

A Sinusoidal Nonlinear Oscillator with Adjustable Frequency

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ABSTRACT

A sinusoidal nonlinear oscillator with adjustable frequency was studied. The frequency of oscillation was adjusted via the change of the center frequency of the second order band-pass filter, while keeping the quality factor constant. The describing function method was used to predict the limit cycles and to analyze the stability of oscillation. The proposed circuit consisted of two important parts in the feedback connection configuration: an operational transconductance amplifier (OTA) as a nonlinear element in feedback path; and a bandpass filter in the feedforward path. The experimental results were simulated by the ORCAD CAPTURE and MATLAB computer programs.

Key words: OTA, describing function, nonlinear oscillator, frequency adjustment

INTRODUCTION

The feedback connection configuration in Figure 1 consists of two parts. $G(s)$ in the upper block is a second order high-Q bandpass filter element and $\psi(\cdot)$ in the lower block is a nonlinear element, in which an operational transconductance amplifier (OTA) was used.

The describing function method is an interesting method to analyze a nonlinear system that oscillates. The describing function method approximates the nonlinear element by a linear

one, as a ratio of the fundamental component to the complex excitation input. This method can be used to analyze magnitude stabilization phenomena in oscillators.

The Nyquist plot is a useful graphical method when used together with the plot of $1/D(a)$ in the complex plane. They provide an efficient way to find the existence of limit cycles and to check the stabilities of these limit cycles. MAPLE and MATLAB can be used to find the numerical integration of the describing function.

MATERIALS AND METHODS

Consider a nonlinear system in the general form of Figure 1. In order to develop the basic version of the describing function method, the system has to satisfy the following four conditions (Khalil, 2000):

1. There is only a single nonlinear component.
2. The nonlinear component is time-invariant.
3. Corresponding to a sinusoidal input $y = a$

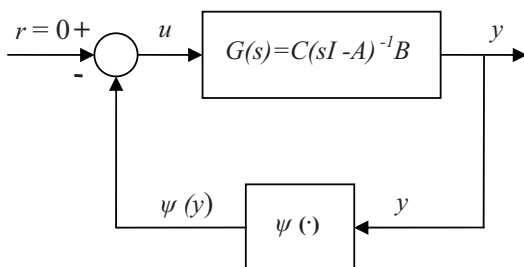


Figure 1 Feedback Connection (Stoll, 2003).

$\sin(\omega_0 t)$, only the fundamental component $w_I(t)$ in the output $w(t) = \psi(a \sin(\omega_0 t))$ has to be considered.

4. The nonlinearity is odd.

1. Periodic function

Let V be a Hilbert space of periodic functions with period $T = \frac{2\pi}{\omega_0}$. The set $\{\frac{1}{\sqrt{2}}, \cos(\omega_0 t), \sin(\omega_0 t), \cos(2\omega_0 t), \sin(2\omega_0 t), \dots\}$ is a complete orthonormal set in V with the inner product defined by Equation 1:

$$\langle f(t), g(t) \rangle := \frac{2}{T} \int_{t_0 - \frac{T}{2}}^{t_0 + \frac{T}{2}} f(t)g(t)dt. \quad \text{for all } f, g \in V.$$

If $\|g(t)\| = 1$, then $|\langle f(t), g(t) \rangle| = \|Proj_g f\|$, when

$$Proj_v u := \frac{\langle u, v \rangle}{\langle v, v \rangle} v.$$

Proposition (Fourier or orthonormal expansion)

Let $\{x_k\}$ be a complete orthonormal set in the Hilbert space V above. Then each x in V can be expressed by Equation 2:

$$x = \sum_{k=1}^{\infty} \langle x, x_k \rangle x_k = \sum_{k=1}^{\infty} Proj_{x_k} x. \quad (2)$$

where, the equality in Equation 2 is in l_2 sense. We call $\langle x, x_k \rangle$, the Fourier series coefficients of

x . Since $\sin(\omega_0 t)$ has period $T = \frac{2\pi}{\omega_0}$, that is $\sin(\omega_0(t+T)) = \sin(\omega_0 t)$, then Equation 3:

