

Right Trapezoid Cover for Triangles of Perimeter Two

Banyat Sroysang

ABSTRACT

A convex region covers a family of arcs if it contains a congruent copy of every arc in the family. In 2000, John E. Wetzel determined the smallest rectangle which covers the family of all triangles with perimeter two. In this paper, we search for the smallest right trapezoid which covers the family of all triangles of perimeter two.

Keywords: worm problem, cover, triangle, perimeter, trapezoid

INTRODUCTION

Moser (1966) posed the question, the worm problem, of what is the region of smallest area which contains a congruent copy of every unit arc on the plane. For a convex region R and a family F of arcs, we say that R **covers** F or R is a **cover** for F , if R contains a congruent copy of each arc in F . A popular shape of arcs is a triangle. Conveniently, we do not consider the problem about the family of all triangles with perimeter one but we consider the problem about the family of all triangles with perimeter two. However, we can scale between triangles with perimeter one and triangles with perimeter two. Wetzel (1997) determined the smallest equilateral triangular cover for the family of all triangles with perimeter two. Wetzel (2000) determined the smallest rectangular cover for the family of all triangles with perimeter two. Furedi and Wetzel (2000) determined the smallest convex cover for the family of all triangles with perimeter two. Zhang and Yuan (2009) determined the smallest regularized parallelogram (whose length of the

smaller diagonal is not less than one) cover for the family of all triangles with perimeter two; this result implies the result in Wetzel (2000). In this paper, we present the smallest regularized right trapezoid (whose length of the smaller diagonal is not less than one) cover for the family of all triangles with perimeter two. This new result also implies the result in Wetzel (2000).

MATERIALS AND METHODS

In this section, we list the basic definitions and a basic result.

Definition 1. A right trapezoid is called a **regularized right trapezoid** if the length of its smaller diagonal is not less than one.

Definition 2. A right trapezoid R is **circumscribed** about a triangle T , if T is a subset of R and each side of R contains a vertex of T .

Conveniently, we let $R(\beta, u, v)$ be the right trapezoid with the smaller angle β and the

lengths u, v ($u \leq v$) of two adjacent sides of the angle β .

Lemma 1. For each triangle T with perimeter two, if $R(\beta, u, v)$ is a circumscribed right trapezoid of T as shown in Figure 1, then

$$\sqrt{u^2 + v^2 - 2uv \cos \beta} \leq 1.$$

Proof. Since each side of any triangle with perimeter two has length less than one, it is clear if Figure 1 (a) occurs. Assume that $R(\beta, u, v)$ is a circumscribed right trapezoid of a triangle T with perimeter two, as shown in Figure 1 (b). We construct a congruent copy of both $R(\beta, u, v)$ and T by a 180° rotation, and label all vertices as shown in Figure 2. Since the angle $\angle EHG$ is at least 90° , we obtain that $|BC'| \leq |BC|$. Then

$$\begin{aligned} 2|FH| &= |FF'| \\ &\leq |FB| + |BC'| + |C'F'| \\ &= |FB| + |BC| + |CF| \\ &\leq |FB| + |BC| + |CF| \\ &= 2. \end{aligned}$$

Thus, $\sqrt{u^2 + v^2 - 2uv \cos \beta} = |FH| \leq 1$.

Definition 3. Let X be a convex set in the plane. For each $\theta \in [0, \pi]$, let $\omega_X(\theta)$ be the distance between the two parallel support lines of X with angle of inclination θ . The minimum of $\omega_X(\theta)$ is

called the **thickness** of X and the maximum of $\omega_X(\theta)$ is called the **diameter** of X .

For each convex set X , we know that $\omega_X(\theta)$ is a continuous function of θ . For each triangle T with perimeter two, the thickness of T is the length of the altitude to the longest side; the diameter of T is the length of the longest side. Since the equilateral triangle with perimeter two has altitude $\frac{1}{\sqrt{3}}$, it follows that, for each triangle T with perimeter two, the thickness of T is at most $\frac{1}{\sqrt{3}}$ and the diameter of T is at least $\frac{2}{3}$.

RESULTS AND DISCUSSION

In this section, we present the main results about the smallest regularized right trapezoid cover for the family of all triangles with perimeter two.

Theorem 1. Let $\beta \in (0, 90^\circ]$. Then

$$R\left(\beta, \frac{\csc \beta}{\sqrt{3}}, \frac{\sqrt{2} + \cot \beta}{\sqrt{3}}\right)$$

is the smallest regularized right trapezoid cover with angle β for the family of all triangles with perimeter two, and the least area is $\frac{2\sqrt{2} + \cot \beta}{6}$.

