Counting Lines and Triangles in the Unit Graphs

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Abstract

Given a group $\mathbf{G}$, define the unit graph $\Gamma_G = \Gamma(\mathbf{G}, \mathbf{E})$ to have the vertex-set $\mathbf{G}$ and the edge-set $\mathbf{E}$ such that for every $x, y \in G$, , $\mathbf{x}$ and $\mathbf{y}$ are adjacent in Γ_G if and only if $xy = e$,and for each $x, y \in G$, , and $\mathbf{x}$ and $\mathbf{y}$ are adjacent in Γ_G where $\mathbf{e}$ is the identity element in a group $\mathbf{G}$. A line in the unit graph Γ_G is an edge $\{\mathbf{a}, \mathbf{b}\} \in \mathbf{E}$ such that the degree of $\mathbf{a}$ is one. A triangle in the unit graph Γ_G is a subgraph which is isomorphic to the cycle of length three. In this paper, we count the number of lines and triangles in the unit graph of some finite groups.

Keywords: group as graphs, graphs of cyclic groups, graphs of dihedral groups, handshaking lemma

1. Introduction

Kandasary and Smarandache [1] represent every finite group in the form of a graph. They call such graphs as identity graph, since the main role in obtaining the graph is played by the identity element of the group.

Given a group $\mathbf{G}$, define the unit graph $\Gamma_G = \Gamma(\mathbf{G}, \mathbf{E})$ to have the vertex-set $\mathbf{G}$ and the edge-set $\mathbf{E}$ such that for every $x, y \in G$, , $\mathbf{x}$ and $\mathbf{y}$ are adjacent in Γ_G if and only if $\mathbf{xy} = \mathbf{e}$,and for each $x, y \in G$, ,and $\mathbf{x}$ and $\mathbf{y}$ are adjacent in Γ_G where $\mathbf{e}$ is the identity element in a group $\mathbf{G}$. A line in the unit graph Γ_G is an edge $\{\mathbf{a}, \mathbf{b}\} \in \mathbf{E}$ such that the degree of $\mathbf{a}$ is one. A triangle in the unit graph Γ_G is a subgraph which is isomorphic to the cycle of length three. Also in 2015, Godase [2] represents graphs of some finite groups, namely the addition group $\mathbb{Z}$ modulo $\mathbf{n}$, the cyclic group C_n of order $\mathbf{n}$ and the dihedral group D_n of order $2\mathbf{n}$ and call such their graphs as the unit graphs. Unfortunately in the studies of Kadasary [1] and Godase [2], the authors only give some results about the number of lines and triangles in the unit graphs. The main results of this paper, we want to count the number of lines and triangles in the unit graph of some finite groups which do not contain in the previous studies of Kadasary [1] and Godase [2].

For the related topic we introduce the reader to the study of DeMeyer [3]. The basic definitions of groups and graphs in general refer any standard book on graph theory, group theory and also can be found more in the previous studies [4-9].

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2. Preliminaries

Now we recall some basic definitions about groups and graphs.

Definition 2.1 A *group* is a nonempty set G under a binary operation $\cdot : G \times G \to G$ that satisfies three properties:

(i) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in G$ (associativity);

(ii) There exists $e \in G$ such that $x \cdot e = e \cdot x = x$ for all $e \in G$ (identity);

(iii) For all $x \in G$, there exists $y \in G$ such that $x \cdot y = y \cdot x = e$ (inverses).

A group $(G, \cdot)$ is called *abelian* if $x \cdot y = y \cdot x$ for all $x, y \in G$

Definition 2.2 We say that a group G is a *finite group* if the cardinality of G is finite. The cardinality of G is called *the order* of G. We say that an element g in G has *finite order* if there exists a positive integer n such that $g^n = e$ The smallest positive integer k such that $g^k = e$ is called *the order* of g such that $g^n = e$ and denoted by $O(g)$

Lemma 2.3 If G be a finite group, then every element g in G has finite order.

Lemma 2.4 Let G be a group of order n and g an element in G. If g has order k, then $k \mid n$.

