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On Sum-Free Arithmetic Sequences

Chitlada Somsup

ABSTRACT

An arithmetic sequence of integers is said to be sum-free if no integer of the sequence is the sum of distinct integers of this sequence. This paper investigated whether $A = \{a, a+d, a+2d, ...\}$, where a and d are positive integers, is sum-free, and then showed that there exists a sum-free subset B of A such that $|B| \geq \frac{1}{2}|A|$. Moreover, it was also shown that if $A = \{a, 2a, ... , T(n)a\}$, where a is a positive integer, then $\frac{3^n - 1}{2} \leq T(n) \leq [n!e]-1$, where $T(n)$ is the largest positive integer such that A can be partitioned into n sum-free subsets.

Key words: sum-free, arithmetic sequence

INTRODUCTION

Schurís theorem says that for any positive integer k, there is a positive integer N such that for any k partitioned subset of $\{1, 2, ..., N\}$, we have $x + y = z$ for some positive integers x, y and z in the same subset (Schur, 1916). From this theorem many mathematicians determined the properties of a subset S of positive integers such that if for any $x, y \in S$ then $x + y \notin S$. Such sets are known as a sum-free set. A set S of positive integers is called *sum-free* if there are not (not necessarily distinct) $x, y, z \in S$ such that $x + y = z$.

The Schur function $f(n)$ is defined to be the largest positive integer such that the set of integers $\{1, 2, ... , f(n)\}$ can be partitioned into n sum-free sets. Only four values of $f(n)$ are known: $f(1) = 1$, $f(2) = 4$ and $f(3) = 13$. The value $f(4) = 44$ was determined by Baumert (1961) as follows :

$S_1 = \{1, 3, 5, 15, 17, 19, 26, 28, 40, 42, 44\}$,
$S_2 = \{2, 7, 8, 18, 21, 24, 27, 33, 37, 38, 43\}$,
$S_3 = \{4, 6, 13, 20, 22, 23, 25, 30, 32, 39, 41\}$,
$S_4 = \{9, 10, 11, 12, 14, 16, 29, 31, 34, 35, 36\}$.

For any positive integer $n \geq 5$, Street (1972) proved that

$$\frac{3^n - 1}{2} \leq f(n) \leq [n!e]-1.$$

Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok 10900, Thailand.
* corresponding author, E-mail: fscichs@ku.ac.th

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In this paper, some sufficient conditions are given for an arithmetic sequence $A = \{a, a+d, a+2d, ...\}$ to be sum-free and it is then shown that if $|A| < \infty$, there exists a sum-free subset B of A such that $|B| \geq \frac{1}{2}|A|$. Finally, bounds are also given for $T(n)$, where $T(n)$ defined to be the largest positive integer such that $\{a, 2a, \ldots, T(n)a\}$, where a is any positive integer, can be partitioned into n sum-free subsets.

MATERIALS AND METHODS

Definition 1. Let S be a subset of positive integers.
We called S is *sum-free,* if for any $x, y \in S$, then $x + y \notin S$.

Definition 2. A *partition* of a set X is a set of nonempty subsets of X such that every element x in X is in exactly one of these subsets.

Equivalently, a set P of nonempty sets is a *partition* of X, if

1. The union of the elements of P is equal to X.
2. The intersection of any two distinct elements of P is empty.

Proposition 3. For any positive integer $n \geq 2$, we have

$$[n!e] = n!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right),$$

where, $[x]$ means the greatest integer less than or equal to x.

RESULTS AND DISCISSION

The following theorems show some arithmetic sequences which are sum-free.

Theorem 1. Let a and d be positive integers such that $a < d$. Then

$$A = \{a, a+d, a+2d, ...\}$$

is a sum-free set.

Proof Let $x, y \in A$. There exist nonnegative integers r, s such that $x = a + rd$ and $y = a + sd$. Then $x + y = (a+rd)+(a+sd)$. Since $a < d$, we get $a+(r+s)d < (a+rd)+(a+sd) < a+(r+s+1)d$. Therefore $x + y \notin A$. Hence A is a sum-free set.

Theorem 2. Let a and d be positive integers such that $a \geq d$. Then

(1) For any positive integer k, $T = \{kd, (k+1)d, \ldots, (2k-1)d\}$ is a sum-free set.

(2) If a is not divisible by d, then $A = \{a, a+d, a+2d, ...\}$ is a sum-free set.

Proof (1) Let $t_1, t_2 \in T$. Then $t_1 = md$ and $t_2 = nd$ for some positive integers m, n where $k \leq m, n \leq 2k-1$. Since $2k \leq m+n \leq 4k-2$, we have $t_1 + t_2 \geq 2kd$. Therefore, $t_1 + t_2 \notin T$. Hence T is a sum-free set.

