Kasetsart J. (Nat. Sci.) 43 : 398 - 401 (2009)

On Derivations of BCC-algebras

Chanwit Prabpayak and Utsanee Leerawat*

ABSTRACT

In this paper, the notions of left-right (resp. right-left) derivations of BCC-algebras are studied and some properties on derivations of BCC-algebras are investigated. This paper also considers regular derivations and the d-invariant on ideals of BCC-algebras.

Key words: derivation, BCC-algebra, BCI-algebra, regular, d-invariant

INTRODUCTION

In the theory of rings, the properties of derivations are important. Several authors (Meng and Xin, 1992; Meng, 1987; Iseki and Tanaka, 1976; Iseki and Tanaka, 1978; Dudek, 1992; Dudek and Zhang, 1998) have studied BCI-algebras, BCK-algebras and BCC-algebras. In 2004, Jun and Xin applied the notion of derivations in ring and near-ring theory to BCI-algebras, and they also introduced a new concept called a regular derivation in BCI-algebras. They investigated some of its properties, defined a d-derivation ideal and gave conditions for an ideal to be d-derivation. Two years later, Hamza and Al-Shehri (2006) studied derivation in BCK-algebras. In 2007, Hamza and Al-Shehri defined a left derivation in BCI-algebras and investigated a regular left derivation. In this paper, the notion of a regular derivation in BCI-algebras is applied to BCC-algebras and some related properties are also investigated.

The algebra $G = (G,\cdot,0)$ defines a non-empty set G, together with a binary operation multiplication and a constant 0. In the sequel, a multiplication will be denoted by juxtaposition.

An algebra $(G,\cdot,0)$ is called a BCC-algebra, if for all $x,y,z \in G$, the following axioms hold:

(1) $((xy)(zy))(xz) = 0$,

(2) $0x = 0$,

(3) $x0 = x$,

(4) $xy = yx = 0$ implies $x = y$

By (1) we get: $(xy)x = 0$ and $x x = 0$ for all $x,y \in G$.

A non-empty subset S of a BCC-algebra G is called a BCC-subalgebra of G, if $xy \in S$ whenever $x,y \in S$. If a binary relation $\leq$ on G is defined by putting $x \leq y$ if and only if $xy = 0$, then $(G, \leq)$ is a partially ordered set. A non-empty subset A of a BCC-algebra G is called a BCC-ideal, if

(5) $0 \in A$,

(6) $(xy)z \in A$ and $y \in A$ imply $xz \in A$.

Putting $z = 0$ in (6) obtains: $x \in y$ A and $y \in A$ implies $x \in A$.

For more details, refer to Dudek (1992) and Dudek and Zhang (1998).

MATERIALS AND METHODS

For elements x and y of a BCC-algebra $G = (G,\cdot,0)$, denote $x \wedge y = y(yx)$.

Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok 10900, Thailand.
* Corresponding author, e-mail: fsciutl@ku.ac.th

Received date : 19/11/08 Accepted date : 29/01/09

Definition Let G be a BCC-algebra. A map $d : G \rightarrow G$ is a left-right derivation (briefly, (l,r)-derivation) of G, if it satisfies the identity $d(xy) = d(x)y \wedge xd(y)$ for all $x,y \in G$. If d satisfies the identity $d(xy) = xd(y) \wedge d(x)y$ for all $x,y \in G$, then d is a right-left derivation (briefly, (r,l)-derivation) of G. Moreover, if d is both a (l,r) and (r,l)-derivation, then d is a derivation of G.

Example Let G = {0,1,2,3} be a BCC-algebra in which the operation $\cdot$ is defined as follows:

$\cdot$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0

Define a map $d : G \rightarrow G$ by

$$d(x) = \begin{cases} 0 & \text{if } x = 0,1,3, \\ 2 & \text{if } x = 2. \end{cases}$$

And define a map $d^* : G \rightarrow G$ by

$$d^*(x) = \begin{cases} 0 & \text{if } x = 0,1, \\ 2 & \text{if } x = 2,3. \end{cases}$$

Then it is easily checked that d is both a (l,r) and (r,l)-derivation of G and d* is a (r,l)-derivation but not a (l,r)-derivation of G.

Definition A derivation d of a BCC-algebra is said to be regular if $d(0) = 0$.

RESULTS AND DISCUSSION

Theorem. A (r,l)-derivation d of a BCC-algebra G is regular.

