A Functional Equation Related to Determinant of Some 3×3 Symmetric Matrices and Its Pexiderized Form

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Abstract

In this work, we present the general solution of a functional equation $f(ux+vy, uy+vx, zw) = f(x,y,z) + f(u,v,w) + f(x,y,z)f(u,v,w)$ for all $x,y,u,v,w,z \in \mathbb{R}$, which arises from determinant of some symmetric 3×3 matrices. We also determine the general solution of its Pexiderized version $f(ux+vy, uy+vx, zw) = g(x,y,z) + h(u,v,w) + \ell(x,y,z)n(u,v,w)$ for all $x,y,u,v,w,z \in \mathbb{R}$, without any regularity assumptions on unknown functions $f,g,h,\ell,n : \mathbb{R}^3 \to \mathbb{R}$.

Keywords: Determinant of matrix, functional equation, logarithmic function, multiplicative function

1. Introduction

By recognizing the identity

$$\det\begin{pmatrix} ux+vy & uy+vx \\ uy+vx & ux+vy \end{pmatrix} = \det\begin{pmatrix} x & y \\ y & x \end{pmatrix}\det\begin{pmatrix} u & v \\ v & u \end{pmatrix},$$

we obtain an interesting functional equation

$$f(ux+vy, uy+vx) = f(x,y)f(u,v) \tag{1.1}$$

for all $x,y,u,v \in \mathbb{R}$. Obviously, $f(x,y) = \det\begin{pmatrix} x & y \\ y & x \end{pmatrix} = x^2 - y^2$ is a solution of (1.1).

In 2002, Chung and Sahoo [1], have found that the general solution of (1.1) for all $x,y,u,v \in \mathbb{R}$ and another functional equation

$$f(ux+vy, uy+vx, zw) = f(x,y,x)f(u,v,w) \tag{1.2}$$

for all $x,y,u,v,w,z \in \mathbb{R}$ are given by

$$f(x,y) = M_1(x+y)M_2(x-y) \tag{1.3}$$

and

$$f(x,y,z) = M_1(x+y)M_2(x-y)M_3(z), \tag{1.4}$$

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respectively, where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions. These two equations are connected with determinant of some symmetric matrices.

In 2008, Houston and Sahoo [2], have shown that the general solutions of the following functional equation

$$f(ux - vy, uy - vx) = f(x, y) + f(u, v) + f(x, y) f(u, v) \tag{1.5}$$

for all $x, y, u, v \in \mathbb{R}$ and another functional equation

$$f(ux + vy, uy - vx) = f(x, y) + f(u, v) + f(x, y) f(u, v) \tag{1.6}$$

for all $x, y, u, v, w, z \in \mathbb{R}$ are given by

$$f(x, y) = M(x^2 - y^2) - 1 \quad \ldots\ldots \tag{1.7}$$

and

$$f(x, y) = M(x^2 + y^2) - 1, \tag{1.8}$$

respectively, where $M : \mathbb{R} \to \mathbb{R}$ is a multiplicative function.

Now we consider the following functional equation:

$$f(ux + vy, uy + vx, zw) = f(x, y, z) + f(u, v, w) + f(x, y, z) f(u, v, w) \tag{1.9}$$

for all $x, y, u, v, w, z \in \mathbb{R}$. Obviously,

$$f(x, y, z) = (x^2 - y^2)z - 1 \tag{1.10}$$

is a solution of (1.9). In this work, we determine the general solution of (1.9) and also treat the functional equation

$$f(ux + vy, uy + vx, zw) = g(x, y, z) + h(u, v, w) + \ell(x, y, z) n(u, v, w) \tag{1.11}$$

for all $x, y, u, v, w, z \in \mathbb{R}$ without any regularity assumptions on unknown functions $f, g, h, \ell, n : \mathbb{R}^3 \to \mathbb{R}$.

Notice that if $g = h = \ell = n = f$, then the functional equation (1.11) is reduced to (1.9) and clear that

$$\begin{cases} f(x, y, z) = (x^2 - y^2)z + 2 \\ g(x, y, z) = 2[(x^2 - y^2)z - 1] + 1 \\ h(x, y, z) = 2[(x^2 - y^2)z - 1] + 1 \\ \ell(x, y, z) = (x^2 - y^2)z - 2 \\ n(x, y, z) = (x^2 - y^2)z - 2 \end{cases} \tag{1.12}$$

for all $x, y, z \in \mathbb{R}$ are solutions of the functional equation (1.11).

2. Preliminaries

Let D be an interval in $\mathbb{R}$ such that whenever $x, y \in D$, then $xy \in D$.

