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# New Classes of Permutation Polynomials Having the Forms $(ax^{p^k} - ax + \delta)^s + x$ and $(ax^{p^j} + bx^{p^k} + cx + \delta)^s + x$ Over $\mathbb{F}_{p^{in}}$

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## Abstract

A class of permutation polynomials of the form $(x^{2^k} + x + \delta)^s + x$ was derived by Zeng-Zhu-Hu in 2010, and this was generalized to similar forms by Zha-Hu in 2012, by Tu-Zeng-Jiang and Tu-Zeng-Li-Helleseth in 2015, and by Zha-Hu in 2016. Using techniques inspired by the work of Zeng-Zhu-Hu, new classes of permutation polynomials of the forms $(ax^{p^k} - ax + \delta)^s + x$ and $(ax^{p^j} + bx^{p^k} + cx + \delta)^s + x$ are derived.

**Keywords:** finite fields, permutation polynomials

## 1. Introduction

Let $\mathbb{F}_q$ denote the finite field of $q$ elements, where $q = p^n$, $p$ is a prime and $n \in \mathbb{N}$ and let $\mathbb{F}_q^* := \mathbb{F}_q \setminus \{0\}$. A polynomial $f(x) \in \mathbb{F}_q[x]$ which induces a bijective map from $\mathbb{F}_q$ to itself is called a permutation polynomial over $\mathbb{F}_q$. Permutation polynomials have been a subject of study for many years, and have applications in coding theory, cryptography, combinatorial designs, and many other areas of mathematics and engineering.

In 2010, Zeng *et al.* [1] provided permutation polynomials over $\mathbb{F}_{2^n}$ in the form $f(x) = (x^{2^k} + x + \delta)^s + x$. Later, Zha and Hu [2] obtained permutation polynomials in very similar form of Zeng-Zhu-Hu but over $\mathbb{F}_{3^n}$ and over $\mathbb{F}_{p^n}$, in 2012. Next, Tu *et al.* [3] provided permutation polynomials in the similar form of Zeng-Zhu-Hu but over $\mathbb{F}_{2^{2m}}$, in 2015 and in the same year, Tu *et al.* [4] obtained permutation polynomials in the form $f(x) = (x^{3^m} - x + \delta)^{3^m+2} + x$ and $f(x) = (x^{3^m} - x + \delta)^{3^{2m}-3^m-1} + x$ over $\mathbb{F}_{3^{2m}}$ and permutation polynomials in the form $f(x) = (x^{p^m} - x + \delta)^{i(p^m-1)+1} + x$ over $\mathbb{F}_{p^{2m}}$.

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Moreover, in 2016, Zha and Hu [5] derived classes of permutation polynomials in the form $f(x)=(x^{2^m}+x+\delta)^{\frac{2^{2m}+2^m+1}{3}}+x$ and $f(x)=(x^{2^m}+x+\delta)^{2^{2m-2}+2^{m-2}+1}+x$ over $\mathbb{F}_{2^{2m}}$ and also the form $f(x)=(x^{3^m}-x+\delta)^{2\cdot 3^{2m-1}+3^{m-1}}+x$ over $\mathbb{F}_{3^{2m}}$. Using techniques inspired by the work of Zeng *et al.* [1], we obtain new classes of permutation polynomials of the following forms:

1. $(ax^{p^k}-ax+\delta)^s+x$ where $a,k,s\in\mathbb{N}$ with $\gcd(n,k)>1$, $s(p^k-1)\equiv 0(\bmod p^n-1)$, $a\in\mathbb{F}_{p^n}^*$ and $\delta\in\mathbb{F}_{p^n}$.

2. $(ax^{p^j}+bx^{p^k}+cx+\delta)^s+x$ where $a,b,c,j,k,s\in\mathbb{N}$ with $j>k,\gcd(n,j,k)=g$, $s(p^g-1)\equiv 0(\bmod p^n-1)$, $a,b,c\in\mathbb{F}_{p^n}^*$, $a+b+c=0$ and $\delta\in\mathbb{F}_{p^n}$.

## 2. Preliminaries

In order to prove the results in this paper, some following basic properties of finite fields are need.

