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Applications of Admitted Lie Group for Stochastic Differential Equations

Atipong Mahanin and Boonlert Srihirun*

ABSTRACT

In this study, the definition of an admitted Lie group for stochastic differential equations was applied to ordinary stochastic differential equations. This approach included the dependent and independent variables in the transformations. The transformations of Brownian motion were defined by the transformation of dependent and independent variables. Admitted Lie group generators for a variety of equations were obtained.

Keywords: stochastic process, determining equations, Lie group of transformations

INTRODUCTION

One of the methods used for finding exact solutions of differential equations is group analysis. A survey of this method can be found in Ovsiannileov (1978), Olver (1986) and Ibragimov (1999).This technique involves the study of symmetries of equations, which involves a local group of transformations mapping a solution of a given system of equations to another solution of the same system. Moreover, symmetries allow finding new solutions of the system. In contrast to deterministic differential equations, there have been only few attempts to apply symmetry techniques to stochastic differential equations. They fall into two groups.

Consider an Itô equation (Equation 1):

$$dX(t,\omega) = f(t,X(t,\omega))dt + g(t,X(t,\omega))\, dB(t,\omega) \quad (1)$$

with initial condition $X(0) = X^{(0)}$ being interpreted in the sense that (Equation 2):

$$X(t,\omega) = X^{(0)}(\omega) + \int_0^t f(s,X(s,\omega))ds + \int_0^t g(s,X(s,\omega))dB(s,\omega), \quad (2)$$

for almost all $\omega \in \Omega$ and each where f(t,x) is a drift function, g(t,x) is a diffusion function and B is one-dimensional Brownian motion. The first integral in this equation is of the Riemann type, while the second integral denotes a sum of integrals.

The first approach (Misawa, 1994; Albeverio and Fei, 1995; Gaeta and Quintero, 1999) considers fiber-preserving transformations

$$\bar{t} = H(t,x,a), \qquad \bar{x} = \varphi(t,x,a)$$

and has been applied to stochastic dynamical system (Misawa, 1994; Albeverio and Fei, 1995) and to the Fokker-Planck equation (Gaeta and Quintero, 1999; Gaeta, 2004). Its weakness is that it can only be applied to fiber-preserving transformations, which form a small subclass of all possible transformations.

Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok 10900, Thailand.
* Corresponding author, e-mail: fscibls@ku.ac.th

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The second approach (Gaeta and Quintero, 1999; Pooe and Wafo, 2001; Unal and Sun, 2004) deals with symmetry transformations including all the dependent variables in the transformation. This approach has been applied to scalar second-order stochastic ordinary differential equations (Gaeta and Quintero, 1999) to the Hamiltonian-Stratonovich dynamic control system (Unal, 2003) and to the Fokker-Planck equation (Unal and Sun, 2004).There have also been attempts to involve Brownian motion in the transformation, without strict proof that Brownian motion is transformed to Brownian motion.

In Srihirun *et al*. (2007), a new definition of an admitted Lie group of transformations for stochastic differential equations was given, including dependent as well as independent variables in the transformation. In particular, the transformation of Brownian motion is defined by transformation of the dependent and independent variables, and there was a strict proof that the transformed Brownian motion satisfies the properties of Brownian motion.

The current study applied the discussion in Srihirun *et al*. (2007) to ordinary stochastic differential equations, and showed how to construct the determining equations for admitted Lie groups of transformations.

MATERIALS AND METHODS

This section provides a review of the theory developed in Srihirun *et al*. (2007) and discusses transformations of stochastic processes and admitted Lie groups.