$$\psi(a \sin(\omega_0(t+T))) = \psi(a \sin(\omega_0 t)) \quad (3)$$

Therefore, $w(t) = \psi(a \sin(\omega_0 t))$ is a periodic function with period T , and with the set

$$\left\{ \frac{1}{\sqrt{2}}, \cos(\omega_0 t), \sin(\omega_0 t), \cos(2\omega_0 t), \sin(2\omega_0 t), \dots \right\}$$

being complete orthonormal, we can write as Equation 4:

$$w(t) = Proj_{x_1} w(t) + Proj_{x_2} w(t) + Proj_{x_3} w(t) + Proj_{x_4} w(t) + \dots$$

$$= Proj_{1/\sqrt{2}} w(t) + Proj_{\cos(\omega_0 t)} w(t) + Proj_{\sin(\omega_0 t)} w(t) + Proj_{\cos(2\omega_0 t)} w(t) + Proj_{\sin(2\omega_0 t)} w(t) \quad (4)$$

or Equation 5:

$$w(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \quad (5)$$

$$\text{where } \frac{a_0}{2} = Proj_I w(t), \quad a_n = Proj_{\cos(n\omega_0 t)} w(t),$$

$$b_n = Proj_{\sin(n\omega_0 t)} w(t)$$

Furthermore, since $G(s)$ has high- Q bandpass characteristics, this implies that only the fundamental component $w_I(t)$ is needed to be considered, namely (Equations 6 and 7):

$$w(t) \approx w_I(t) = Proj_{\cos(\omega_0 t)} w(t) + Proj_{\sin(\omega_0 t)} w(t) \quad (6)$$

$$= M \sin(\omega_0 t + \phi) \quad (7)$$

where

$$M(a, \omega_0) = \sqrt{\|Proj_{\cos(\omega_0 t)} w(t)\|^2 + \|Proj_{\sin(\omega_0 t)} w(t)\|^2} = \sqrt{\langle w(t), \cos(\omega_0 t), \sin(\omega_0 t) \rangle} \quad (8)$$

$$\phi(a, \omega_0) = \tan^{-1}(\langle w(t), \cos(\omega_0 t) \rangle / \langle w(t), \sin(\omega_0 t) \rangle) \quad (9)$$

This sinusoidal can be written in complex form as Equation 10:

$$w_I(t) = M e^{j(\omega_0 t + \phi)} = (\langle w(t), \sin(\omega_0 t) \rangle + j \langle w(t), \cos(\omega_0 t) \rangle) e^{j(\omega_0 t)} \quad (10)$$

2. Describing function method

The describing function of the nonlinear element is defined to be the complex ratio of the fundamental component of the output of the nonlinear element to the input sinusoidal, by Equation 11:

$$D(a, \omega_0) = M e^{j(\omega_0 t + \phi)} = \frac{(b_1 + j a_1) e^{j(\omega_0 t)}}{a e^{j(\omega_0 t)}} \quad (11)$$

and for the case of single-valued nonlinearity as Equation 12:

$$D(a) = \frac{b_I}{a} = \frac{\langle w(t), \sin(\omega_0 t) \rangle}{a} \quad (12)$$

Since $w(t)$ is an odd function, a_I is zero.

In order for a self-sustained oscillation of amplitude a and frequency ω_0 in the system of Figure 1 to exist, the variables in the loop must satisfy the following relations:

$$\begin{aligned} w &= D(a)y \\ y &= -G(j\omega_0)w \end{aligned}$$

Therefore, we have $y = -G(j\omega_0)D(a)y$. Because $y \neq 0$, this implies Equation 13:

$$G(j\omega_0)D(a) + 1 = 0 \quad (13)$$

which can be written as Equation 14:

$$G(j\omega_0) = \frac{-1}{D(a)} \quad (14)$$

Equation 14 is the condition of oscillation or so-called condition for existence of limit cycles.

Plots of both the frequency response function $G(j\omega)$ (varying ω) and the negative inverse describing function $(-1/D(a))$ (varying a) in the complex plane can be investigated. If the two curves intersect, then there exist limit cycles.

