The Cut Locus of Riemannian Manifolds: a Surface of Revolution

Minoru Tanaka*

Department of Mathematics, Tokai University, Kanagawa, Japan

Abstract

This article reviews the structure theorems of the cut locus for very familiar surfaces of revolution. Some properties of the cut locus of a point of a Riemannian manifold are also discussed.

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1. Definition of a cut point and the cut locus

Let $\gamma : [0, a] \to M$ denote a minimal geodesic segment emanating from a point p on a complete connected Riemannian manifold M. The end point γ (a) is called a cut point of p along the minimal geodesic segment γ if any geodesic extension $\tilde{\gamma} : [0, b] \to M$, where b > a, of γ is not minimal anymore.

Definition 1.1 The cut locus C_p of a point p is the set of all cut points of p along minimal geodesic segments emanating from p.

It is very difficult to determine the structure of the cut locus of a point in a Riemannian manifold. The cut locus for a smooth surface is not a graph anymore, although it was proved by Myers in [1] and [2] that the cut locus of a point in a compact real analytic surface is a finite graph. In fact, Gluck and Singer [3] proved that there exists a 2-sphere of revolution admitting a cut locus with infinitely many branches. Their result implies that one cannot improve the following Theorem 1.3 without any additional assumption.

Theorem 1.2 [3] There exists a 2-sphere of revolution with positive Gaussian curvature such that the cut locus of a point admits an infinitely many branches.

Hebda proved in [4] that the distance function ρ to the cut locus of a point (in a complete 2-dimensional Riemannian manifold) is absolutely continuous where ρ is finite. Hence, for any pair of cut points of a point p can be connected by a rectifiable curve in C_p if the pair is in the same connected component.

*Corresponding author: E-mail: tanaka@tokai-u.jp

Theorem 1.3 [4] The cut locus of a point of a complete 2-dimensional (smooth) Riemannian manifold is a local tree, and the distance function to the cut locus is absolutely continuous where ρ is finite. In particular, the cut locus has a natural interior metric.

Remark 1.4 A topological space T is called a tree if for any two points p, q in T can be joined by a unique continuous curve. A topological space X is called a local tree if for any point $x \in X$ and any neighborhood V of x there exists a neighborhood $U \subset V$ of x which is a tree.

Remark 1.5 Hartman [5] studied detail differentiable structures of the cut locus of a simply closed smooth curve in a complete Riemannian manifold homeomorphic to Euclidean plane. His work was generalized to a simply closed curve in a 2-dimensional Riemannian manifold [6-8].

The cut locus of a point in a smooth Reimannian manifold cannot be a fractal set, i.e., the Hausdorff dimension is an integer see [9].

2. A surface of revolution homeomorphic toEuclidean plane

Definition 2.1 A complete Riemannian manifold (M, g) homeomorphic to Euclidean plane is called a surface of revolution if the manifold admits a point *p*, with $C_p = \phi$, such that the Riemannian metric g is expressed as

$$g = dr^2 + m(r)^2 d\theta^2$$

by making use of geodesic polar coordinates (r, θ) around *p*. The point *p* is called the vertex of the manifold.

It is known that a complete Riemannian manifold M homeomorphic to Euclidean plane is a surface of revolution with vertex p if and only if for each t > 0 the Gaussian curvature G is constant on $\mathbf{S}_\mathbf{p}$ (t) := {q ∈ M |d(p, q) = t}.

Definition 2.2 A complete Riemannian manifold homeomorphic to Euclidean plane is called a von Mangoldt surface of revolution if the manifold admits a point *p* such that for any pair of points x, y with d(p, x) ≥ d(p, y), G(y) ≥G(x) holds. Here G denotes the Gaussian curvature of *M*.

Remark 2.3 A von Mangoldt surface of revolution is actually a surface of revolution, and a surface of revolution with vertex p is a von Mangoldt surface of revolution if and only if the Gaussian curvature is decreasing along each meridian, which means a geodesic emanating from the vertex *p*.