- A function $M : D \to \mathbb{R}$ is said to be a *multiplicative function* if and only if $M(xy) = M(x)M(y)$ for all $x, y \in D$.
- A function $L : D \to \mathbb{R}$ is said to be a *logarithmic function* if and only if $L(xy) = L(x) + L(y)$ for all $x, y \in D$.

Remark 1.

1. If M is a constant function, then $M \equiv 0$ or $M \equiv 1$.
2. If $0 \in D$, then $L \equiv 0$.

Lemma 2.1 *Let $D \subseteq \mathbb{R}$ be an interval such that whenever $x, y \in D$, then $xy \in D$. The general solution $f : D^3 \to \mathbb{R}$ of the functional equation*

$$f(x_1x_2, y_1y_2, zw) = f(x_1, y_1, z) + f(x_2, y_2, w) \tag{2.1}$$

for all $x_1, x_2, y_1, y_2, w, z \in D$ is given by

$$f(x, y, z) = L_1(x) + L_2(y) + L_3(z), \tag{2.2}$$

where $L_1, L_2, L_3 : D \to \mathbb{R}$ are logarithmic functions.

Proof. It is easy to check that the solution (2.2) satisfies the functional equation (2.1).

Next, let $f : D^3 \to \mathbb{R}$. Suppose that f is a constant function, say $f \equiv c$, where c is an arbitrary constant. Then from (2.1) we have $c = 0$, so the constant solution of (2.1) is $f(x, y, z) = 0$ for all $x, y, z \in D$, which is included in (2.2).

From now on we suppose that f is a non-constant function. Fix $a \in D$. Then

$$\begin{aligned}
f(x, y, z) &= f(x, y, z) + f(a, a, a) + 2f(a, a, a) - 3f(a, a, a) \\
&= f(xa, ya, za) + f(a, a, a) + f(a, a, a) - 3f(a, a, a) \\
&= f((xa)a, (ya)a, (za)a) + f(a, a, a) - 3f(a, a, a) \\
&= f((xaa)a, (yaa)a, (zaa)a) - 3f(a, a, a) \\
&= f(xa(aa), a(yaa), a(zaa)) - 3f(a, a, a) \\
&= f(xa, a, a) + f(aa, yaa, zaa) - 3f(a, a, a) \\
&= f(xa, a, a) + f(a, ya, a) + f(a, a, za) - 3f(a, a, a) \\
&= L_1(x) + L_2(y) + L_3(z)
\end{aligned}$$

for all $x, y, z \in D$, where

$$L_1(x) := f(xa, a, a) - f(a, a, a),$$

$$L_2(y) := f(a, \mathrm{y}\,a, a) - f(a, a, a),$$

and

$$L_3(z) := f(a, a, \mathrm{z}\,a) - f(a, a, a).$$

Next, we will show that L_1, L_2 and L_3 are logarithmic functions in D. Consider

$$\begin{aligned}
L_1(xy) &= f((xy)a, a, a) - f(a, a, a) \\
&= f((xy)a, a, a) + f(a, a, a) - 2f(a, a, a) \\
&= f((xya)a, aa, aa) - 2f(a, a, a) \\
&= f((xa)(ya), aa, aa) - 2f(a, a, a) \\
&= f(xa, a, a) + f(ya, a, a) - 2f(a, a, a) \\
&= f(xa, a, a) - f(a, a, a) + f(ya, a, a) - f(a, a, a) \\
&= L_1(x) + L_1(y).
\end{aligned}$$

Thus, L_1 is a logarithmic function.Similarly, L_2 and L_3 are logarithmic functions.

Remark 2. Notice that if $D = \mathbb{R}$, then $f(x,y,z) = 0$ is the only solution of the functional equation (2.1).

Lemma 2.2 *Let $D \subseteq \mathbb{R}$ be an interval such that whenever $x, y \in D$, then $xy \in D$. The general solution $f : D^3 \to \mathbb{R}$ of the functional equation*

$$f(x_1x_2, y_1y_2, zw) = f(x_1, y_1, z) f(x_2, y_2, w) \tag{2.3}$$

for all $x_1, x_2, y_1, y_2, w, z \in D$ is given by

$$f(x,y,z) = M_1(x)M_2(y)M_3(z), \tag{2.4}$$

where $M_1, M_2, M_3 : D \to \mathbb{R}$ are multiplicative functions.

Proof. It is easy to check that the solution (2.4) satisfies the functional equation (2.3).

Assume that f is a constant function, say $f \equiv c$, where c is an arbitrary constant. Then from (2.3) we have $c = 0$ or $c = 1$, so the constant solutions of (2.3) are $f(x,y,z) = 0$ or $f(x,y,z) = 1$ for all $x, y, z \in D$, which are included in (2.4).