Each intersection point of the curve $G(j\omega)$ and the curve $(-1/D(a))$ corresponds to a limit cycle. If points near the intersection and along the increasing- a side of the curve $(-1/D(a))$ are not encircled by the curve $G(j\omega)$ then the corresponding limit cycle is stable. Otherwise, the limit cycle is unstable. For example, in Figure 2, there are two limit cycles. Points P_1 and P_2

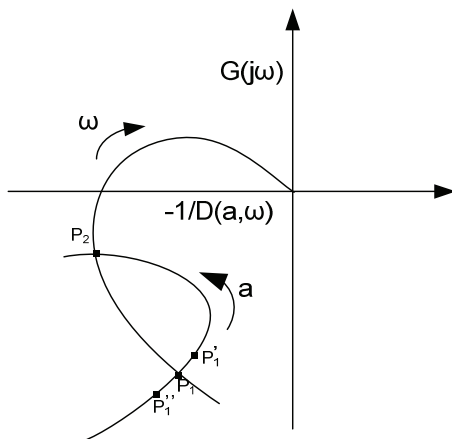


Figure 2 Detection of limit cycles.

correspond to the existing limit cycles of a system. P_1 corresponds to the unstable limit cycle, while P_2 refers to the stable one.

3. OTA nonlinear behavior

The simplified equivalent circuit diagram of an OTA is given in Figure 3. The input-output relation is described by Equation 15:

$$i_{out} = I \frac{\exp\left(\frac{qv}{kT}\right) - 1}{\exp\left(\frac{qv}{kT}\right) + 1} = I \tanh\left(\frac{v}{2V_T}\right) \quad (15)$$

where, $v = v^+ - v^-$ and $V_T = \frac{kT}{q}$.

4. Obtaining a mathematical model of an OTA from an experiment.

The OTA IC LM13600 was used as a nonlinear element for a nonlinear oscillator system in Figure 1. An OTA was tested for its input-output characteristic by applying a triangular signal to the input of the OTA and measuring its output as shown in Figure 4.

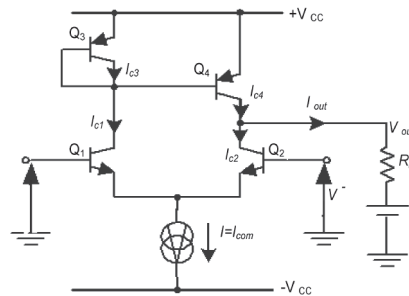


Figure 3 OTA equivalent circuit.

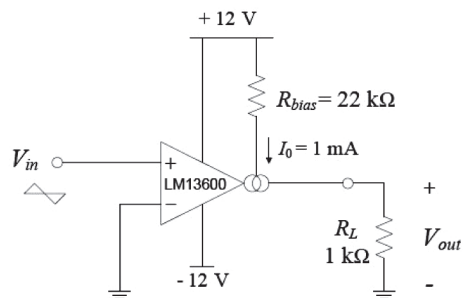


Figure 4 OTA test circuit.

The input-output relationship obtained as an xy-graph is shown in Figure 5, where the x-axis and y-axis represents V_{in} and V_{out} respectively. Measured data from the graph of Figure 5 are shown as follows.

$$\begin{bmatrix} t_i \\ y_i \end{bmatrix} = \begin{bmatrix} v_{in} \\ v_{out} \end{bmatrix} = \begin{bmatrix} -0.5 & -0.4 & -0.3 & -0.2 & -0.1 & -0.08 & -0.06 & -0.04 & -0.02 & 0 & 0.02 & 0.04 \\ -1.3 & -1.3 & -1.3 & -1.3 & -1.25 & -1.18 & -1.08 & -0.85 & -0.55 & 0 & 0.55 & 0.85 \\ 0.06 & 0.08 & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 \\ 1.08 & 1.18 & 1.25 & 1.3 & 1.3 & 1.3 & 1.3 \end{bmatrix}$$