Typical examples of a von Mangoldt surface of revolution are paraboloids and 2-sheeted hyperboloids. Elerath [10] determined the structure of the cut locus for special classical surfaces of revolution.

Theorem 2.4 Let M(f) denote a surface of revolution defined by $z = f(\sqrt{x^2 + y^2})$, where f : R → (0,∞) denotes a smooth even function. If the Gaussian curvature is decreasing along each meridian, then for each point q of M (f), the cut locus C_q of *q* is empty or a subset of the meridian opposite to q.

Remark 2.5 The Sturm comparison theorem is a key tool in the proof of Theorem 2.4. Typical examples of a von Mangoldt surface are paraboloids $(z = a(x^2 + y^2))$ and a connected component of 2-sheeted hyperboloids $(z = a\sqrt{x^2 + y^2 + 1})$

Theorem 2.6 Let $(M, dr^2 + m(r)^2 d\theta^2)$ be a von Mangoldt surface of revolution with vertex p. Then the cut locus of a point q in M is either empty or a subset of the meridian opposite to *q*. More precisely, either $C_q = \phi$ or there exists a positive number t0 satisfying $C_p = \{(r,\theta) \mid \mathrm{r} \geq \mathrm{t0}, \theta = \pi + \theta(\mathrm{q})\}$

Definition 2.7 A point q of a surface of revolution $(M, dr^2 + d\theta^2)$ homeomorphic to Euclidean plane is called a *pole* if ex $\mathbf{p_q} : \mathbf{T_q}\ M \to M$ is injective (or equivalently $C_q = \phi$.

It is trivial that the vertex *p* of a surface of revolution is a pole. It is known that the set of poles of a surface of revolution forms a closed ball centered at the vertex and furthermore, we obtain

Theorem 2.8 Let $(M, dr^2 + d\theta^2)$ denote a surface of revolution with vertex p. Then the set of poles on M equals a closed ball centered at p and M admits a non-trivial pole if and only if

lim $\mathbf{inf}_{\mathbf{r}\to\infty}\mathbf{m(r)}$ is non-zero and $\int_1^\infty m(r)^{-2}$ dr is *finite.*

Example 2 .9 For a paraboloid of revolution, $\lim_{r\to\infty} m(r) = \int_1^\infty m(r)^{-2} dr = \infty$ Thus the vertex is a unique pole.

Example 2.10 For a 2-sheeted hyperboloid of revolution, $\lim_{r\to\infty} m(r) = \infty$ and $\int_1^\infty m(r)^{-2} dr$ is finite.

Hence any point sufficiently close to the vertex is a pole.

3. A surface of revolution homeomorphic to a 2-sphere

Definition 3.1 A Riemannian manifold (M, g) homeomorphic to a 2-sphere is called a 2-sphere of revolution if M admits a point p with a single cut point q such that the Riemannian metric g is expressed as $g = dr^2 + m(r)^2 d\theta^2$ on M \ {p, q} by using geodesic polar coordinates (r, θ) around p. The point p and it's unique cut point is called a pair of poles of the 2-sphere.

Theorem 3.2 Let $(M, dr^2 + m(r)^2 d\theta^2)$ denote a 2-sphere of revolution with a pair of poles p; q satisfying the following two properties.

(3.1) M is symmetric with respect to the reflection fixing the equator r = 1/2•d (p, q).

(3.2) The Gaussian curvature of M is decreasing along a meridian from the point p to the point on the equator.

Then the cut locus of a point $x \in M \setminus \{p, q\}$, *with* $\theta(x) = 0$ is either a subarc of the open half opposite meridian $\theta^{-1}(\pi)$ to x or a single point on the open half opposite meridian. Moreover, if the cut locus of x is a single point, then the Gaussian curvature of *M* is constant.

Remark 3.3 A meridian of M means a periodic geodesic passing through *p* and *q*. For example, $\theta^{-1}(0) \cup \theta^{-1}(\pi) \cup \{p, q\}$ is a meridian.