Next, suppose that f is a non-constant function and fix an element $a \in D$. Let f be such that it satisfies (2.3) with $f(a,a,a) \neq 0$. Then

$$\begin{aligned}
f(x,y,z) &= f(x,y,z)\, f(a,a,a)\, f(a,a,a)^2\, f(a,a,a)^{-3} \\
&= f(xa, ya, za)\, f(a,a,a)\, f(a,a,a)\, f(a,a,a)^{-3} \\
&= f((xa)a, (ya)a, (za)a)\, f(a,a,a)\, f(a,a,a)^{-3} \\
&= f((xaa)a, (yaa)a, (zaa)a)\, f(a,a,a)^{-3} \\
&= f(xa(aa), a(yaa), a(zaa))\, f(a,a,a)^{-3} \\
&= f(xa, a, a)\, f(aa, yaa, zaa)\, f(a,a,a)^{-3} \\
&= f(xa, a, a)\, f(aa, (ya)a, a(za))\, f(a,a,a)^{-3} \\
&= f(xa, a, a)\, f(a, ya, a)\, f(a, a, za)\, f(a,a,a)^{-3} \\
&= M_1(x)M_2(y)M_3(z)
\end{aligned}$$

for all $x, y, z \in D$, where

$$M_1(x) := f(xa, a, a)\, f(a,a,a)^{-1},$$

$$M_2(y) := f(a, \mathrm{y}\, a, a)\, f(a,a,a)^{-1},$$

and

$$M_3(z) := f(a, a, z\, a)\, f(a,a,a)^{-1}.$$

Now we will show that M_1, M_2 and M_3 are multiplicative functions in D. Consider

$$
\begin{aligned}
M_1(xy) &= f((xy)a, a, a)\, f(a, a, a)^{-1} \\
&= f((xy)a, a, a)\, f(a, a, a)\, f(a, a, a)^{-2} \\
&= f((xya)a, aa, aa)\, f(a, a, a)^{-2} \\
&= f((xa)(ya), aa, aa)\, f(a, a, a)^{-2} \\
&= f(xa, a, a)\, f(ya, a, a)\, f(a, a, a)^{-2} \\
&= f(xa, a, a)\, f(a, a, a)^{-1}\, f(ya, a, a)\, f(a, a, a)^{-1} \\
&= M_1(x) M_1(y).
\end{aligned}
$$

Thus, M_1 is a multiplicative function. Similarly, M_2 and M_3 are multiplicative functions.

Lemma 2.3 *The general solution $f, \ell : \mathbb{R}^3 \to \mathbb{R}$ of the functional equation*

$$f(x_1x_2, y_1y_2, zw) = f(x_1, y_1, z) + f(x_2, y_2, w) + \ell(x_1, y_1, z)\ell(x_2, y_2, w) \tag{2.5}$$

for all $x_1, x_2, y_1, y_2, w, z \in \mathbb{R}$ is given by

$$\begin{cases} f(x, y, z) = \delta^2 \left[M_1(x) M_2(y) M_3(z) - 1 \right] \\ \ell(x, y, z) = \delta \left[M_1(x) M_2(y) M_3(z) - 1 \right], \end{cases} \tag{2.6}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions and δ is an arbitrary constant.

Proof. It is easy to check that the solution (2.6) satisfies the functional equation (2.5).

Next, suppose that ℓ is a constant function, say $\ell \equiv -\delta$, where δ is an arbitrary constant. Then from (2.5) we have

$$f(x_1x_2, y_1y_2, zw) = f(x_1, y_1, z) + f(x_2, y_2, w) + \delta^2 \tag{2.7}$$

for all $x_1, x_2, y_1, y_2, w, z \in \mathbb{R}$. Define a function $F : \mathbb{R}^3 \to \mathbb{R}$ by

$$F(x, y, z) = f(x, y, z) + \delta^2 \tag{2.8}$$

for all $x, y, z \in \mathbb{R}$. Using (2.7) and (2.8) we have

$$F(x_1x_2, y_1y_2, zw) = F(x_1, y_1, z) + F(x_2, y_2, w) \tag{2.9}$$

for all $x_1, x_2, y_1, y_2, w, z \in \mathbb{R}$. By Remark 2, we obtain that $F(x, y, z) \equiv 0$ is the only solution of the functional equation (2.9). From (2.8), hence $f(x, y, z) = -\delta^2$ for all $x, y, z \in \mathbb{R}$, which is included in (2.6).