Fitting data using the least squares method (Equation 16):

$$\min_x \sum_{i=1}^m (y_i - f(t_i, x))^2 \quad (16)$$

where, $x = (a, b)$ and

$$f(t, a, b) = a \tanh(bt); \quad a = 1.3 \text{ mA (bias current)}$$

$$f(t, b) = 1.3 \tanh(bt). \quad (17)$$

With the data from the matrix above, Equation 16 can be expanded as:

$$S = (y_1 - f(t_1, b))^2 + (y_2 - f(t_2, b))^2 + \dots + (y_{19} - f(t_{19}, b))^2$$

$$= (-1.3 - f(-0.5, b))^2 + (-1.3 - f(-0.4, b))^2 + \dots + (1.3 - f(0.5, b))^2 \quad (18)$$

Then differentiate Equation 18 to find a minimum point, using the MAPLE software

$$\frac{dS(b)}{db} = 0 \Rightarrow b \approx 20$$

$$\frac{d^2S(b)}{d^2b} = 0.008308718086 > 0$$

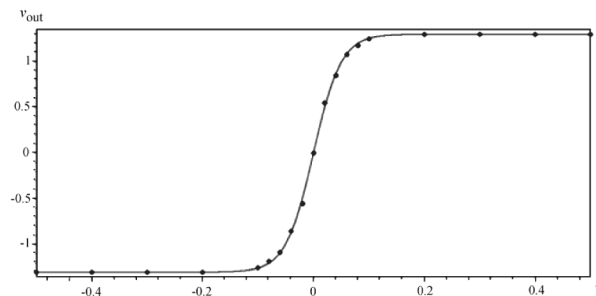


Figure 5 Data fitting using a least squares method (Slotine and Li, 1991.).

Note that the second derivative test above confirms that $b = 20$ is the minimum point.

By fitting the data to a graph of the tanh function, and from the derivation above, we obtain a practical OTA characteristic as Equation 19:

$$v_{out} = 1.3 \tanh(20v) \quad (19)$$

So, we can obtain our desired transfer function of $\psi(y) = \frac{1}{50} \tanh(20y)$ by cascading the OTA with an amplifier with gain 20. The reality system is disturbing, so the gain is 15.385.

5. Bandpass filter design

The second order bandpass filter in Figure 6 has bandpass filter characteristics of the form (Equations 20-22):

$$G(s) = \frac{-\omega_0 s}{s^2 + 2\alpha s + \omega_0^2} \quad (20)$$

$$\omega_0 = \frac{1}{CR} \quad (21)$$

$$Q = \frac{\omega_0}{2\alpha} = \frac{1}{3} \left(\frac{R_3 + R_4}{R_3} \right) \quad (22)$$

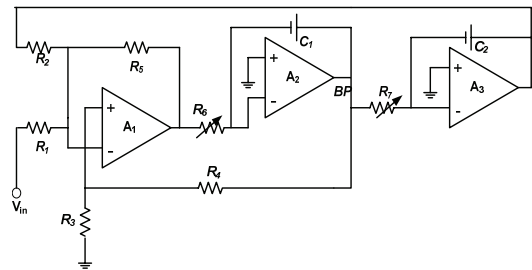


Figure 6 Second order bandpass filter circuit (St Kevin).

The passive component values were chosen as $R_1 = R_2 = R_5 = 100 \text{ k}\Omega$, $C_1 = C_2 = 4.7 \mu\text{F}$, $R_3 = 1 \text{ k}\Omega$, $R_4 = 34.76 \text{ k}\Omega$ (fixed $Q = 4.464$) and the center of frequency can be adjusted by varying the value of $R_6 = R_7 = R$.

6. Design procedure

Consider the linear system in Figure 1, having bandpass filter characteristics of the form

$G(s) = \frac{-\omega_0 s}{s^2 + 2\alpha s + \omega_0^2}$ where ω_0 is the resonant frequency and 2α is the bandwidth. If a self-excited oscillation exists with the frequency of oscillation at $\omega = \omega_0$, then Equation 23:

$$G(j\omega_0) = \frac{\omega_0}{2\alpha} = -Q \quad (23)$$

Therefore, the condition of oscillation, or the condition for existence of the limit cycle of Equation 14 becomes Equation 24:

$$\frac{1}{Q} = \frac{2\alpha}{\omega_0} = -\frac{1}{G(j\omega_0)} = D(a) \quad (24)$$

where, Q is the quality factor of the bandpass filter.