Proof Since d is (r,l)-derivation of G, $d(0) = d(0x) = 0d(x) \wedge d(0)x = 0 \wedge d(0)x = (d(0)x)((d(0)x)(0)) = (d(0)x)(d(0)x) = 0$. #

Corollary A derivation d of a BCC-algebra G is regular.

Using regular derivations, some properties of derivations of BCC-algebra can be obtained.

Proposition Let d be a self-map of a BCC-algebra G.

1. If d is a (l,r)-derivation of G, then $d(x) = d(x) \wedge x$ for all $x \in G$.
2. If d is a (r,l)-derivation of G, then for all $d(x) = x \wedge d(x)$ for all $x \in G$.

Proof 1. Let d be a (l,r)-derivation of G. Then $d(x) = d(x0) = d(x)0 \wedge xd(0) = d(x) \wedge x0 = d(x) \wedge x$.

2. Let d be a (r,l)-derivation of G. Then $d(x) = d(x0) = xd(0) \wedge d(x)0 = x0 \wedge d(x) = x \wedge d(x)$. #

Proposition Let G be a BCC-algebra with partial order $\leq$, and let d be a derivation of G. Then the following hold for all $x, y \in G$:

1. $d(x) \leq x$,
2. $d(xy) \leq d(x)y$,
3. $d(xy) \leq xd(y)$,
4. $d(xd(x)) = 0$,
5. $d(d(x)) \leq x$,
6. $d^{-1}(0) := \{x \in G \mid d(x) = 0\}$ is a BCC-subalgebra of G.

Proof 1. By the previous Proposition, $d(x) = x(xd(x))$. Then $d(x)x = 0$. Thus $d(x) \leq x$.

2. $d(xy) = xd(y) \wedge d(x)y = (d(x)y)((d(x)y)(xd(y)))$, then $d(xy)(d(x)y) = 0$. Thus $d(xy) \leq d(x)y$.

3. $d(xy) = d(x)y \wedge xd(y) = (xd(y))((xd(y))(d(x)y))$. Thus $d(xy) \leq xd(y)$.

4. $d(xd(x)) = xd(d(x)) \wedge d(x)d(x) = xd(d(x)) \wedge 0 = 0(0(xd(d(x)))) = 0$

5. $d(d(x)) = d(x(xd(x))) = d(x)(xd(x)) \wedge xd(xd(x)) = d(x)(xd(x)) \wedge x = x(x(d(x)(xd(x))))$ Thus $d(d(x)) \leq x$.

6. Since d is regular, $d^{-1}(0) \neq \phi$. Let x, y

$\in d^{-1}(0)$. Since $d(xy) = xd(y) \wedge d(x)y = x0 \wedge 0y = x \wedge 0 = 0$, we get $xy \in d^{-1}(0)$. Hence $d^{-1}(0)$ is a BCC-subalgebra of G. #

Note that $d^{-1}(0)$ is, in general, not an ideal of G, as seen in the previous example, $(23)1 \in d^{-1}(0)$ and $3 \in d^{-1}(0)$ but $21 \notin d^{-1}(0)$.

Proposition Let G be a BCC-algebra. Then $d_n(d_{n-1}(...(d_2(d_1(x)))...)) \leq x$ for $n \in \mathbf{N}$, where $d_1, d_2, ..., d_n$ are derivations of G.

Proof For n=1. $d_1(x) = d_1(x0) = d_1(x)0 \wedge xd_1(0) = d_1(x) \wedge x = x(xd_1(x)) \leq x$. Then $d_1(x)x = 0$. That is $d_1(x) \leq x$.

Let $n \in \mathbf{N}$ and assume that $d_n(d_{n-1}(...(d_2(d_1(x))...)) \leq x$. For simplicity, let $D_n = d_n(d_{n-1}(...(d_2(d_1(x))...))$. Then $d_{n+1}(D_n) = d_{n+1}(D_n 0) = d_{n+1}(D_n)0 \wedge D_n d_{n+1}(0) = d_{n+1}(D_n) \wedge D_n = D_n(D_n d_{n+1}(d_n))$. Thus $d_{n+1}(D_n)D_n = 0$. Hence $d_{n+1}(D_n) \leq D_n$. By assumption, $d_{n+1}(D_n) \leq D_n \leq x$. #

Definition. Let d be a derivation of a BCC-algebra G. An ideal A of G is said to be d-invariant if $d(A) \subseteq A$, where $d(A) = \{d(x) \mid x \in A\}$.