From now on we suppose that ℓ is a non-constant function. Substituting $x_2 = y_2 = w = 0$ in (2.5) we have

$$\ell(x, y, z) = \alpha f(x, y, z) \tag{2.10}$$

for all $x, y, z \in \mathbb{R}$ and for some $\alpha \in \mathbb{R}$. Notice that if $\alpha = 0$, then ℓ is a constant function. Hence $\alpha \neq 0$. Next, using (2.10) back into (2.5) we have

$$f(x_1x_2, y_1y_2, zw) = f(x_1, y_1, z) + f(x_2, y_2, w) + \alpha^2 f(x_1, y_1, z) f(x_2, y_2, w) \tag{2.11}$$

for all $x_1, x_2, y_1, y_2, w, z \in \mathbb{R}$.

Define a function $F_1 : \mathbb{R}^3 \to \mathbb{R}$ by

$$F_1(x, y, z) = \alpha^2 f(x, y, z) + 1 \tag{2.12}$$

for all $x, y, z \in \mathbb{R}$. Using (2.11) and (2.12) we obtain

$$F_1(x_1x_2, y_1y_2, zw) = F_1(x_1, y_1, z)F_1(x_2, y_2, w) \tag{2.13}$$

for all $x_1, x_2, y_1, y_2, w, z \in \mathbb{R}$. By Lemma 2.2, we have

$$F_1(x, y, z) = M_1(x)M_2(y)M_3(z) \tag{2.14}$$

for all $x, y, z \in \mathbb{R}$, where $M_1, M_2, M_3(z) : \mathbb{R} \to \mathbb{R}$ are multiplicative functions.

Finally, using (2.10), (2.12) and (2.14) we obtain

$$\begin{cases} f(x, y, z) = \dfrac{1}{\alpha^2}\left[M_1(x)M_2(y)M_3(z) - 1\right] \\ \ell(x, y, z) = \dfrac{1}{\alpha}\left[M_1(x)M_2(y)M_3(z) - 1\right] \end{cases} \tag{2.15}$$

for all $x, y, z \in \mathbb{R}$, which are the asserted solutions.

Lemma 2.4 *The general solution $f, \ell, n : \mathbb{R}^3 \to \mathbb{R}$ of the functional equation*

$$f(x_1x_2, y_1y_2, zw) = f(x_1, y_1, z) + f(x_2, y_2, w) + \ell(x_1, y_1, z)n(x_2, y_2, w) \tag{2.16}$$

for all $x_1, x_2, y_1, y_2, w, z \in \mathbb{R}$ are given by

$$\begin{cases} f(x, y, z) = 0, \\ \ell(x, y, z)n(u, v, w) = 0, \end{cases} \tag{2.17}$$

or

$$\begin{cases} f(x, y, z) = \dfrac{1}{k_1k_2}\left[M_1(x)M_2(y)M_3(z) - 1\right] \\ \ell(x, y, z) = \dfrac{1}{k_2}\left[M_1(x)M_2(y)M_3(z) - 1\right] \\ n(x, y, z) = \dfrac{1}{k_1}\left[M_1(x)M_2(y)M_3(z) - 1\right], \end{cases} \tag{2.18}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions and k_1, k_2 are arbitrary nonzero constants.

Proof. It is easy to check that the solutions (2.17) and (2.18) satisfy the functional equation (2.16).

Next, if $\ell(x, y, z)n(u, v, w) = 0$, then from (2.16) we have

$$f(x_1x_2, y_1y_2, zw) = f(x_1, y_1, z) + f(x_2, y_2, w) \tag{2.19}$$

for all $x_1, x_2, y_1, y_2, w, z \in \mathbb{R}$. By Remark 2, $f(x, y, z) \equiv 0$ is the only solution of the functional equation (2.16).

If $\ell(x, y, z) = n(u, v, w)$ for all $x, y, z \in \mathbb{R}$, then from (2.16) we have

$$f(x_1x_2, y_1y_2, zw) = f(x_1, y_1, z) + f(x_2, y_2, w) + n(x_1, y_1, z)n(x_2, y_2, w) \tag{2.20}$$

for all $x_1, x_2, y_1, y_2, w, z \in \mathbb{R}$. By Lemma 2.3, we obtain

$$\begin{cases} f(x,y,z) = \delta^2 \left[M_1(x)M_2(y)M_3(z) - 1 \right] \\ \ell(x,y,z) = \delta \left[M_1(x)M_2(y)M_3(z) - 1 \right], \end{cases} \tag{2.21}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions and δ is an arbitrary constant.