The far right-hand side of Equation 24 is the describing function of the function $\psi(\cdot)$, and for a fixed $\psi(\cdot)$, it depends only on the magnitude of oscillation (Equation 26):

$$D(a) = \frac{\langle w(t), \sin(\omega_0 t) \rangle}{\alpha} = \frac{2}{a\pi} \int_0^\pi \psi(a \sin \theta) \sin \theta d(\theta) \quad (25)$$

For known values of α and ω_0 , $D(a)$ is an equation in only one unknown variable, a , and with the aid of modern computers, this can be solved numerically, given the condition for the existence of a limit cycle, or one can plot $D(a)$ against a . From Equation 24, in order for the limit cycle to exist Equation 26 is required:

$$0 < \frac{1}{Q} = \frac{2\alpha}{\omega_0} < D_{\max} = D(0) \quad (26)$$

as can be seen in Figure 7.

Example

The example attempts to design an OTA-based sinusoidal nonlinear oscillator of constant amplitude 0.1 volts and adjust the frequency of oscillation between 120 rad/s to 12,000 rad/s.

Step1. Considering the OTA nonlinear behavior in Equation 15 with $I = 1 \text{ mA}$, we should obtain $i_{out} = 0.001 \tanh(20v)$, but instead we have obtained from Figure 4 and Equation 19 by experiment $i_{out} = 0.013 \tanh(20v)$, Then $\psi(y) = \frac{1}{50} \tanh(20y)$ can be obtained with an additional amplifier of gain 15.385 V/A at the output of the OTA.

If there is no error in the equivalent circuit of Figure 3, $i_{out} = 0.001 \tanh(20v)$ Therefore $\psi(y) = \frac{1}{50} \tanh(20y)$ can be obtained with additional amplifier of gain 20 V/A at the output of the OTA instead.

Step2. The plot $D(a)$ with respect to a , for $\psi(y) = \frac{1}{50} \tanh(20y)$, is shown in Figure 7 (Pranayanuntana P, Kaewsaiha. 2006).

Step3. With $a = 0.1$ volt, we have $Q = \frac{1}{D(0.1)} = \frac{1}{0.224} = 4.464$, $D(0.1)$ was evaluated numerically by mathematical software, such as MAPLE. From Equation 22, we can set $Q = 4.464$ by fitting $R_3 = 1 \text{ k}\Omega$, $R_4 = 12.392 \text{ k}\Omega$. We can vary the frequency of oscillation by changing the center frequency in Equation 21, as shown in Figure 8 and Table1, to achieve the desired specification.

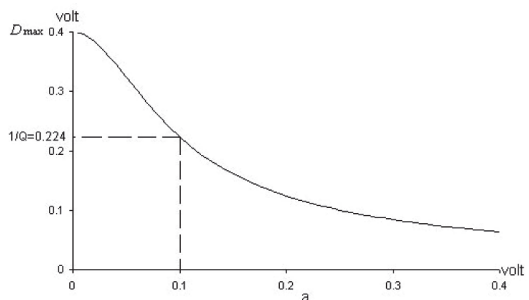


Figure 7 $D(a)$ for $\psi(y) = \frac{1}{50} \tanh(20y)$.

Step4. From Figure 9, the starting point of $-1/D(a)$ is $-1/D(0) = -2.5$ and the locus of $-1/D(0.1)$ intersects with the locus of $G(j\omega)$ with $\omega=120$ rad/s, when $R = 1773.0496 \Omega$, $\omega=1200$ rad/s when $R = 177.30496 \Omega$, $\omega=12000$ rad/s, when $R = 17.730496 \Omega$ at the point $(-4.464, 0)$, therefore there exist a limit cycle. The point near the intersection and along the increasing- a side of the curve $-1/D(a)$ are not encircled by the curve $G(j\omega)$. This implies that the limit cycle is stable.

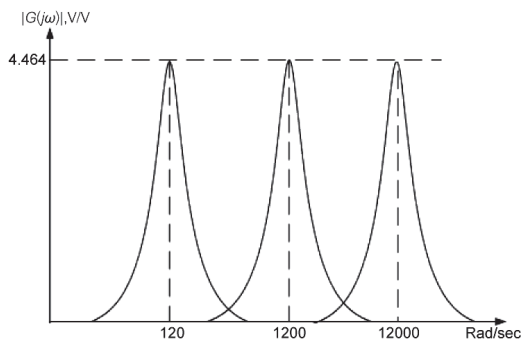


Figure 8 Bandpass Filter with different ω_0 .