(2) Since $a > d$ and $d \nmid a$, we have $a - d > 0$. So, we let $a_1 = a - d$. Then $B_1 = \{a_1, a_1 + d, \ldots\} = \{a - d, a, a + d, \ldots\}$. If $a_1 > d$, let $a_2 = a_1 - d$. Hence $B_2 = \{a_2, a_2 + d, \ldots\}$ $= \{a - 2d, a - d, a, a + d, \ldots\}$. If $a_2 > d$, let $a_3 = a_2 - d$. Therefore $B_3 = \{a_3, a_3 + d, \ldots\}$ $= \{a - 3d, a - 2d, a - d, a, a + d, \ldots\}$. Continuing this way until we get $a_k < d$, it is true that we use $d \nmid a$ and $a_{k-1} > d$. By Theorem 1, we have $B_k = \{a_k, a_k + d, \ldots\}$ is a sum-free set. Since $A \subseteq B_1 \subseteq B_2 \subseteq \ldots \subseteq B_k$ and B_k is a sum-free set, so A is a sum-free set.

Proposition 3. Suppose that $A = \{x_1 a, x_2 a, \ldots, x_n a\}$ is a sum-free set, where a and $x_1, \ldots, x_n$ are positive integers. Then,

$$B = \{(3x_1 - 1)a, 3x_1 a, (3x_2 - 1)a, 3x_2 a, \ldots, (3x_n - 1)a, 3x_n a\}$$

is a sum-free set.

Proof Suppose that B is not sum free. Then, there exist $\alpha, \beta \in B$, such that $\alpha + \beta \in B$.

Case $\alpha + \beta = 3x_k a$ *for some* $k \in \{1, 2, \ldots, n\}$. If $\alpha = 3x_i a$ and $\beta = 3x_j a$ for some $i, j \in \{1, 2, \ldots, n\}$, then $3x_k a = \alpha + \beta = 3(x_i a + x_j a)$. Therefore, $x_i a + x_j a = x_k a$, which is a contradiction because $x_i a, x_j a, x_k a \in A$ and A is a sum-free set. If $\alpha = (3x_i - 1)a$ and $\beta = (3x_j - 1)a$ for some $i, j \in \{1, 2, \ldots, n\}$, then $3x_k a = \alpha + \beta = (3(x_i + x_j) - 2)a$. Thus $2 = 3(x_i + x_j - x_k)$, which is impossible. If $\alpha = (3x_i - 1)a$ and $\beta = 3x_j a$ for some $i, j \in \{1, 2, \ldots, n\}$, then $3x_k a = \alpha + \beta = (3(x_i + x_j) - 1)a$. Therefore $1 = 3(x_i + x_j - x_k)$, which is impossible.

Case $\alpha + \beta = (3x_k - 1)a$ *for some* $k \in \{1, 2, \ldots, n\}$. If $\alpha = 3x_i a$ and $\beta = 3x_j a$ for some $i, j \in \{1, 2, \ldots, n\}$, then $(3x_k - 1)a = \alpha + \beta = 3(x_i + x_j)a$. Thus $-1 = 3(x_i + x_j - x_k)$, which is impossible. If $\alpha = (3x_i - 1)a$ and $\beta = (3x_j - 1)a$ for some $i, j \in \{1, 2, \ldots, n\}$, then $(3x_k - 1)a = \alpha + \beta = (3(x_i + x_j) - 2)a$. Therefore $1 = 3(x_i + x_j - x_k)$, which is impossible. If $\alpha = (3x_i - 1)a$ and $\beta = 3x_j a$ for some $i, j \in \{1, 2, \ldots, n\}$, then $(3x_k - 1)a = \alpha + \beta = (3x_i - 1)a + 3x_j a$. Therefore $x_i a + x_j a = x_k a$, which is a contradiction because $x_i a, x_j a, x_k a \in A$ and A is a sum-free set. Hence B is a sum-free set.

In the next theorem, we will consider the lower bound of a sum-free subset in a set of arithmetic sequence.

Theorem 4. Let $A = \{a, a + d, \ldots, a + nd\}$ be an arithmetic sequence, where a and d are positive integers and n is a nonnegative integer. Then there exists a sum-free subset $B \subseteq A$ such that $|B| \geq \frac{1}{2}|A|$.

Proof If $n = 0$, then $A = \{a\}$. Let $B = \{a\} = A$. Then B is a sum-free set and $|B| = |A| \geq \frac{1}{2}|A|$.