If $\ell(x,y,z) \neq n(x,y,z)$ and ℓ, n are nonzero constant functions, then setting $x_2 = y_2 = w = 0$ in (2.16) we have

$$\ell(x,y,z) = k_1 f(x,y,z) \tag{2.22}$$

for all $x, y, z \in \mathbb{R}$ and for some $k_1 \in \mathbb{R}$. Clear that if $k_1 = 0$, then ℓ is a constant function. Hence $k_1 \neq 0$. Similarly, substituting $x_1 = y_1 = z = 0$ in (2.16), we get

$$n(x,y,z) = k_2 f(x,y,z) \tag{2.23}$$

for all $x, y, z \in \mathbb{R}$ and for some $k_2 \neq 0$. Using (2.22) and (2.23) back into (2.16), we have

$$f(x_1x_2, y_1y_2, zw) = f(x_1, y_1, z) + f(x_2, y_2, w) + k_1k_2 f(x_1, y_1, z) f(x_2, y_2, w) \tag{2.24}$$

for all $x_1, x_2, y_1, y_2, w, z \in \mathbb{R}$.

Define a function $F : \mathbb{R}^3 \to \mathbb{R}$ by

$$F(x,y,z) = k_1k_2 f(x,y,z) + 1 \tag{2.25}$$

for all $x, y, z \in \mathbb{R}$. Using (2.24) and (2.25), we have

$$F(x_1x_2, y_1y_2, zw) = F(x_1, y_1, z)F(x_2, y_2, w) \tag{2.26}$$

for all $x_1, x_2, y_1, y_2, w, z \in \mathbb{R}$. By Lemma 2.2, we obtain

$$F(x,y,z) = M_1(x)M_2(y)M_3(z) \tag{2.27}$$

for all $x, y, z \in \mathbb{R}$, where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions.

Finally, using (2.22), (2.23), (2.25), and (2.27), we obtain

$$\begin{cases} f(x,y,z) = \dfrac{1}{k_1k_2} \left[M_1(x)M_2(y)M_3(z) - 1 \right] \\ \ell(x,y,z) = \dfrac{1}{k_2} \left[M_1(x)M_2(y)M_3(z) - 1 \right] = k_1 f(x,y,z) \\ n(x,y,z) = \dfrac{1}{k_1} \left[M_1(x)M_2(y)M_3(z) - 1 \right] = k_2 f(x,y,z) \end{cases} \tag{2.28}$$

for all $x, y, z \in \mathbb{R}$, which are the asserted solutions.

Next, we will determine the general solution of the functional equation (1.9) and also determine the general solution of its Pexiderized form.

3. Main Results

Theorem 3.1 *The general solution of the functional equation* (1.9) *is given by*

$$f(x,y,z)=M_1(x+y)M_2(x-y)M_3(z)-1 \tag{3.1}$$

for all $x,y,z\in\mathbb{R}$, *where* $M_1,M_2,M_3:\mathbb{R}\to\mathbb{R}$ *are multiplicative functions.*

Proof. It is easy to check that the solutions (3.1) satisfies the functional equation (1.9).

Next, suppose that f is a constant function, say $f\equiv c$, where c is an arbitrary constant. Then from (1.9) we have $c=0$ or $c=-1$, so the constant solutions of (1.9) are $f(x,y,z)=0$ or $f(x,y,z)=-1$ for all $x,y,z\in\mathbb{R}$, which are included in (3.1).

From now on we suppose that f is a non-constant function. Define a function $F:\mathbb{R}^3\to\mathbb{R}$ by

$$F(x,y,z)=f\left(\frac{x+y}{2},\frac{x-y}{2},z\right)+1 \tag{3.2}$$

for all $x,y,z\in\mathbb{R}$. Then from (3.2) we have

$$f(x,y,z)=F(x+y,x-y,z)-1 \tag{3.3}$$

for all $x,y,z\in\mathbb{R}$. Next, using (3.3) back into (1.9) we obtain

$$F((x+y)(u+v),(x-y)(u-v),zw)=F(x+y,x-y,z)F(u+v,u-v,w) \tag{3.4}$$

for all $x,y,u,v,w,z\in\mathbb{R}$.

Substituting $x_1=x+y$, $y_1=x-y$, $x_2=u+v$ and $y_2=u-v$ in (3.4), we have

$$F(x_1x_2,y_1y_2,zw)=F(x_1,y_1,z)F(x_2,y_2,w) \tag{3.5}$$

for all $x_1,x_2,y_1,y_2,w,z\in\mathbb{R}$. Next, setting $w=z=1$ in (3.5), we get

$$F(x_1x_2,y_1y_2,1)=F(x_1,y_1,1)F(x_2,y_2,1) \tag{3.6}$$

for all $x_1,x_2,y_1,y_2\in\mathbb{R}$. Letting $y_1=y_2=1$ in (3.6), we have

$$F(x_1x_2,1,1)=F(x_1,1,1)F(x_2,1,1) \tag{3.7}$$

for all $x_1,x_2\in\mathbb{R}$.