**Theorem 2.1** [6] *Let $R$ be a commutative ring of prime characteristic $p$. Then*

$$(a+b)^{p^n}=a^{p^n}+b^{p^n} \text{ and } (a-b)^{p^n}=a^{p^n}-b^{p^n}$$

*for all $a,b\in\mathbb{R}$ and $n\in\mathbb{N}$.*

**Theorem 2.2** [6] *If $F$ is a finite field with $q$ elements, then every $a\in F$ satisfies $a^q=a$.*

**Corollary 2.3** [7] *Let $F$ be a finite field with $q$ elements and $E$ be a field which contains $F$ as a subfield. Then $a^q=a$ for all $a\in F$ and, moreover, for any $\alpha\in E$, $\alpha^q=\alpha$ implies $\alpha\in F$.*

A permutation polynomial over a finite field $\mathbb{F}_{p^n}$ is a polynomial that can be induced to be a permutation of $\mathbb{F}_{p^n}$. Precisely, $f(x)$ is a permutation polynomial over $\mathbb{F}_{p^n}$ if and only if $f$ is a bijective map from $\mathbb{F}_{p^n}$ to itself. However, there are several equivalent ways to determine such polynomials, which we collect from [6] as:

**Lemma 2.4** [6] *A polynomial $f(x)\in\mathbb{F}_q[x]$ is said to be a permutation polynomial over $\mathbb{F}_q$ if and only if one of the following conditions holds*:

(1) *$f:c\mapsto f(c)$ is onto;*

(2) *$f:c\mapsto f(c)$ is one-to-one;*

(3) *$f(x)=a$ has a solution in $\mathbb{F}_q$ for each $a\in\mathbb{F}_q$;*

(4) *$f(x)=a$ has a unique solution in $\mathbb{F}_q$ for each $a\in\mathbb{F}_q$.*

Moreover, the main results in [1] and [8] are also required which we record them as:

**Proposition 2.5** [1] *For any $n$ and $k$ with $\gcd(n,k)>1$, let $s$ be a positive integer with $s(2^k-1)\equiv 0(\bmod 2^n-1)$ and $\delta\in\mathbb{F}_{2^n}$. Then $f(x)=(x^{2^k}+x+\delta)^s+x$ is a permutation polynomial over $\mathbb{F}_{2^n}$.*

**Proposition 2.6** [8] *For any $a\in\mathbb{F}_q^*$ and $i,j\in\mathbb{N}$, we have $a^i=a^j$ if and only if $i\equiv j(\bmod q-1)$.*

## 3. Main Results

**Theorem 3.1** *Let* $n,a,k,s \in \mathbb{N}$ *with* $\gcd(n,k)>1$, $s(p^k-1)\equiv 0 \pmod{p^n-1}$, $a \in \mathbb{F}_{p^n}^*$ *and* $\delta \in \mathbb{F}_{p^n}$. *Then*

$$f(x)=(ax^{p^k}-ax+\delta)^s+x$$

*is a permutation polynomial over* $\mathbb{F}_{p^n}$.

*Proof.* By Lemma 2.4, the polynomial $f(x)\in \mathbb{F}_{p^n}[x]$ is a permutation polynomial over $\mathbb{F}_{p^n}$ if and only if $f(x)=d$ has a solution in $\mathbb{F}_{p^n}$ for each $d \in \mathbb{F}_{p^n}$, it is enough to show that the equation

$$(ax^{p^k}-ax+\delta)^s+x = d \tag{3.1}$$

has a solution in $\mathbb{F}_{p^n}$. From (3.1), we have

$$\begin{aligned}(ax^{p^k}-ax+\delta)^s &= d-x,\\ (ax^{p^k}-ax+\delta)^{s(p^k-1)} &= (d-x)^{p^k-1}.\end{aligned} \tag{3.2}$$

*Case I.* $ad^{p^k}-ad+\delta=0$. Then (3.1) has a solution $x=d$.

*Case II.* $ad^{p^k}-ad+\delta\neq 0$. We claim that $d$ is not a solution of (3.1). Suppose that $d$ is a solution of (3.1). Then

$$\begin{aligned}(ad^{p^k}-ad+\delta)^s+d&=d,\\ (ad^{p^k}-ad+\delta)^s&=0,\\ ad^{p^k}-ad+\delta&=0,\end{aligned}$$

which is a contradiction. Thus $d$ is not a solution of (3.1).

If $x_0$ is a solution of (3.1), then $x_0\neq d$ and $ax_0^{p^k}-ax_0+\delta\neq 0$. From (3.2), we have

$$(ax_0^{p^k}-ax_0+\delta)^{s(p^k-1)}=(d-x_0)^{p^k-1}.$$

Since $s(p^k-1)\equiv 0 \pmod{p^n-1}$, by Proposition 2.6,

$$(d-x_0)^{p^k-1}=(ax_0^{p^k}-ax_0+\delta)^{s(p^k-1)}=1,$$

so $(d-x_0)^{p^k}=d-x_0$. By Corollary 2.3, $d-x_0\in\mathbb{F}_{p^k}^*$. Let $d-x_0=\beta$ for some $\beta\in\mathbb{F}_{p^k}^*$. Hence $x_0=d-\beta$.