Definition 2.5 A group G is called cyclic if there exists $a \in G$ such that

$$G = \langle a \rangle = \left\{ a^n \mid n \in \mathbb{Z} \right\}$$

We say a is a generator of G. A cyclic group may have many generators. Although the list $..., a^{-2}, a^{-1}, a^0 a^1, a^2, ...$ has infinitely many entries, the set $\left\{ a^n \mid n \in \mathbb{Z} \right\}$ may have only finitely many elements. Also, since

$$a^i \cdot a^j = a^{i+j} = a^{j+i} = a^j \cdot a^i,$$

every cyclic group is abelian.

Example 2.6

1. $\mathbb{Z}$ under addition is an infinite cyclic group which is generated by 1 or -1.
2. $\mathbb{Z}_n = \{\bar{0}, \bar{1}, \bar{2}, ..., \overline{n-1}\}$ with addition modulo n is a finite cyclic group where $\bar{a} = \{b \in \mathbb{Z} \mid a \equiv b \bmod n\}$ the equivalent class with addition modulo n. For the convenience, throughout this paper, we write $\mathbb{Z}_n = \{0, 1, 2, ..., n-1\}$.

Definition 2.7 A (simple) *graph* $\Gamma_G = \Gamma(V, E)$ is a pair of vertex-set V and edge-set E where V is a non-empty set and E is a collection of subsets of $V \times V$. An element in V is called a *vertex* in the graph Γ. An element in E is called an *edge* in the graph Γ. We say that two vertices x and y in a graph Γ are *adjacent* if there is an edge between x and y, and denoted by $\{x, y\}$.

Definition 2.8 For each vertex x in a graph Γ, the number of vertices adjacent (or the number of edges incident to) the vertex x is called the *degree* of x, denoted by $\deg(x)$.

Lemma 2.9 (***Handshaking lemma***) *In a graph* $\Gamma(V, E)$, *we have*

$$\sum_{x \in V} \deg(x) = 2|E|$$

where $|A|$ *is the cardinality of a set* A

Now we recall the definition of the unit graph as follows.

Definition 2.10 Given a group G with e as the identity element, define the *unit graph* $\Gamma_G = \Gamma(G, E)$ to have the vertex-set G and the edge-set E satisfying two conditions:

(i) For every $x, y \in G$, x and y are adjacent in Γ_G if and only if $xy = e$;

(ii) For each $x, y \in G$, x and e are adjacent in Γ_G.

Definition 2.11 Given a group G, a *line* in the unit graph Γ_G is an edge $\{x,e\}$ such that the degree of a vertex x is one. The number of lines in the unit graph Γ_G is denoted by $\mathbf{line}(G)$. A *triangle* in the unit graph Γ_G is a subgraph which is isomorphic to the cycle of length three. The number of triangles in the unit graph Γ_G is denoted by $\mathbf{tri}(G)$.

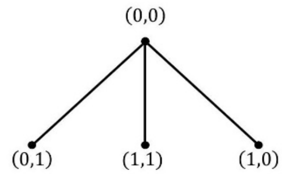

Figure 1. The unit graph of Klien four group

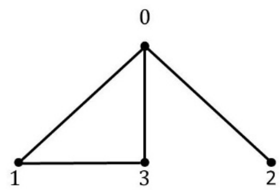

Figure 2. The unit graph of cyclic group of order 4

Example 2.12 Consider the unit graphs of groups of order 4.
For the Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$ under the addition modulo 2 (Figure 1), we have

$$E_{\mathbb{Z}_2\times\mathbb{Z}_2} = \{\{(1,0),(0,0)\},\{(0,1),(0,0)\},\{(1,1),(0,0)\}\}.$$

So that $\mathbf{line}(\mathbb{Z}_2 \times \mathbb{Z}_2) = 3$ and $\mathbf{tri}(\mathbb{Z}_2 \times \mathbb{Z}_2) = 0$.
For the cyclic group $\mathbb{Z}_4$ under the addition modulo 4 (Figure 2), we have

$$E_{\mathbb{Z}_2\times\mathbb{Z}_2} = \{\{1,0\},\{2,0\},\{3,0\},\{1,3\}\}.$$

So that $\mathbf{line}(\mathbb{Z}_4) = 1$ and $\mathbf{tri}(\mathbb{Z}_2) = 1$.