Assume that $n \geq 1$. If $a < d$, by Theorem 1, we get A is a sum-free set. So, we let $B = A$. Then B is a sum-free set and $|B| = |A| \geq \frac{1}{2}|A|$. If $a \geq d$ and $d \nmid a$, by Theorem 2, we get A is a sum-free set. So, we let $B = A$. Then B is a sum-free set and $|B| = |A| \geq \frac{1}{2}|A|$. If $a \geq d$ and $d \mid a$, then there exists a positive integer k such that $a = kd$. Then $A = \{kd, (k+1)d, \ldots, (k+n)d\}$ and $|A| = n+1$. If $\left[\frac{k+n}{2}\right]+1 \leq k$, then let $B=A$. So, we get $|B| \geq \frac{1}{2}|A|$. Now, if $\left[\frac{k+n}{2}\right]+1 > k$, then let $B = \left\{\left(\left[\frac{k+n}{2}\right]+1\right)d, \ldots, (k+n)d\right\}$.

Next, we will show that $B \subseteq A$.

Case k+n is an even number, we have $k+n-\left[\frac{k+n}{2}\right] = \left[\frac{k+n}{2}\right]$, and then $\left[\frac{k+n}{2}\right]+1 = k+\left(n-\left[\frac{k+n}{2}\right]+1\right)$. Therefore $n-\left[\frac{k+n}{2}\right]+1 = \frac{n-k}{2}+1$. If $n=k$, then $\frac{n-k}{2}+1 \leq n$. If $n<k$, then $\frac{n-k}{2}+1 < 1 \leq n$. If $n>k$, then $\frac{n-k}{2}+1 \leq n$. So $\left(\left[\frac{k+n}{2}\right]+1\right)d = \left(k+\left(n-\left[\frac{k+n}{2}\right]+1\right)\right)d \leq (k+n)d$.

Case $k+n$ is an odd number, we have $\left[\frac{k+n}{2}\right] = \frac{k+n-1}{2}$. Therefore $k+n-\left[\frac{k+n}{2}\right] = \left[\frac{k+n}{2}\right]+1$. So $\left[\frac{k+n}{2}\right]+1 = k+\left(n-\left[\frac{k+n}{2}\right]\right)$. Then $n-\left[\frac{k+n}{2}\right] = \frac{n-k+1}{2}$. Since $k+n$ is an odd number, $k \neq n$. If $n<k$, then $\frac{n-k+1}{2} \leq n$. If $n>k$, then $\frac{n-k+1}{2} \leq n$. So $\left(\left[\frac{k+n}{2}\right]+1\right)d = \left(k+\left(n-\left[\frac{k+n}{2}\right]+1\right)\right)d \leq (k+n)d$. Hence $B \subseteq A$.

Next, we will show that B is a sum-free set. Let $x, y \in B$. Then there exist r, s such that $x=rd$ and $y=sd$, where $\left[\frac{k+n}{2}\right]+1 \leq r, s \leq k+n$. Since $2\left[\frac{k+n}{2}\right]+2 \leq r+s \leq 2(k+n)$, we get $x+y \geq (k+n)d$ This implies that $x+y \notin B$. Thus, B is a sum-free set. So $|B| = (k+n)-\left[\frac{k+n}{2}\right] \geq \frac{k+n}{2}$. Since $k \geq 1$ and $|A| = n+1$, we have $|B| \geq \frac{1}{2}|A|$.

Example 5. Assume that $A = \{3, 4, 5, 6, 7, 8, 9, 10\}$. So $|A| = 8$.

Let $B = \{6, 7, 8, 9, 10\} \subseteq A$. Then, we see that B is a sum-free set and $|B| = 5$. Hence, $5 = |B| > \frac{1}{2}|A| = 4$.

Definition 6. Given an arithmetic sequence $A = \{a, 2a, 3a, \ldots, T(n)a\}$, where a is a positive integer. Define $T(n)$ to be the largest positive integer such that A can be partitioned into n sum-free subsets.

Theorem 7. Let a be a positive integer. Suppose that $A = \{a, 2a, \ldots, T(n)a\}$ can be partitioned into n sum-free subsets. Then,

$$\frac{3^n - 1}{2} \leq T(n) \leq [n!e] - 1.$$

Proof Suppose that A can be partitioned into n sum-free subsets, say $S_1, S_2, \cdots, S_n$. Hence, $A = S_1 \cup S_2 \cup \ldots \cup S_n$ and $S_i \cap S_j = \varnothing$ for all $i \neq j$.