Substituting $x_0$ into (3.1), we have

$$\begin{aligned}(a(d-\beta)^{p^k}-a(d-\beta)+\delta)^s+(d-\beta)&=d,\\ (a(d^{p^k}-\beta^{p^k})-ad+a\beta+\delta)^s&=d-d+\beta \qquad \text{[ by Theorem 2.1]},\\ (ad^{p^k}-a\beta^{p^k}-ad+a\beta+\delta)^s&=\beta,\\ (ad^{p^k}-a\beta-ad+a\beta+\delta)^s&=\beta \qquad [\because \beta\in\mathbb{F}_{p^k}^*],\\ (ad^{p^k}-ad+\delta)^s&=\beta.\end{aligned}$$

Hence $x_0 = d - \beta = d - (ad^{p^k} - ad + \delta)^s$.

Next, we shall show that $x = d - (ad^{p^k} - ad + \delta)^s$ is a solution of (3.1).

Let $B = ad^{p^k} - ad + \delta$. Then $x = d - B^s$.

Since $(B^s)^{p^k} = (B^s)^{p^k - 1} \cdot (B^s) = 1 \cdot B^s = B^s$, $B^s \in \mathbb{F}_{p^k}^*$, consider

$$\begin{aligned}
[a(d - B^s)^{p^k} - a(d - B^s) + \delta]^s + (d - B^s) &= [a(d^{p^k} - (B^s)^{p^k}) - ad + aB^s + \delta]^s + d - B^s \\
&= [a(d^{p^k} - B^s) - ad + aB^s + \delta]^s + d - B^s \\
&= [ad^{p^k} - aB^s - ad + aB^s + \delta]^s + d - B^s \\
&= [ad^{p^k} - ad + \delta]^s + d - B^s \\
&= B^s + d - B^s = d.
\end{aligned}$$

Thus (3.1) has a solution $x = d - (ad^{p^k} - ad + \delta)^s$. Hence $f(x)$ is a permutation polynomial over $\mathbb{F}_{p^n}$.

Choosing $p = 2$ in Theorem 3.1, we get:

**Corollary 3.2** *Let $n, a, k$ and $s$ be positive integers with $s(2^k - 1) \equiv 0 (\mathrm{mod}\, 2^n - 1)$ and $\delta \in \mathbb{F}_{2^n}$. Then*

$$f(x) = (ax^{2^k} + ax + \delta)^s + x$$

*is a permutation polynomial over $\mathbb{F}_{2^n}$.*

Choosing $a = 1$ in Corollary 3.2, we obtain the following corollary which is Proposition 2.5

**Corollary 3.3** *For any $n$ and $k$ with $\gcd(n,k) > 1$, let $s$ be a positive integer with $s(2^k - 1) \equiv 0 (\mathrm{mod}\, 2^n - 1)$ and $\delta \in \mathbb{F}_{2^n}$. Then*

$$f(x) = (x^{2^k} + x + \delta)^s + x$$

*is a permutation polynomial over $\mathbb{F}_{2^n}$.*

**Theorem 3.4** *Let $n, a, b, c, j, k, s \in \mathbb{N}$ with $g = \gcd(n, j, k)$, $j \geq k$, $s(p^g - 1) \equiv 0 (\mathrm{mod}\, p^n - 1)$, $a, b, c \in \mathbb{F}_{p^n}^*$ and $a + b + c = 0$. Let $\delta \in \mathbb{F}_{p^n}$. Then*

$$f(x) = (ax^{p^j} + bx^{p^k} + cx + \delta)^s + x$$

*is a permutation polynomial over $\mathbb{F}_{p^n}$.*

*Proof.* By Lemma 2.4, the polynomial $f(x) \in \mathbb{F}_{p^n}[x]$ is a permutation polynomial over $\mathbb{F}_{p^n}$ if and only if $f(x) = d$ has a solution in $\mathbb{F}_{p^n}$ for each $d \in \mathbb{F}_{p^n}$, it is enough to show that the equation

$$(ax^{p^j} + bx^{p^k} + cx + \delta)^s + x = d \tag{3.3}$$

has a solution in $\mathbb{F}_{p^n}$. From (3.3), we have

$$\begin{aligned}
(ax^{p^j} + bx^{p^k} + cx + \delta)^s &= d - x, \\
(ax^{p^j} + bx^{p^k} + cx + \delta)^{s(p^g - 1)} &= (d - x)^{p^g - 1}.
\end{aligned} \tag{3.4}$$

*Case I.* $ad^{p^j}+bd^{p^k}+cd+\delta=0$. Then (3.3) has a solution $x=d$.