Example. Let $G = \{0,1,2,3,4,5\}$ and the multiplication be defined as follows:

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Then G is a BCC-algebra and $A = \{0,1,2,3,4\}$ is a BCC-ideal of G (Dudek and Zhang, 1998). Define $d : G \rightarrow G$ by

$$d(x) = \begin{cases} 0 & \text{if } x = 0,1,2,3,4, \\ 5 & \text{if } x = 5. \end{cases}$$

Then it is easily checked that d is a derivation of G. And $d(A) = \{0\} \subseteq A$. Thus A is d-invariant.

The following theorem shows that every ideal in BCC-algebra is d-invariant.

Theorem Let d be a derivation of a BCC-algebra G. Then every ideal A of G is d-invariant.

Proof Let A be an ideal of a BCC-algebra G. Let $y \in d(A)$. Then $y = d(x)$ for some $x \in A$. It follows that $yx = d(x)x = 0 \in A$, which implies $y \in A$. Thus $d(A) \subseteq A$. Hence A is d-invariant. #

CONCLUSION

The BCC-algebra $G = (G, \cdot, 0)$ defines a non-empty set G with a constant 0 and a binary operation denoted by juxtaposition satisfying the following axioms for all $x, y, z \in G$:

(1) $((xy)(zy))(xz) = 0$,

(2) $0x = 0$,

(3) $x0 = x$,

(4) $xy = yx = 0$ implies $x = y$

For elements x and y of a BCC-algebra G, denote $x \wedge y = y(yx)$ and define a map $d : G \rightarrow G$. Then d is a (l,r)-derivation of G, if it satisfies the identity $d(xy) = d(x)y \wedge xd(y)$ for all $x, y \in G$. If d satisfies the identity $d(xy) = xd(y) \wedge d(x)y$ for all $x, y \in G$, then d is a (r,l)-derivation of G. And if d is both (l,r)- and (r,l)-derivations, d is a derivation of G.

A derivation d of a BCC-algebra is said to be regular if $d(0) = 0$. An ideal A of G is said to be d-invariant if $d(A) \subseteq A$.

The results of this paper show that:

1. A derivation of BCC-algebra is regular.

2. If d is a (l,r)-derivation of a BCC-algebra G, then $d(x) = d(x) \wedge x$ for all $x \in G$. If d is a (r,l)-derivation of a BCC-algebra G, then $d(x) = x \wedge d(x)$ for all $x \in G$.

3. In BCC-algebra, the following hold:
   3.1 $d(x) \leq x$,
   3.2 $d(xy) \leq d(x)y$,
   3.3 $d(xy) \leq xd(y)$,
   3.4 $d(xd(x)) = 0$,
   3.5 $d(d(x)) \leq x$ and
   3.6 $d^{-1}(0) := \{x \in G \mid d(x) = 0\}$ is a BCC-subalgebra of G.
4. $d_n(d_{n-1}(...(d_2(d_1(x)))...)) \leq x$ where $d_1, d_2, \ldots, d_n$ are derivations of a BCC-algebra G.
5. If d is a derivation of a BCC-algebra G, then any ideal of G is d-invariant.

ACKNOWLEDGEMENT

This work was supported by a grant from the Graduate School, Kasetsart University.

LITERATURE CITED

Dudek, W. A. 1992. On constructions of BCC-algebras. **Selected Papers on BCK- and BCC-algebras** 1: 93-96.

Dudek, W. A. and X. Zhang. 1998. On ideals and congruences in BCC-algebras. **Czechoslovak Math. Journal.** 48(123): 21-29.

Hamza, A. S. A. and N. O. Al-Shehri. 2006. Some results on derivations of BCI-algebras. **Coden Jnsmac** 46: 13-19.

Hamza, A. S. A. and N. O. Al-Shehri. 2007. On left derivations of BCI-algebras. **Soochow Journal of Mathematics** 33(3): 435-444.

Iseki, K. and S. Tanaka. 1976. Ideal theory of BCK-algebras. **Math. Japonica.** 21: 351-366.

Iseki, K. and S. Tanaka. 1978. An introduction to the theory of BCK-algebras. **Math. Japonica.** 23: 1-26.

Jun, Y. B. and X. L. Xin. 2004. On derivations of BCI-algebras. **Information Sciences** 159: 167-176.

Meng, D. J. 1987. BCI-algebras and abelian groups. **Math. Japonica.** 47(1): 693-696.

Meng, J. and X. L. Xin. 1992. Commutative BCI-algebras. **Math. Japonica.** 37(3): 569-572.