Corollary 2.13 [1, 2] *If G is a cyclic group of odd order then also G has the identity graph G_i which is formed only by triangles with no lines.*

Theorem 2.14 [1] *If $G = \langle g \rangle$ is a cyclic group of order n, n an odd number then the identity graph G_i of G is formed with $\frac{n-1}{2}$ triangles.*

Theorem 2.15 [1] *If $G = \langle g \rangle$ is a cyclic group of order n, n an even number then the identity graph G_i of G is formed with $\frac{n-2}{2}$ triangles.*

3. Main Results

In this paper, we count the number of lines and triangles in the unit graph of finite groups.

Lemma 3.1 For a group G of order n, we have

$$\mathbf{line}(G) + 2\mathbf{tri}(G) = n - 1.$$

Proof. Consider the degree of each vertex $\mathbf{n}$, in Γ_G. It is clear that $\deg(\mathbf{e})=\mathbf{n}-1$ by the definition of unit graphs. For any vertex $\mathbf{v}\in\mathbf{G}$ and $\mathbf{O}(\mathbf{v})=2$, then $\{0,\mathbf{v}\}$ will be a line in Γ_G. Thus the number of line must be equal to the number of vertex $\mathbf{v}$ with $\mathbf{O}(\mathbf{v})=2$. In other case the vertex will be in a triangle. That is for any two vertices $\mathbf{u},\mathbf{v}$ such that $\mathbf{u}\neq\mathbf{e}$ and $\mathbf{v}\neq\mathbf{e}$ in a triangle must be adjacent to each other, so $\deg(\mathbf{u})=2=\deg(\mathbf{v})$. We now get that

$$\sum_{\mathbf{x}\in\mathbf{G}}\deg(\mathbf{x})=\deg(\mathbf{e})+\mathbf{line}(\mathbf{e})+4\mathbf{tri}(\mathbf{G}).$$

Now consider the number of edges in Γ_G. It can be seen that the number of edges in one triangle equals 3 and the number of edges in one line equals 1. Then $|\mathbf{E}|=\mathbf{line}(\mathbf{G})+3\mathbf{tri}(\mathbf{G})$. By Lemma 2.9., we conclude that

$$\deg(\mathbf{e})+\mathbf{line}(\mathbf{G})+4\mathbf{tri}(\mathbf{G})=\sum_{\mathbf{x}\in\mathbf{G}}\deg(\mathbf{x})=2|\mathbf{E}|=2(\mathbf{line}(\mathbf{e})+3\mathbf{tri}(\mathbf{G})).$$

Then $\mathbf{n}-1+\mathbf{line}(\mathbf{G})+4\mathbf{tri}(\mathbf{G})=2\mathbf{line}(\mathbf{G})+6\mathbf{tri}(\mathbf{G})$. This implies that

$$\mathbf{n}-1=\mathbf{line}(\mathbf{G})+2\mathbf{tri}(\mathbf{G}).$$

This completes the proof.

Proposition 3.2 (Cyclic groups) For a cyclic groups $\mathbb{Z}_\mathbf{n}$ with the addition operation of order $\mathbf{n}$, we have

$$\mathbf{line}\left(\mathbb{Z}_\mathbf{n}\right)=\begin{cases}0, \mathbf{n}=2\mathbf{k}+1\\ 1, \mathbf{n}=2\mathbf{k}+2\end{cases}$$

and

$$\mathbf{tri}\left(\mathbb{Z}_\mathbf{n}\right)=\begin{cases}\dfrac{\mathbf{n}-1}{2}, \mathbf{n}=2\mathbf{k}+1\\ \dfrac{\mathbf{n}-2}{2}, \mathbf{n}=2\mathbf{k}+2.\end{cases}$$

Proof. For $\mathbf{n}$ is odd, it follows from Corollary 2.13., Theorem 2.14.
For $\mathbf{n}$ is even, it follows from Theorem 2.15.

Example 3.3 Consider the unit graph of the group $\mathbb{Z}_{13}=\{0,1,2,...,12\}$ of order 13 (Figure 3). $\mathbf{E}_{\mathbb{Z}_{13}}=\{\{0,\mathbf{i}\}\,|\,\mathbf{i}=1,2,3,...,12\}\cup\{\{\mathbf{i},\mathbf{j}\}\,|\,\mathbf{j}+\mathbf{k}=13,\ \mathbf{j}\neq\mathbf{k}$ and $\mathbf{j},\mathbf{k}\in\{1,2,3,...,12\}\}$. So that $\mathbf{line}\left(\mathbb{Z}_{13}\right)=0$ and $\mathbf{tri}\left(\mathbb{Z}_{13}\right)=0$.