Table 1 Value of center frequency ω_0 and R_4 .

ω_0 (rad/s)	R
120	1773.0496 Ω
1200	177.30496 Ω
12000	17.730496 Ω

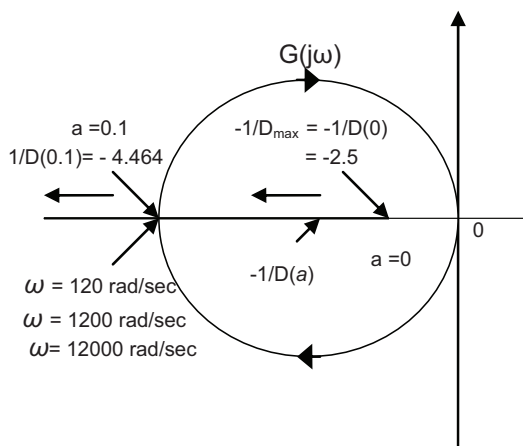


Figure 9 Plots of $G(j\omega)$ and $-1/D(a)$.

RESULTS

The output of a sinusoidal signal with different frequencies is shown in Figures 10-12.

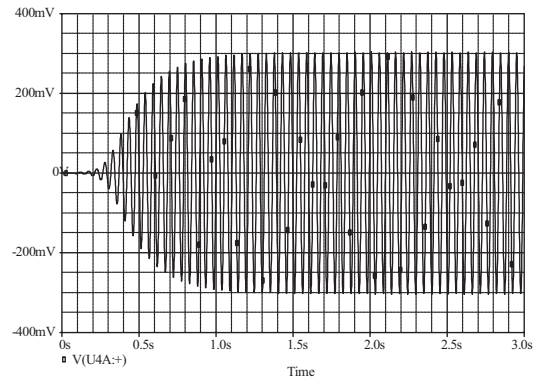


Figure 10 The signal at $\omega_0 = 120$ rad/s.

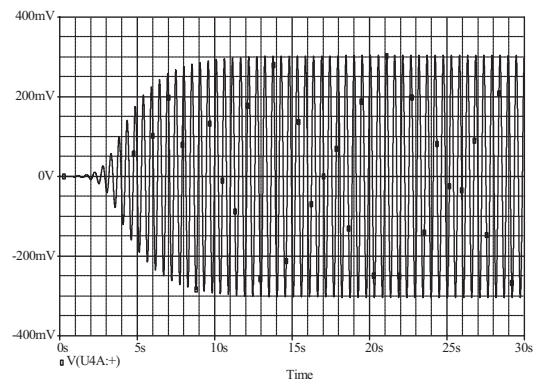


Figure 11 The signal at $\omega_0 = 1,200$ rad/s.

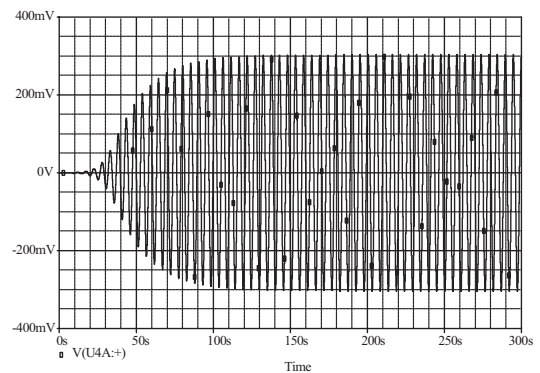


Figure 12 The signal at $\omega_0 = 12,000$ rad/s.

CONCLUSION

The presentation in this paper of an OTA-based sinusoidal nonlinear oscillator with independently adjustable frequency of oscillation has confirmed the theoretical result. ORCAD CAPTURE, MAPLE and MATLAB were useful computer programs that helped in circuit simulation, numerical integration and graphical plotting. The describing function method was a very useful method to forecast the existence of limit cycles and to determine the stability of the limit cycles.

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