Define a function $M_1:\mathbb{R}\to\mathbb{R}$ by

$$M_1(x)=F(x,1,1) \tag{3.8}$$

for all $x\in\mathbb{R}$. Then using (3.7) and (3.8), we obtain

$$M_1(x_1x_2)=M_1(x_1)M_1(x_2) \tag{3.9}$$

for all $x_1,x_2\in\mathbb{R}$, so $M_1:\mathbb{R}\to\mathbb{R}$ is a multiplicative.

Similarly, setting $x_1=x_2=1$ in (3.6), we get

$$F(1,y_1y_2,1)=F(1,y_1,1)F(1,y_2,1) \tag{3.10}$$

for all $y_1,y_2\in\mathbb{R}$. Define a function $M_2:\mathbb{R}\to\mathbb{R}$ by

$$M_2(y)=F(1,y,1) \tag{3.11}$$

for all $y\in\mathbb{R}$. Then using (3.10) and (3.11), we obtain that $M_2:\mathbb{R}\to\mathbb{R}$ is a multiplicative.

Next, letting $x_2=y_1=1$ in (3.6), we have

$$F(x_1,y_2,1)=F(x_1,1,1)F(1,y_2,1) \tag{3.12}$$

for all $x_1, y_2 \in \mathbb{R}$. Using (3.8), (3.11) and (3.12), we obtain

$$F(x_1, y_2, 1) = M_1(x_1)M_2(y_2) \tag{3.13}$$

for all $x_1, y_2 \in \mathbb{R}$.

Setting $x_1 = x_2 = y_1 = y_2 = 1$ in (3.5), we have

$$F(1,1,zw) = F(1,1,z)F(1,1,w) \tag{3.14}$$

for all $w, z \in \mathbb{R}$. Define a function $M_3 : \mathbb{R} \to \mathbb{R}$ by

$$M_3(z) = F(1,1,z) \tag{3.15}$$

for all $z \in \mathbb{R}$. Then we obtain that $M_3 : \mathbb{R} \to \mathbb{R}$ is a multiplicative.

Letting $x_2 = y_2 = z = 1$ in (3.5), we have

$$F(x_1, y_1, w) = F(x_1, y_1, 1)F(1,1,w) \tag{3.16}$$

for all $x_1, y_1, w \in \mathbb{R}$ and using (3.13), (3.15) and (3.16), we obtain

$$F(x_1, y_1, w) = M_1(x_1)M_2(y_1)M_3(w) \tag{3.17}$$

for all $x_1, y_1, w \in \mathbb{R}$.

Finally, using (3.3) and (3.17), we have

$$f(x,y,z) = M_1(x+y)M_2(x-y)M_3(z) - 1 \tag{3.18}$$

for all $x, y, z \in \mathbb{R}$, which is a general solution of (1.9).

Now we will determine the general solution of the functional equation (1.11).

Theorem 3.2 *The general solution of the functional equation (1.11) are given by*

$$\begin{cases} f(x,y,z) = \alpha_2\beta_2 + \alpha_1 + \beta_1 \\ g(x,y,z) = \alpha_1 \\ h(x,y,z) = \alpha_2\beta_2 + \beta_1 - \alpha_2 n(x,y,z) \\ \ell(x,y,z) = \alpha_2 \\ n(x,y,z) \text{ is arbitrary}, \end{cases} \tag{3.19}$$

or

$$\begin{cases} f(x,y,z) = \alpha_2\beta_2 + \alpha_1 + \beta_1 \\ g(x,y,z) = \alpha_2\beta_2 + \alpha_1 - \beta_2 \ell(x,y,z) \\ h(x,y,z) = \beta_1 \\ \ell(x,y,z) \text{ is arbitrary} \\ n(x,y,z) = \beta_2, \end{cases} \tag{3.20}$$

or

$$\begin{cases} f(x,y,z) = \dfrac{1}{k_1k_2}\Big[M_1(x+y)M_2(x-y)M_3(z)-1\Big]+\alpha_1+\beta_1+\alpha_2\beta_2 \\ g(x,y,z) = \dfrac{1}{k_1k_2}\Big[M_1(x+y)M_2(x-y)M_3(z)-1\Big] \\ \qquad\qquad -\dfrac{\beta_2}{k_2}\Big[M_1(x+y)M_2(x-y)M_3(z)-1\Big]+\alpha_1 \\ h(x,y,z) = \dfrac{1}{k_1k_2}\Big[M_1(x+y)M_2(x-y)M_3(z)-1\Big] \\ \qquad\qquad -\dfrac{\alpha_2}{k_1}\Big[M_1(x+y)M_2(x-y)M_3(z)-1\Big]+\beta_1 \\ \ell(x,y,z) = \dfrac{1}{k_2}\Big[M_1(x+y)M_2(x-y)M_3(z)-1\Big]+\alpha_2 \\ n(x,y,z) = \dfrac{1}{k_1}\Big[M_1(x+y)M_2(x-y)M_3(z)-1\Big]+\beta_2, \end{cases} \tag{3.21}$$

where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ *are multiplicative functions,* $\alpha_1, \alpha_2, \beta_1$ *and* β_2 *are arbitrary constants, and* k_1, k_2 *are nonzero arbitrary constants.*