Assume that the set of transformations in Equation 3:

$$\bar{t} = H(t,x,a), \qquad \bar{x} = \varphi(t,x,a) \qquad (3)$$

composes a Lie group. Let $h(t, x) = \frac{\partial H}{\partial a}(t,x,0)$, $\xi(t, x, 0) = \frac{\partial \varphi}{\partial a}(t,x,0)$ be the coefficients of the infinitesimal generator

$$h(t, x)\partial_t + \xi(t, x)\partial_x.$$

According to Lie's theorem, the functions H(t,x,a) and $\varphi(t, x, a)$ satisfy the Lie Equations in 4:

$$\frac{\partial H}{\partial a} = h(H,\varphi), \qquad \frac{\partial \varphi}{\partial a} = \xi(h,\varphi) \qquad (4)$$

and the initial conditions for a = 0 are given in 5:

$$H = t, \qquad \varphi = x. \qquad (5)$$

Since $\frac{\partial H}{\partial t}(t,x,0) = 1$, then $\frac{\partial H}{\partial t}(t,x,0) > 0$ in a neighborhood of a = 0, where one can find a function $\eta(t, x, a)$ such that

$$\eta^2(t,x,a) = \frac{\partial H}{\partial t}(t,x,a).$$

Using the function $\eta(t, x, a)$, one can find a transformation of a stochastic process $X(t, \omega)$ by Equation 6:

$$\bar{X}(\bar{t},\omega) = \varphi(\alpha(\bar{t}), X(\alpha(\bar{t}),\omega), a), \qquad (6)$$

where

$$\beta(t) = \int_0^t \eta^2(s, X(s,\omega), a)ds, \qquad t \geq 0,$$

and $\alpha(t)$ is the converse function of $\beta(t)$. This gives an action of Lie group (Equation 3) on the set of stochastic process. Replacing $\bar{t}$ by in Equation 6, produces

$$\bar{X}(\beta(t),\omega) = \varphi(t, X(t,\omega), a).$$

It is useful to introduce the function

$$\tau(t,x) = \frac{\partial \eta}{\partial a}(t,x,0).$$

Notice that the function h(t, x) and $\tau(t, x)$ are related by the formulae

$$\tau(t,x) = \frac{\frac{\partial h}{\partial t}(t,x)}{2}, \qquad h(t,x) = 2\int_0^t \tau(s,x)ds.$$

The notions of the admitted Lie group and determining equations can now be presented.

Definition 1. A Lie group of transformations (Equation 3) is called admitted by the stochastic differential Equation 2, if for any solution X(t, ω) of Equation 2, the functions ξ(t, x) and τ(t, x) satisfy the following determining Equations 7:

$$\xi_t(t,X(t,\omega))+f\xi_x(t,X(t,\omega))+\frac{1}{2}g^2\xi_{xx}(t,X(t,\omega))$$
$$-2f_t(t,X(t,\omega))\int_0^t\tau(s,X(s,\omega))ds-f_x\xi(t,X(t,\omega))$$
$$-2f\tau(t,X(t,\omega))=0, \quad (7)$$

$$g\xi_x(t,X(t,\omega))-2g_t(t,X(t,\omega))\int_0^t\tau(s,X(s,\omega))ds-$$
$$g\tau(t,X(t,\omega))-g_x\xi(t,X(t,\omega))=0.$$

Note that the determining Equations 7 were constructed under the assumption that the Lie group of transformations in Equation 3 transforms any solution of Equation 2 to a solution of the same equation.

RESULTS AND DISCUSSION

Ornstein-Uhlenbeck process (Oksendal, 1998)

Consider Equation 8:

$$dx = \mu x dt + \sigma dBt, \quad (8)$$

where μ, σ are real constants. The solution of Equation 8 with the initial condition $X(0) = X_0$ is called the Ornstein-Uhlenbeck process. The system of determining Equation 7 for Equation 8 becomes Equation 9:

$$\xi_t+\mu x\xi_x+\frac{1}{2}\sigma^2\xi_{xx}-\mu\xi-2\mu x\tau=0, \quad (9)$$
$$\sigma\xi_x-\sigma\tau=0.$$

A particular class solution defined by the additional assumption ξ(t, x) = F(t), is

$$\xi = C_1e^{\mu t} \text{ and } \tau = 0.$$

Then h(t, x) = 0, so the admitted generator is

$$e^{\mu t}\partial_x.$$

The mean-reverting Ornstein-Uhlenbeck process (Oksendal, 1998)