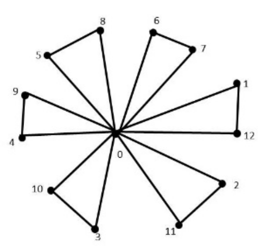

Figure 3. The unit graph of the cyclic group of order 13

Example 3.4 Consider the unit graph of the group $\mathbb{Z}_8 = \{0,1,2,...,7\}$ of order 8 (Figure 4).
$\mathbf{E}_{\mathbb{Z}_8} = \{\{0,\mathbf{i}\} \mid \mathbf{i} = 1,2,3,...,7\} \cup \{\{\mathbf{i},\mathbf{j}\} \mid \mathbf{j}+\mathbf{k}=7, \mathbf{j}\neq\mathbf{k}$ and $\mathbf{j},\mathbf{k}\in\{1,2,3,...,7\}\}$. So that $\mathbf{line}(\mathbb{Z}_8)=1$ and $\mathbf{tri}(\mathbb{Z}_{13})=3$.

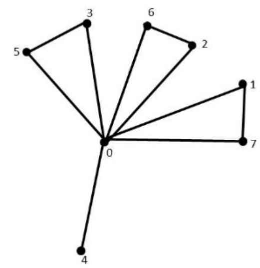

Figure 4. The unit graph of the cyclic group of order 8

Proposition 3.5 (Product of two cyclic groups) *For a group* $\mathbb{Z}_\mathbf{m}\times\mathbb{Z}_\mathbf{n}$ *with the addition operations are defined component-wise, where* $\mathbf{m},\mathbf{n}$ *are positive integers. Then*

$$\mathbf{line}(\mathbb{Z}_\mathbf{m}\times\mathbb{Z}_\mathbf{n}) = \begin{cases} 0, \mathbf{m}=2\mathbf{k}+1, \mathbf{n}=2\mathbf{k}+1 \\ 1, \mathbf{m}=2\mathbf{k}+1, \mathbf{n}=2\mathbf{k}+2 \\ 1, \mathbf{m}=2\mathbf{k}+2, \mathbf{n}=2\mathbf{k}+1 \\ 3, \mathbf{m}=2\mathbf{k}+2, \mathbf{n}=2\mathbf{k}+2 \end{cases}$$

and

$$\mathbf{tri}(\mathbb{Z}_\mathbf{m}\times\mathbb{Z}_\mathbf{n}) = \begin{cases} \frac{\mathbf{mn}-1}{2}, \mathbf{m}=2\mathbf{k}+1, \mathbf{n}=2\mathbf{k}+1 \\ \frac{\mathbf{mn}-2}{2}, \mathbf{m}=2\mathbf{k}+1, \mathbf{n}=2\mathbf{k}+2 \\ \frac{\mathbf{mn}-2}{2}, \mathbf{m}=2\mathbf{k}+2, \mathbf{n}=2\mathbf{k}+1 \\ \frac{\mathbf{mn}-4}{2}, \mathbf{m}=2\mathbf{k}+2, \mathbf{n}=2\mathbf{k}+2 \end{cases}$$

Proof. If $\mathbf{m}$ and $\mathbf{n}$ are odd then $\mathbf{mn}$ is also odd. Thus there is no $(\mathbf{x},\mathbf{y})\in\mathbb{Z}_\mathbf{m}\times\mathbb{Z}_\mathbf{n}$ such that $(\mathbf{x},\mathbf{y})^2=(0,0)$. Thus there is no line in $\Gamma_{\mathbb{Z}_\mathbf{m}\times\mathbb{Z}_\mathbf{n}}$. By Lemma 3.1., $\mathbf{tri}(\mathbb{Z}_\mathbf{m}\times\mathbb{Z}_\mathbf{n})=\frac{\mathbf{m}-\mathbf{n}}{2}$.