Without loss of generality, we assume that $m_1 = |S_1| \geq |S_i|$, for $i = 1, 2, \ldots, n$. Hence $T(n) \leq (m_1 n)$. Let $S_1 = \{x_1 a, x_2 a, \ldots, x_{m_1} a\}$, where $x_1 a < x_2 a < \ldots < x_{m_1} a$. Let $B_1 = \{(x_2 - x_1)a, (x_3 - x_1)a, \ldots, (x_{m_1} - x_1)a\}$. It is obvious that $B_1 \subseteq A$. Since S_1 is a sum-free set, $B_1 \not\subset S_1$. Hence $B_1 \subseteq A - S_1$ or $B_1 \subseteq S_2 \cup S_3 \cup \ldots \cup S_n$. Let S_2 be such that $m_2 = |S_2 \cap B_1| \geq |S_j \cap B_1|$ for $j = 2, 3, \ldots, n$. That is $\{(x_{i_1} - x_1)a, \cdots, (x_{m_2} - x_1)a\} \subseteq S_2$, where $i_1 < i_2 < \ldots < i_{m_2}$. Then $m_1 - 1 \leq m_2(n-1)$. Hence $B_2 = \{(x_{i_2} - x_{i_1})a, \ldots, (x_{i_{m_2}} - x_{i_1})a\} \subseteq S_3 \cup \ldots \cup S_n$. Let S_3 be such that $m_3 = |S_3 \cap B_2| \geq |S_j \cap B_2|$ for $j = 3, 4, \ldots, n$. Then $m_2 - 1 \leq m_3(n-2)$**.** Continuing in this way, we get $m_r - 1 \leq m_{r+1}(n-r)$ for $r = 1, 2, \ldots, k$ and $m_k = 1$, where $1 \leq k \leq n$. Hence

$$T(n) \leq n!\left(\frac{1}{(n-1)!} + \cdots + \frac{1}{(n-k)!}\right) \leq [n!e] - 1.$$

Next we will show that $T(n) \geq \left(\frac{3^n - 1}{2}\right)$. Suppose that $\{a, 2a, \ldots, T(n)a\}$ can be partitioned into n sum-free subsets, namely

$$S_1 = \{x_{11}a, x_{12}a, \ldots, x_{1t_1}a\},$$
$$S_2 = \{x_{21}a, x_{22}a, \ldots, x_{2t_2}a\},$$
$$\vdots$$
$$S_n = \{x_{n1}a, x_{n2}a, \ldots, x_{nt_n}a\}.$$

Then $\{a, 2a,$ Ö $,(3T(n)+1)a\}$ can be partitioned into $n+1$ subsets, namely

$$\overline{S}_1 = \{(3x_{11} - 1)a, 3x_{11}a, (3x_{12} - 1)a, 3x_{12}a, \ldots, (3x_{1t_1} - 1)a, 3x_{1t_1}a\},$$
$$\overline{S}_2 = \{(3x_{21} - 1)a, 3x_{21}a, (3x_{22} - 1)a, 3x_{22}a, \ldots, (3x_{2t_2} - 1)a, 3x_{2t_2}a\},$$
$$\vdots$$
$$\overline{S}_n = \{(3x_{n1} - 1)a, 3x_{n1}a, (3x_{n2} - 1)a, 3x_{n2}a, \ldots, (3x_{nt_n} - 1)a, 3x_{nt_n}a\},$$
$$\overline{S}_{n+1} = \{a, 4a, 7a, \ldots, (3T(n)+1)a\}.$$

By Theorem 1 and Theorem 3, then $\overline{S}_1, \overline{S}_2, \ldots, \overline{S}_{n+1}$ are sum-free sets. Therefore $T(n+1) \geq 3T(n)+1$. Since $T(1) = 1$, this yields $T(n) \geq \dfrac{3^n - 1}{2}$. This completes the proof of this theorem.

CONCLUSIONS

The results of this paper show that:

(1) For an arithmetic sequence $A = \{a, a+d, a+2d, ...\}$ where *a* and *d* are positive integers,

1.1 if $a < d$, then *A* is a sum-free set;

1.2 if $a \geq d$ and $d \mid a$, then $T = \{kd, (k+1)d, ..., (2k-1)d\}$ is a sum-free set for all positive integer k;

1.3 if $a > d$ and $d \nmid a$, then *A* is a sum-free set.

(2) Let $A = \{a, a+d, ..., a+nd\}$ be an arithmetic sequence, where *a* and *d* are positive integers and *n* is a nonnegative integer. Then, there exists a sum-free subset $B \subseteq A$, such that $|B| \geq \frac{1}{2}|A|$.

(3) Let *a* be a positive integer. Suppose that $A = \{a, 2a, ..., T(n)a\}$ can be partitioned into *n* sum-free subsets. Then,

$$\frac{3^n - 1}{2} \leq T(n) \leq [n!e] - 1 .$$

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