Consider Equation 10:

$$dx = (m - x)dt + \sigma dB_t, \quad (10)$$

where are real constants. The solution of Equation 10 with the initial condition $X(0) = X_0$ is called the mean-reverting Ornstein-Uhlenbeck process. The system of determining Equation 7 for Equation 10 becomes Equation 11:

$$\xi_t+(m-x)\xi_x+\frac{1}{2}\sigma^2\xi_{xx}+\xi-2(m-x)\tau=0,$$
$$\sigma\xi_x-\sigma\tau=0 \quad (11)$$

A particular class of solution defined by the additional assumption ξ(t, x) = F(t), is

$$\xi = C_1e^{-t} \text{ and } \tau = 0.$$

Then h(t, x) = 0, so the admitted generator is

$$e^{-t}\partial_x.$$

Bessel process (Hendersen and Plaschko, 2006)

Consider Equation 12:

$$dx=\frac{\alpha-1}{2x}dt+dB_t, \quad \alpha=2,3,... \quad (12)$$

The solution of Equation 12 with the initial condition $X(0) = X_0$ is called the Bessel process. The system of determining Equation 7 for Equation 12 becomes Equation 13:

$$\xi_t+(\frac{\alpha-1}{2x})\xi_x+\frac{1}{2}\xi_{xx}+(\frac{\alpha-1}{2x^2})\xi-(\frac{\alpha-1}{x})\tau=0,$$
$$\xi_x-\tau=0 \quad (13)$$

A particular class of solution defined by the additional assumption ξ(t, x) = F(x), is

$$\xi=C_1x+C_2x^{\frac{1}{2}(-2+2\alpha)} \text{ and } \tau=C_1+C_2(\alpha-1)x^{\alpha-2}.$$

Then $h(t, x) = 2C_1t + 2C_2(\alpha-1)x^{\alpha-2}t$, so the admitted generators are

$$x\partial_x + 2t\partial_t, \quad x^{a-1}\partial_x + 2(\alpha-1)x^{\alpha-2}t\partial_t.$$

The Lie group of transformations corresponding to the first admitted generator is Equation 14:

$$\bar{x}=xe^a \qquad \bar{t}=te^{2a}. \quad (14)$$

It is easy to check that this Lie group transforms a solution of Equation 12 into a solution of the same equation. Let us construct the Lie group of transformations corresponding to the second admitted generator. One has to solve the Lie equations

$$\frac{\partial H}{\partial a}(t,x,a) = 2(\alpha-1)\varphi^{\alpha-2}H,$$

$$\frac{\partial \varphi}{\partial a}(t,x,a) = \varphi^{\alpha-1}$$

with the initial conditions

$$H(t,x,0) = t, \quad \varphi(t,x,0) = x.$$

The solution of this Cauchy problem is

$$H(t,x,a) = \frac{t\left[a(2-\alpha)+x^{2-\alpha}\right]^{\frac{2(\alpha-1)}{2-\alpha}}}{x^{2(\alpha-1)}},$$

$$\varphi(t,x,a) = \left[a(2-\alpha)+x^{2-\alpha}\right]^{\frac{1}{2-\alpha}}.$$

Then, $\eta^2 = H_t = \dfrac{\left[a(2-\alpha)+X^{2-\alpha}\right]^{\frac{2(\alpha-1)}{2-\alpha}}}{X^{2(\alpha-1)}}$. The transformations of the independent

Variable, t and the dependent variable, x are

$$\bar{x} = \varphi(t,x,a) = \left[a(2-\alpha)+x^{2-\alpha}\right]^{\frac{1}{2-\alpha}},$$

$$\bar{t} = H(t,x,a) = \frac{t\left[a(2-\alpha)+x^{2-\alpha}\right]^{\frac{2(\alpha-1)}{2-\alpha}}}{x^{2(\alpha-1)}}. \tag{15}$$