If $\mathbf{m}$ is odd and $\mathbf{n}$ is even (or $\mathbf{m}$ is even and $\mathbf{n}$ is odd). Then there is only $\{(0,0),(\frac{\mathbf{m}}{2},0)\}\in\mathbf{E}(\mathbb{Z}_\mathbf{m}\times\mathbb{Z}_\mathbf{n})$. This implies $\mathbf{line}(\mathbb{Z}_\mathbf{m}\times\mathbb{Z}_\mathbf{n})=1$. By Lemma 3.1. $\mathbf{tri}(\mathbb{Z}_\mathbf{m}\times\mathbb{Z}_\mathbf{n})=\frac{\mathbf{mn}-2}{2}$

If $\mathbf{m}$ and $\mathbf{n}$ are even then $\{(0,0),(\frac{\mathbf{m}}{2},0)\},\{(0,0),(\frac{\mathbf{m}}{2},\frac{\mathbf{m}}{2})\}$ and $\{(0,0),(\frac{\mathbf{m}}{2},\frac{\mathbf{n}}{2})\}$ are edges in $\Gamma_{\mathbb{Z}_\mathbf{m}\times\mathbb{Z}_\mathbf{n}}$. Then $\mathbf{line}(\mathbb{Z}_\mathbf{m}\times\mathbb{Z}_\mathbf{n})=3$. By Lemma 3.1. $\mathbf{tri}(\mathbb{Z}_\mathbf{m}\times\mathbb{Z}_\mathbf{n})=\frac{\mathbf{mn}-4}{2}$.

Corollary 3.6 (Product of cyclic groups of even order) *For a group* $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ *with the addition operations are defined component-wise where* $n_1, \ldots, n_k$ *are even positive integers, we have*

$$\mathbf{line}\left(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}\right) = 2^k - 1$$

and

$$\mathbf{tri}\left(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}\right) = \frac{(n_1 \cdots n_k) - 2^k}{2}.$$

Proposition 3.7 (Dihedral groups) For the dihedral group $(D_n, \cdot)$ of order $2n$ where $D_n = \left\langle a, b \mid a^n = b^n = 1, bab = a \right\rangle$, we have

$$\mathbf{line}\left(D_n\right) = \begin{cases} n, n = 2k+1 \\ n+1, n = 2k+2 \end{cases}$$

and

$$\mathbf{tri}\left(D_n\right) = \begin{cases} \dfrac{n-1}{2}, n = 2k+1 \\ \dfrac{n-2}{2}, n = 2k+2. \end{cases}$$

Proof. It can be seen that the set of all elements of order two is $\{a, ab, ab^2, \ldots., ab^{n-1}\}$ if n is odd and $\{a, b^n, ab, ab^2, \ldots., ab^{n-1}\}$ if n is even, so we get $\mathbf{line}(D_n)$ is n and $n+1$, respectively. By using Lemma 3.1., we get $\mathbf{tri}(D_n) = \frac{n-1}{2}$ if n is odd and $\mathbf{tri}(D_n) = \frac{n-2}{2}$ if n is even. This completes the proof.

Lemma 3.8 Let S_n be the symmetric group of order $n!$ for $n \neq 1$ and $S = \{\alpha \in S_n \mid \alpha^2 = e\}$ where $e = (1)$ the identity element in S_n. Then $\alpha \in S$ if and only if α is a transposition or α is a product of disjoint transposition.

Finally, we give some results of the number of lines and triangles in the unit graphs of the symmetric group S_n when $n \neq 1$.

Proposition 3.9 (Symmetric groups) *For the symmetric group* $(S_n, \circ)$, *we have* $\mathbf{line}(S_1) = 0 = \mathbf{tri}(S_1), \mathbf{tri}(S_2) = 0,$ *and for* $n \geq 3$ *we get*

$$\mathbf{line}(S_n) = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \frac{\binom{n}{2}\binom{n-2}{2}\cdots\binom{n-2(j-1)}{2}}{j!}$$

and

$$2\mathbf{tri}(S_n) = n! - 1 - \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \frac{\binom{n}{2}\binom{n-2}{2}\cdots\binom{n-2(j-1)}{2}}{j!}$$

Where $\lfloor n \rfloor$ the greatest integer that is less than or equal to n.

Proof. Using Lemmas 3.1 and 3.8

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