Proof. First, we will show that

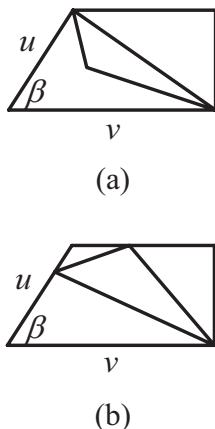


Figure 1 Two circumscribed right trapezoids.

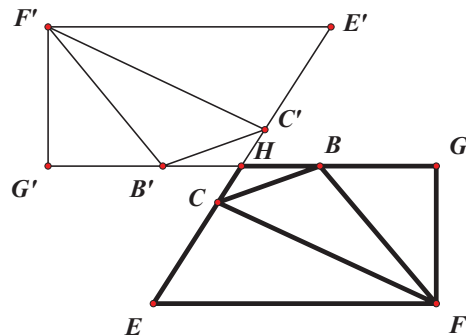


Figure 2 A 180° -rotation of Figure 1 (b).

$$R\left(\beta, \frac{\csc \beta}{\sqrt{3}}, \frac{\sqrt{2} + \cot \beta}{\sqrt{3}}\right)$$

is a cover for the family of all triangles with perimeter two. Let T be a triangle with perimeter two. Then there are angles $\theta_1, \theta_2 \in [0, 360^\circ]$ such that the thickness and the diameter of T are $\omega_T(\theta_1)$ and $\omega_T(\theta_2)$, respectively. Then $\omega_T(\theta_1) \leq \frac{1}{\sqrt{3}}$ and

$\omega_T(\theta_2) \geq \frac{2}{3}$. Since ω_T is continuous, by the intermediate value theorem, there is an angle $\theta \in [\theta_1, \theta_2]$ such that $\omega_T(\theta) = \frac{1}{\sqrt{3}}$. Let L_1 and L_2 be the two parallel support lines of T with angle of inclination θ . Then L_1 contains a vertex of T and L_2 contains a vertex of T . Let A, B and C be vertices of T . Under the rigid motions of T , without loss of generality, we can assume that T put on the position as shown in Figure 3; the angle α is at most 90° .

Let D be a vertex in L_2 such that the segment CD is perpendicular to L_1 . Then $|CD| = \frac{1}{\sqrt{3}}$. If $180^\circ - \angle BAD \leq \beta$, then we through B draw the segment EF with $\angle CFE = \beta$ as shown in Figure 3 (a); otherwise, we through A draw the segment EF with $\angle CFE = \beta$ as shown in Figure 3 (b). Then the right trapezoid $CDEF$ covers T . Note that $|EF| = \frac{\csc \beta}{\sqrt{3}}$.

By Lemma 1, we have

$$|CE|^2 = |CF|^2 + |EF|^2 - 2|CF||EF|\cos\beta \leq 1.$$

$$\begin{aligned} \text{Then } |CF| &= \frac{\cot \beta}{\sqrt{3}} + \sqrt{|CE|^2 - \frac{1}{3}} \\ &< \frac{\cot \beta}{\sqrt{3}} + \sqrt{1 - \frac{1}{3}} \\ &= \frac{\sqrt{2} + \cot \beta}{\sqrt{3}}. \end{aligned}$$

This implies that $R\left(\beta, \frac{\csc \beta}{\sqrt{3}}, \frac{\sqrt{2} + \cot \beta}{\sqrt{3}}\right)$ also covers T .

Next, we suppose that a regularized right trapezoid $R_0(\beta, u, v)$ is a cover for the family of

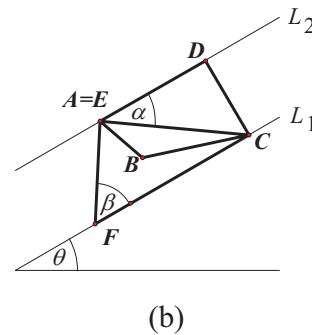
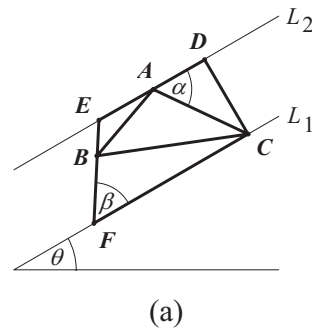


Figure 3 The right trapezoid $CDEF$ is a circumscribed right trapezoid of the triangle ABC .

all triangles with perimeter two but $R_0(\beta, u, v)$ is not a congruent copy of $R\left(\beta, \frac{\csc \beta}{\sqrt{3}}, \frac{\sqrt{2} + \cot \beta}{\sqrt{3}}\right)$. Then

$$\sqrt{u^2 + v^2 - 2uv \cos \beta} \geq 1.$$

Since $R_0(\beta, u, v)$ covers the equilateral triangle with perimeter two, we have the length $u \geq \frac{\csc \beta}{\sqrt{3}}$.