It is possible to show that the Lie group of transformations (Equation 15) transforms every solution of Equation 12 into a solution of the same equation. Assume that X(t) is a solution of Equation 12. In Srihirun *et al.* (2007), it was proven that the Brownian motion B(t) is transformed to the Brownian motion (Equation 16):

$$\bar{B}(t) = \int_0^{\alpha(t)} \frac{\left[a(2-\alpha)+X^{2-\alpha}\right]^{\frac{(\alpha-1)}{2-\alpha}}}{X^{(\alpha-1)}}dB(s), \tag{16}$$

where

$$\beta(t) = \int_0^t \frac{\left[a(2-\alpha)+X^{2-\alpha}\right]^{\frac{2(\alpha-1)}{2-\alpha}}}{X^{2(\alpha-1)}}ds, \quad \alpha(t) =$$

$$\inf_{s\geq 0}\{s : \beta(s) > t\}, \quad t \geq 0.$$

Applying Itô's formula to the function $\varphi(t,x,a) = \left[a(2-\alpha)+x^{2-\alpha}\right]^{\frac{1}{2-\alpha}}$, produces Equation 17:

$$\varphi(t,X(t,\omega),a) = \varphi(t,X(0,\omega),a) + \frac{(\alpha-1)}{2}$$

$$\int_0^t X^{2(1-\alpha)}(s)\left[a(2-\alpha)+X^{2-\alpha}(s)\right]^{\frac{2\alpha-3}{2-\alpha}}ds$$

$$+\int_0^t X^{1-\alpha}(s)\left[a(2-\alpha)+X^{2-\alpha}(s)\right]^{\frac{\alpha-1}{2-\alpha}}dB(s). \tag{17}$$

Changing the variable of the integral $s = \alpha(\bar{s})$ in the Riemann integral in Equation 16, it becomes

$$\frac{(\alpha-1)}{2}\int_0^t X^{2(1-\alpha)}(s)\left[a(2-\alpha)+X^{2-\alpha}(s)\right]^{\frac{2\alpha-3}{2-\alpha}}ds$$

$$= \int_0^{\beta(t)} \frac{\alpha-1}{2\bar{X}(\alpha(\bar{s}))}d(\alpha(\bar{s})).$$

Because of the transformations of the Brownian motions (Equation 16), the Itô's integral in Equation 17 becomes

$$\int_0^t X^{1-\alpha}(s)\left[a(2-\alpha)+X^{2-\alpha}(s)\right]^{\frac{\alpha-1}{2-\alpha}}dB(s)$$

$$= \int_0^{\beta(t)} X^{1-\alpha}(s)\left[a(2-\alpha)+X^{2-\alpha}(s)\right]^{\frac{\alpha-1}{2-\alpha}}\cdot\left[a(2-\alpha)\cdot\right.$$

$$+X^{2-\alpha}(s)\Big]^{\frac{\alpha-1}{2-\alpha}} \cdot X^{1-\alpha}(s)d\bar{B}(\bar{s}) = \int_0^{\beta(t)} d\bar{B}(\bar{s}).$$

Because $\bar{X}(\beta(t),\omega) = \varphi(t,X(t,\omega),a)$ and $\bar{X}(0,\omega) = \varphi(0,X(0,\omega),a)$, one has

$$\bar{X}(\beta(t),\omega) = \bar{X}(0,\omega) + \int_0^{\beta(t)} \frac{\alpha-1}{2\bar{X}} ds + \int_0^{\beta(t)} d\bar{B}(s).$$

This confirms that the Lie group of transformations (Equation 15) transforms any solution of Equation 12 into a solution of the same equation.

In Srihirun *et al.* (2007), the theory developed was applied to two equations, with one describing geometric Brownian motion and the other describing Brownian motion with drift. Now, it has been applied to three stochastic differential equations, with one equation representing an Ornstein-Uhlenbeck process, one equation describing mean-reverting an Ornstein-Uhlenbeck process, and the last one describing a Bessel process.

CONCLUSION

The definition of an admitted Lie group of transformations for stochastic differential equations given in Srihirun *et al.* (2007) was applied to stochastic differential equations. This approach included the dependent and independent variables in the transformations. The transformations of Brownian motion were defined by the transformation of dependent and independent variables. The developed theory was applied to three stochastic differential equations, one equation representing an Ornstein-Uhlenbeck process, one equation describing mean-reverting an Ornstein-Uhlenbeck process, and the last one describing a Bessel process.

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