*Case II.* $ad^{p^j}+bd^{p^k}+cd+\delta\neq 0$. We claim that $d$ is not a solution of (3.3). Suppose that $d$ is a solution of (3.3). Then

$$\begin{aligned}(ad^{p^j}+bd^{p^k}+cd+\delta)^s+d&=d,\\ (ad^{p^j}+bd^{p^k}+cd+\delta)^s&=0,\\ ad^{p^j}+bd^{p^k}+cd+\delta&=0,\end{aligned}$$

which is a contradiction. Thus $d$ is not a solution of (3.3).

If $x_0$ is a solution of (3.3), then $x_0\neq d$ and $ax_0^{p^j}+bx_0^{p^k}+cx_0+\delta\neq 0$. From (3.4), we have

$$(ax_0^{p^j}+bx_0^{p^k}+cx_0+\delta)^{s(p^g-1)}=(d-x_0)^{p^g-1}.$$

Since $s(p^g-1)\equiv 0 \pmod{p^n-1}$, by Proposition 2.6,

$$(d-x_0)^{p^g-1}=(ax_0^{p^j}+bx_0^{p^k}+cx_0+\delta)^{s(p^g-1)}=1,$$

so $(d-x_0)^{p^g}=d-x_0$. By Corollary 2.3, $d-x_0\in\mathbb{F}_{p^g}^*$. Let $d-x_0=\beta$ for some $\beta\in\mathbb{F}_{p^g}^*$. Hence $x_0=d-\beta$.

Substituting $x_0$ into (3.3), we have

$$\begin{aligned}(a(d-\beta)^{p^j}+b(d-\beta)^{p^k}+c(d-\beta)+\delta)^s+(d-\beta)&=d,\\ (a(d^{p^j}-\beta^{p^j})+b(d^{p^k}-\beta^{p^k})+cd-c\beta+\delta)^s&=d-d+\beta \qquad [\text{ by Theorem 2.1}],\\ (ad^{p^j}-a\beta^{p^j}+bd^{p^k}-b\beta^{p^k}+cd-c\beta+\delta)^s&=\beta,\\ (ad^{p^j}-a\beta+bd^{p^k}-b\beta+cd-c\beta+\delta)^s&=\beta \qquad [\because \beta\in\mathbb{F}_{p^k}^*\subseteq\mathbb{F}_{p^j}^*],\\ (ad^{p^j}+bd^{p^k}+cd+\delta-(a+b+c)\beta)^s&=\beta,\\ (ad^{p^j}+bd^{p^k}+cd+\delta)^s&=\beta \qquad [\because a+b+c=0].\end{aligned}$$

Hence $x_0=d-\beta=d-(ad^{p^j}+bd^{p^k}+cd+\delta)^s$.

Next, we shall show that

$$x=d-(ad^{p^j}+bd^{p^k}+cd+\delta)^s$$

is a solution of (3.3).

Let $B=ad^{p^j}+bd^{p^k}+cd+\delta$. Then $x=d-B^s$.

Since $(B^s)^{p^g}=(B^s)^{p^g-1}\cdot(B^s)=1\cdot B^s=B^s$, $B^s\in\mathbb{F}_{p^g}^*$, consider

$$
\begin{aligned}
&[a(d-B^s)^{p^j}+b(d-B^s)^{p^k}+c(d-B^s)+\delta]^s+(d-B^s)\\
&=[a(d^{p^j}-(B^s)^{p^j})+b(d^{p^k}-(B^s)^{p^k})+cd-cB^s+\delta]^s+d-B^s\\
&=[a(d^{p^j}-B^s)+b(d^{p^k}-B^s)+cd-cB^s+\delta]^s+d-B^s\\
&=[ad^{p^j}-aB^s+bd^{p^k}-bB^s+cd-cB^s+\delta]^s+d-B^s\\
&=[ad^{p^j}+bd^{p^k}+cd+\delta-(a+b+c)B^s]^s+d-B^s\\
&=[ad^{p^j}+bd^{p^k}+cd+\delta]^s+d-B^s\\
&=B^s+d-B^s=d.
\end{aligned}
$$

Thus (3.3) has a solutin $x=d-(ad^{p^j}+bd^{p^k}+cd+\delta)^s$. Hence $f(x)$ is a permutation polynomial over $\mathbb{F}_{p^n}$.

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