A typical 2-sphere of revolution satisfying (1.1) and (1.2) is an ellipsoid de-fined by

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1, (0 < a < b)$$

Theorem 3.4 Let $(M, dr^2 + m(r)^2 d\theta^2)$ denote a 2-sphere of revolution with a pair of poles *p, q* satisfying (1.1) such that

(3.3) the Gaussian curvature of *M* is increasing along a meridian from the point *p* to the point on the equator.

Then, the cut locus of a point $x \in M \setminus \{p, q\}$ is either a single point or a subarc of the antipodal parallel $r = d(p,q) - r(x)$ to *x*. Moreover, if the cut locus of *x* is a single point, then the Gaussian curvature of *M* is constant.

A typical example of a 2-sphere of revolution satisfying (3.1) and (3.3) is an ellipsoid defined by

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1, (0 < a < b)$$

The structure of the cut locus of a general ellipsoid, i.e., a surface defined by $x^2 / a^2 + y^2 / b^2 + z^2 / c^2 = 1$, where 0 < a < b < c, has been determined by Itoh-Kiyohara [11]. The cut locus of a generic point of the ellipsoid is an arc. In [12] and [13], this result was generalized to a Liouville surface and Liouville manifolds.

Open Problem Let $(M, dr^2 + m(r)^2 d\theta^2)$ denote a 2-sphere of revolution with a pair of poles p, q. Suppose that a point $x \in M \setminus \{p, q\}$ *is a pole, i.e.,* C_x is a single point. Then, is any point $y \in M$ with $d(p, y) \leq d(p, x)$ a pole?

Remark 3.5 This would be true if M satisfies (3.1), and m is strictly increasing on (0, m(x)). The first claim of Theorem 2.8 is a non-compact version of this problem. From Theorem 3.4, it follows that the cut locus of a point on the equator $r = 1/2 \cdot d(p, q)$ is a subset of the equator. This theorem was generalized to a wider class of 2-spheres of revolution by Bonnard-Caillau-Sinclair –Tanaka [1].

Theorem 3.6 Let $(M, dr^2 + m(r)^2 d\theta^2)$ denote a 2-sphere of revolution satisfying (3.1). Suppose that the cut locus of a point on the equator $r = 1/2 \cdot d(p, q)$ is a subset of the equator. Then, the cut locus of a point $x \in M \setminus \{p, q\}$ is either a subarc of the antipodal parallel $r = d(p,q) - r(x)$ to x or a single point on the antipodal parallel.

Remark 3.7 Theorems 3.4 and 3.6 were generalized for a class of cylinders of revolution by *P*. Chitsakul [14], and [15] respectively.

Example 3.8 There exists a family $\{\mathbf{M}_\lambda\}_\lambda$ of 2-spheres of revolution satisfying both properties in Theorem 3 .6 , but the Gaussian curvature is not mono-tonic along the meridian. By using geodesic polar coordinates *(r, θ)* around a point p of the unit sphere $\mathbf{S}^2(1)$ we give a family of Riemannian metrics

$g_\lambda := dr^2 + m\lambda(r)^2 d\theta^2$, where $(\lambda \geq 0)$ is a parameter, on the unit sphere. Here $m_k := \sqrt{\lambda+1}\sin r / \sqrt{1+\lambda\cos^2 r}$. Then $M_k := (S^2(1), dr^r + m_\lambda^2(r)d\theta^2)$

satisfies both properties in Theorem 3 .6 , but the Gaussian curvature is not monotonic along a meridian if $\lambda > 2$.

4. A surface of revolution homeomorphic to a 2-torus

Let M be a standard torus in 3-dimensional Euclidean space defined by

$$(\sqrt{x^2 + y^2 - R})^2 + z^2 = r^2 (R > r > 0) \text{.}$$

The surface M is given by rotating the *(x, z)*-plane curve $\{(x,0,z) / (x - R^2 + z^2 = r^2)\}$ around the *z*-axis.