Case 1. $u = \frac{\csc \beta}{\sqrt{3}}$ and

$$\sqrt{u^2 + v^2 - 2uv \cos \beta} > 1.$$

Then
$$\sqrt{(u^2 + v^2 - 2uv \cos \beta) - (u \sin \beta)^2} > \sqrt{1 - (u \sin \beta)^2}$$

$$= \sqrt{\frac{2}{3}}.$$

Then the length $v > u \cos \beta + \sqrt{\frac{2}{3}}$.

Then the area is

$$\begin{aligned} & \frac{1}{2}(2v - u \cos \beta)(u \sin \beta) \\ & > \frac{1}{2}\left(u \cos \beta + 2\sqrt{\frac{2}{3}}\right)(u \sin \beta) \\ & = \frac{1}{2}\left(\frac{\cos \beta}{\sqrt{3}} + 2\sqrt{\frac{2}{3}}\right)\frac{1}{\sqrt{3}} \\ & = \frac{2\sqrt{2} + \cot \beta}{6}. \end{aligned}$$

Thus, the area of $R(\beta, \frac{\csc \beta}{\sqrt{3}}, \frac{\sqrt{2} + \cot \beta}{\sqrt{3}})$ is less than the area $R_0(\beta, u, v)$.

Case 2. $u > \frac{\csc \beta}{\sqrt{3}}$ and

$$\sqrt{u^2 + v^2 - 2uv \cos \beta} = 1.$$

Then $2uv(1 - \cos \beta) = 1 - (u - v)^2 \geq 1$.

Then $uv \geq \frac{1}{2(1 - \cos \beta)}$.

Then the area is

$$\begin{aligned} & \frac{1}{2}(2v - u \cos \beta)(u \sin \beta) \\ & \geq \frac{1}{2}(2v - v \cos \beta)(u \sin \beta) \\ & = \left(1 - \frac{\cos \beta}{2}\right)uv \sin \beta \\ & \geq \left(1 - \frac{\cos \beta}{2}\right)\frac{\sin \beta}{2(1 - \cos \beta)} \\ & > \frac{2\sqrt{2} + \cot \beta}{6}, \end{aligned}$$

due to their derivatives on $(0, 90^\circ]$.

Thus, the area of $R(\beta, \frac{\csc \beta}{\sqrt{3}}, \frac{\sqrt{2} + \cot \beta}{\sqrt{3}})$ is less than the area $R_0(\beta, u, v)$.

Case 3. $u > \frac{\csc \beta}{\sqrt{3}}$ and

$$\sqrt{u^2 + v^2 - 2uv \cos \beta} > 1.$$

We scale $R_0(\beta, u, v)$ to a smaller similar regularized right trapezoid $R_1(\beta, x, y)$, for which at least one condition holds:

$$\begin{aligned} \text{(i)} \quad & x = \frac{\csc \beta}{\sqrt{3}} \text{ and } \sqrt{x^2 + y^2 - 2xy \cos \beta} \geq 1, \\ \text{(ii)} \quad & x \geq \frac{\csc \beta}{\sqrt{3}} \text{ and } \sqrt{x^2 + y^2 - 2xy \cos \beta} = 1. \end{aligned}$$

Then the area of $R(\beta, \frac{\csc \beta}{\sqrt{3}}, \frac{\sqrt{2} + \cot \beta}{\sqrt{3}})$ is less than or equal to the area $R_1(\beta, x, y)$. Thus, the area of $R(\beta, \frac{\csc \beta}{\sqrt{3}}, \frac{\sqrt{2} + \cot \beta}{\sqrt{3}})$ is less than the area $R_0(\beta, u, v)$.

Therefore, $R(\beta, \frac{\csc \beta}{\sqrt{3}}, \frac{\sqrt{2} + \cot \beta}{\sqrt{3}})$ is the smallest regularized right trapezoid cover with angle β for the family of all triangles with perimeter two.

Corollary 1. (Wetzel, 2000)

The rectangle $R(90^\circ, \frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}})$ is the smallest rectangle cover for the family of all triangles with perimeter two, and the area is $\frac{\sqrt{2}}{3}$.

Proof. This follows Theorem 1 in case $\beta = 90^\circ$.

Theorem 2. Let $\beta \in (0, 90^\circ]$. Assume that $R_0(\beta, u_0, v_0)$ is a regularized right trapezoid. Then, the smallest regularized right trapezoid $R_1(\beta, u, v)$ similar to $R_0(\beta, u_0, v_0)$ that covers the family of all triangles with perimeter two is determined as follows:

(a) if $\frac{v_0}{u_0} \geq \cos \beta + \sqrt{2} \sin \beta$, then the length $u = \frac{\csc \beta}{\sqrt{3}}$ and the length $v = \left(\frac{v_0}{u_0}\right)u$;

