\delta}) \left[\frac{1 - p}{x^2} \left(\frac{1}{z} + \frac{1}{y} \right) + \frac{1}{x} \left(\frac{\tilde{\alpha}y}{z^2} + \frac{\tilde{\alpha}}{z} + \frac{p}{y^2} \right) \right] < 0. \end{aligned}$$

From Lemma 3 in Moghadas and Gumel (2002), we know the system (5) has no periodic orbits, homoclinic orbits or polygons in Γ^* .

Since Γ^* is a positively invariant set, it follows from Theorem 1 that the ω -limit set of each solution of the system (1) must be a single point P^* in Γ^* . Therefore, we have established the following theorem.

Theorem 3 If $R_0 > 1$, the endemic equilibrium P^* of the system (1) is globally stable when $p \geq 0$.

CONCLUSION

In this paper, we discuss the SIQ model with constant immigration. We derive a basic reproduction number R_0 and that it determines the global dynamics of (1); if $R_0 > 1$, a unique endemic equilibrium P^* is globally asymptotically stable in the interior of the feasible region so that the disease persists at the endemic equilibrium level

if it is initially present.

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