This surface has the following two properties. (4.4) It is symmetric with respect to the *(x, y)*-plane, i.e., it has a reflective symmetry with respect to the plane.

(4.5) The Gaussian curvature is increasing from the point $(\mathbf{R} - \mathbf{r}, 0, 0)$ to the point $(R + r, 0, 0)$ along the meridian defined by $y = 0$.

The structure of the cut locus for this torus is topologically complicated (see Figure 1). If we state it roughly,

Theorem 4.1 A cut point of a point $p = (x_0, 0, z_0)$, $x_0 > 0$, on the torus is a point on the meridian $\{(x, (x,0, z) \in M \mid x < 0\}$ opposite to p, a point on the antipodal parallel $\{(x, y, z) \in M \mid z = -z_0\}$, or a point on a (piecewise C1) Jordan curve which intersects the meridian opposite to p at a single point and is freely homotopic to each parallel.

Remark 4.2 The structure of the cut locus is determined for a class of 2-torus of revolution which contains all standard tori in Euclidean space [16]. More precisely, let $(S^1 \times S^1, dt^2 + m(t)^2 d\theta^2)$ denote a torus with warped product Riemannian metric $dt^2 + m(t)^2 d\theta^2$, where dt^2 and $d\theta^2$ denote the Riemannian metric of a circle with length 2 a and 2 b respectively and m denotes a positive smooth warping function on *R* satisfying the following two properties:

(4.4) $\mathbf{m}(-\mathbf{t}) = \mathbf{m}(\mathbf{t}) = \mathbf{m}(\mathbf{t} + 2\mathbf{a})$ for any real number t.

(4.5) The Gaussian curvature $-\frac{m''}{m}(t)$ is increasing on $[0, a]$.

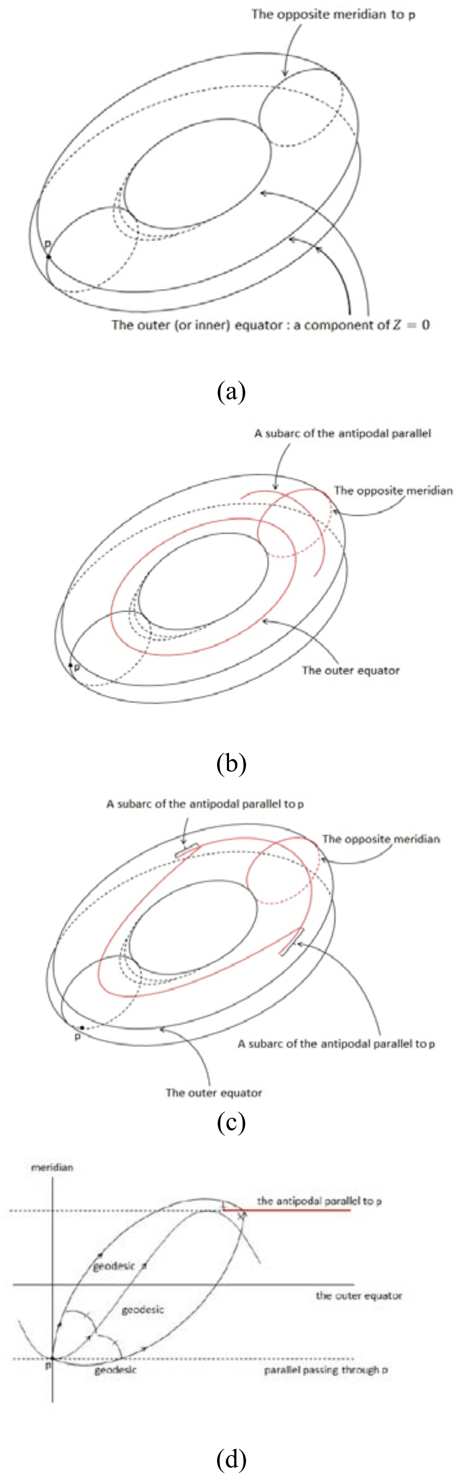

Figure 1. The structure of the cut locus

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