(b) if $\frac{v_0}{u_0} < \cos \beta + \sqrt{2} \sin \beta$, then the

$$\text{length } u = \frac{u_0}{\sqrt{u_0^2 + v_0^2 - 2u_0v_0 \cos \beta}}$$

and

$$\text{the length } v = \frac{v_0}{\sqrt{u_0^2 + v_0^2 - 2u_0v_0 \cos \beta}}.$$

Proof. (a) Assume that

$$\frac{v_0}{u_0} \geq \cos \beta + \sqrt{2} \sin \beta.$$

$$\begin{aligned} \text{Then } v &= \left(\frac{v_0}{u_0}\right) u \\ &\geq (\cos \beta + \sqrt{2} \sin \beta) \left(\frac{\csc \beta}{\sqrt{3}}\right) \\ &= \frac{\sqrt{2} + \cot \beta}{\sqrt{3}}. \end{aligned}$$

Then $R_1(\beta, u, v)$ contains a congruent copy of $R(\beta,$

$\frac{\csc \beta}{\sqrt{3}}, \frac{\sqrt{2} + \cot \beta}{\sqrt{3}})$. Thus $R_1(\beta, u, v)$ is a cover for the family of all triangles with perimeter two. Since the equilateral triangle with perimeter two must be covered, there is no smaller regularized right trapezoid cover. This proves (a).

(b) Assume that

$$\frac{v_0}{u_0} < \cos \beta + \sqrt{2} \sin \beta.$$

$$\text{Then } (v_0 - u_0 \cos \beta)^2 \leq 2(u_0 \sin \beta)^2.$$

$$\begin{aligned} \text{Then } u_0^2 + v_0^2 - 2v_0u_0 \cos \beta &\leq u_0^2 + 2(u_0 \sin \beta)^2 - (u_0 \cos \beta)^2 \\ &= u_0^2 + 3(u_0 \sin \beta)^2 - u_0^2 \\ &= 3(u_0 \sin \beta)^2. \end{aligned}$$

$$\text{Then } v \geq u$$

$$\begin{aligned} &= \frac{u_0}{\sqrt{u_0^2 + v_0^2 - 2u_0v_0 \cos \beta}} \\ &\geq \frac{u_0}{\sqrt{3(u_0 \sin \beta)^2}} \\ &= \frac{\csc \beta}{\sqrt{3}}. \end{aligned}$$

$$\text{Moreover, } \sqrt{u^2 + v^2 - 2uv \cos \beta} = 1.$$

$$\begin{aligned} \text{Then } v &= u \cos \beta + \sqrt{1 - (u \sin \beta)^2} \\ &= \frac{v_0}{\sqrt{u_0^2 + v_0^2 - 2u_0v_0 \cos \beta}} \end{aligned}$$

and no smaller regularized right trapezoid similar to that covers the family of all triangles with perimeter two.

Next, we will show that $R_1(\beta, u, v)$ is a cover for the family of all triangles with perimeter two. Let T be a triangle ABC with perimeter two and longest side $a = |BC|$. Note that

$$a \leq 1 = \sqrt{u^2 + v^2 - 2uv \cos \beta}.$$

Case 1. $u \sin \beta \leq a$. Then there is a direction θ such that $\omega_{R_1}(\theta) = u \sin \beta$. Similar to the proof of theorem 1, the regularized right trapezoid $R_1(\beta, u, v)$ covers T .

Case 2. $u \sin \beta > a$. Then

$$u \sin \beta \geq \frac{1}{\sqrt{3}} \geq h$$

where h is the distance between A and BC .

Thus $v \geq u \geq u \sin \beta \geq \max\{a, h\}$. It is easy to see that the regularized right trapezoid $R_1(\beta, u, v)$ covers T .

This proves (b).

Corollary 2. (Wetzel, 2000)

Let $R_0(90^\circ, u_0, v_0)$ be a regularized right trapezoid. Then, the smallest regularized right trapezoid $R_1(90^\circ, u, v)$ similar to $R_0(90^\circ, u_0, v_0)$ that covers the family of all triangles with perimeter two has sides:

$$(a) \frac{1}{\sqrt{3}} \text{ and } \frac{1}{\sqrt{3}} \left(\frac{v_0}{u_0}\right) \text{ if } v_0 \geq \sqrt{2} u_0;$$

$$(b) \frac{u_0}{\sqrt{u_0^2 + v_0^2}} \text{ and } \frac{v_0}{\sqrt{u_0^2 + v_0^2}} \text{ if } v_0 < \sqrt{2} u_0.$$

Proof. This follows Theorem 2 in case $\beta = 90^\circ$.

CONCLUSION

We summarize, for $\beta \in (0, 90^\circ]$, that the smallest regularized right trapezoid cover for the family of all triangles with perimeter two has area $\frac{2\sqrt{2} + \cot \beta}{6}$. This result implies the result in Wetzel (2000) about the rectangular cover for the family of all triangles with perimeter two.

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