Proof. It is easy to check that the solutions $(3.19)-(3.21)$ satisfy the functional equation (1.11). Next, we will show that the general solutions of (1.11) have above forms.

First, we define functions $F, G, H, L, N : \mathbb{R}^3 \to \mathbb{R}$ by

$$\begin{cases} F(x,y,z) = f\left(\dfrac{x+y}{2}, \dfrac{x-y}{2}, z\right) \\ G(x,y,z) = g\left(\dfrac{x+y}{2}, \dfrac{x-y}{2}, z\right) \\ H(x,y,z) = h\left(\dfrac{x+y}{2}, \dfrac{x-y}{2}, z\right) \\ L(x,y,z) = \ell\left(\dfrac{x+y}{2}, \dfrac{x-y}{2}, z\right) \\ N(x,y,z) = n\left(\dfrac{x+y}{2}, \dfrac{x-y}{2}, z\right) \end{cases} \tag{3.22}$$

for all $x, y, z \in \mathbb{R}$. Then we have

$$\begin{cases} f(x,y,z) = F(x+y, x-y, z) \\ g(x,y,z) = G(x+y, x-y, z) \\ h(x,y,z) = H(x+y, x-y, z) \\ \ell(x,y,z) = L(x+y, x-y, z) \\ n(x,y,z) = N(x+y, x-y, z) \end{cases} \tag{3.23}$$

for all $x, y, z \in \mathbb{R}$. Next, using (3.23) back into (1.11), we obtain

$$F((x+y)(u+v),(x-y)(u-v),zw)=G(x+y,x-y,z)+H(u+v,u-v,w)+L(x+y,x-y,z)N(u+v,u-v,w) \quad (3.24)$$

for all $x,y,u,v,w,z\in\mathbb{R}$.

Substituting $x_1=x+y$, $y_1=x-y$, $x_2=u+v$ and $y_2=u-v$ in (3.24), we have

$$F(x_1x_2,y_1y_2,zw)=G(x_1,y_1,z)+H(x_2,y_2,w)+L(x_1,y_1,z)N(x_2,y_2,w) \quad (3.25)$$

for all $x_1,x_2,y_1,y_2,w,z\in\mathbb{R}$. Letting $x_1=y_1=z=1$ in (3.25), we get

$$F(x_2,y_2,w)=G(1,1,1)+H(x_2,y_2,w)+L(1,1,1)N(x_2,y_2,w) \quad (3.26)$$

for all $x_2,y_2,w\in\mathbb{R}$. Setting $G(1,1,1)=\alpha_1$ and $L(1,1,1)=\alpha_2$ in (3.26), we obtain

$$F(x_2,y_2,w)=H(x_2,y_2,w)+\alpha_2N(x_2,y_2,w)+\alpha_1 \quad (3.27)$$

for all $x_2,y_2,w\in\mathbb{R}$.

Similarly, setting $x_2=y_2=w=1$ in (3.25), we get

$$F(x_1,y_1,z)=G(x_1,y_1,z)+H(1,1,1)+L(x_1,y_1,z)N(1,1,1) \quad (3.28)$$

for all $x_1,y_1,z\in\mathbb{R}$. Letting $H(1,1,1)=\beta_1$ and $N(1,1,1)=\beta_2$ in (3.28), we obtain

$$F(x_1,y_1,z)=G(x_1,y_1,z)+\beta_2L(x_1,y_1,z)+\beta_1 \quad (3.29)$$

for all $x_1,y_1,z\in\mathbb{R}$. Next, using (3.27) and (3.29) back into (3.25), we have

$$\begin{aligned}F(x_1x_2,y_1y_2,zw)&=F(x_1,y_1,z)+F(x_2,y_2,w)+L(x_1,y_1,z)N(x_2,y_2,w)\\&\quad-\beta_2L(x_1,y_1,z)-\alpha_2N(x_2,y_2,w)-\alpha_1-\beta_1\end{aligned} \quad (3.30)$$

for all $x_1,x_2,y_1,y_2,w,z\in\mathbb{R}$. Define functions $F_1,L_1,N_1:\mathbb{R}^3\to\mathbb{R}$ by

$$\begin{cases}F_1(x,y,z)=F(x,y,z)-\alpha_1-\beta_1-\alpha_2\beta_2\\ L_1(x,y,z)=L(x,y,z)-\alpha_2\\ N_1(x,y,z)=N(x,y,z)-\beta_2\end{cases} \quad (3.31)$$

for all $x,y,z\in\mathbb{R}$. Next, using (3.31) back into (3.30), we get

$$F_1(x_1x_2,y_1y_2,zw)=F_1(x_1,y_1,z)+F_1(x_2,y_2,w)+L_1(x_1,y_1,z)N_1(x_2,y_2,w) \quad (3.32)$$

for all $x_1,x_2,y_1,y_2,w,z\in\mathbb{R}$. By Lemma 2.4, we obtain

$$\begin{cases}F_1(x,y,z)=0,\\ L_1(x,y,z)N_1(u,v,w)=0,\end{cases} \quad (3.33)$$

or

$$\begin{cases} F_1(x,y,z) = \dfrac{1}{k_1k_2}\Big[M_1(x)M_2(y)M_3(z)-1\Big] \\ L_1(x,y,z) = \dfrac{1}{k_2}\Big[M_1(x)M_2(y)M_3(z)-1\Big] \\ N_1(x,y,z) = \dfrac{1}{k_1}\Big[M_1(x)M_2(y)M_3(z)-1\Big], \end{cases} \qquad (3.34)$$

Where $M_1, M_2, M_3 : \mathbb{R} \to \mathbb{R}$ are multiplicative functions and k_1, k_2 are arbitrary nonzero constants.

Next, using (3.22), (3.27), (3.29), (3.31), and (3.33), we have

$$\begin{cases} f(x,y,z) = \alpha_2\beta_2 + \alpha_1 + \beta_1 \\ g(x,y,z) = \alpha_1 \\ h(x,y,z) = \alpha_2\beta_2 + \beta_1 - \alpha_2 n(x,y,z) \\ \ell(x,y,z) = \alpha_2 \\ n(x,y,z) \ \text{is arbitrary}, \end{cases} \qquad (3.35)$$

or

$$\begin{cases} f(x,y,z) = \alpha_2\beta_2 + \alpha_1 + \beta_1 \\ g(x,y,z) = \alpha_2\beta_2 + \alpha_1 - \beta_2\ell(x,y,z) \\ h(x,y,z) = \beta_1 \\ \ell(x,y,z) \ \text{is arbitrary} \\ n(x,y,z) = \beta_2, \end{cases} \qquad (3.36)$$

or

$$\begin{cases} f(x,y,z) = \alpha_2\beta_2 + \alpha_1 + \beta_1 \\ g(x,y,z) = \alpha_1 \\ h(x,y,z) = \beta_1 \\ \ell(x,y,z) = \alpha_2 \\ n(x,y,z) = \beta_2, \end{cases} \qquad (3.37)$$

for all $x, y, z \in \mathbb{R}$. Notice that the equation (3.37) is included in (3.35) and (3.36).

Finally, using (3.22), (3.27), (3.29), (3.31) and (3.34), we obtain

$$\begin{cases} f(x,y,z) = \dfrac{1}{k_1k_2}\Big[M_1(x+y)M_2(x-y)M_3(z)-1\Big]+\alpha_1+\beta_1+\alpha_2\beta_2 \\ g(x,y,z) = \dfrac{1}{k_1k_2}\Big[M_1(x+y)M_2(x-y)M_3(z)-1\Big] \\ \qquad\qquad -\dfrac{\beta_2}{k_2}\Big[M_1(x+y)M_2(x-y)M_3(z)-1\Big]+\alpha_1 \\ h(x,y,z) = \dfrac{1}{k_1k_2}\Big[M_1(x+y)M_2(x-y)M_3(z)-1\Big] \\ \qquad\qquad -\dfrac{\alpha_2}{k_1}\Big[M_1(x+y)M_2(x-y)M_3(z)-1\Big]+\beta_1 \\ \ell(x,y,z) = \dfrac{1}{k_2}\Big[M_1(x+y)M_2(x-y)M_3(z)-1\Big]+\alpha_2 \\ n(x,y,z) = \dfrac{1}{k_1}\Big[M_1(x+y)M_2(x-y)M_3(z)-1\Big]+\beta_2, \end{cases} \tag{3.38}$$

for all $x, y, z \in \mathbb{R}$. Notice that the equations (3.35) and (3.36) are included in (3.38), which are the asserted solutions.

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