Current Applied Science and Technology Vol. 18 No. 2 May-August 2018
DOI: 10.14456/cast.2018.2

Lower Bound for *p*-Adic Exponential Polynomials Evaluated at Some Integer Points

Vichian Laohakosol and Angkana Sripayap*

Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok, Thailand

Received: 29 March 2018, Revised: 1 July 2018, Accepted: 4 July 2018

Abstract

In 1981, a *p*-adic interpolation method based on divided differences was derived and was applied to derive, among other things, results on the number of zeros and the bound of certain *p*- adic exponential polynomials. Here, lower bounds for a *p*-adic exponential polynomial evaluated over some rational integers are derived using a method of van der Poorten.

Keywords: *p*-adic exponential polynomial, Turán's theorem

1. Introduction

In the paper , a *p*- adic interpolation method using divided differences [1] was developed, and was then applied to obtain results on the number of zeros and the bounds of the coefficients of *p*- adic exponential polynomials, as well as to obtain a *p*- adic analogue of Turán's first main theorem on sums of powers. In this paper, we continue their investigation on *p*-adic exponential polynomials in the spirit of Turán's main theorems. We shall establish lower bounds for a *p*- adic exponential polynomial, evaluated over some rational integers, in terms of its derivatives at the origin, its functional values at some other rational integral points, and its coefficients. Since the technique employed in Laohakosol and Pitman [1] to derive *p*- adic Turán's first main theorem does not generalize to exponential polynomials, we thus have to use different methods. The approach we adopt here is van der Poorten's method of evaluating determinants as appeared in van der Poorten [2,3].

Notation. The following will be standard throughout the entire paper:
1. p a fixed rational prime,
2. $|\cdot|_p$ the *p*-adic valuation so normalized that $|p|_p = 1/p$,
3. $\mathbb{Q}_p$ the field of *p*-adic numbers, that is the completion of $\mathbb{Q}$ (the field of rational numbers) with respect to the *p*-adic valuation $|\cdot|_p$,
4. $\mathbb{C}_p$ the completion of the algebraic closure of $\mathbb{Q}_p$

*Corresponding author: Tel.:662-5625555 ext 647086
E-mail: fscianr@ku.ac.th

We shall always be working in $\mathbb{C}_p$. In a few places, we find it useful to derive certain estimates via Schnirelman integrals. We use the symbol $\int_{a,R}$ to denote the Schnirelman integral over the circle in $\mathbb{C}_p$ with center *a* and radius *R*. For the definition and basis properties of Schnirelman integral, we refer to the appendix of Adams [4].

2. Preliminaries

The following lemma, due essentially to van der Poorten [2], is simple but quite fundamental for our investigation.

Lemma 1. *Let E be a function of the form*

$$E(z) = \sum_{k=1}^{M} b_k g_k(z)$$

where $b_1, b_2, \ldots, b_M$ *are constants in* $\mathbb{C}_p$ *and* $g_1, g_2, \ldots, g_M$ *are functions analytic on some domain G of* $\mathbb{C}_p$. *Further let* $z_1, z_2, \ldots, z_M$ *be points of G; let* $s_1, s_2, \ldots, s_M$ *be non-negative integers; and let* $H(Y_1, Y_2, \ldots, Y_L)$ *be a form linear in* $Y_1, Y_2, \ldots, Y_L$ $(1 \le L \le M)$. *Finally denote by* Δ_{ij} $(1 \le i, j \le M)$ *the cofactor of* $g_j^{(s_i)}(z_i)$ *in the determinant*

$$\Delta = \left| g_j^{(s_i)}(z_i) \right|_{1 \le j, i \le M}.$$

Then there is an integer v such that $1 \le v \le M$ *and*

$$\left| E^{(s_v)}(z_v) \right|_p \ge \frac{\left| H\left(b_{j(1)}, b_{j(2)}, \ldots, b_{j(L)}\right) \right|_p}{\max_{1 \le i \le M} \left| H\left(\Delta_{ij(1)}, \Delta_{ij(2)}, \ldots, \Delta_{ij(L)}\right) / \Delta \right|_p}$$

where $1 \le j(1) \le \ldots \le j(L) \le M$.

Proof By differentiating at $z_1, z_2, \ldots, z_M$, we obtain a system of *M* linear equations in $b_1, b_2, \ldots, b_M$ of the forms

$$\sum_{k=1}^{M} b_k g_k^{(s_i)}(z_i) = E^{(s_i)}(z_i) \quad (i = 1, 2, \ldots, M),$$

which we may solve by Cramer's rule to obtain

$$b_k \Delta = \sum_{i=1}^{M} \Delta_{ik} E^{(s_i)}(z_i) \quad (k = 1, 2, \ldots, M).$$

Thus

$$H(b_{j(1)}, b_{j(2)}, \ldots, b_{j(L)})\Delta = \sum_{i=1}^{M} H(\Delta_{ij(1)}, \Delta_{ij(2)}, \ldots, \Delta_{ij(L)}) E^{(s_i)}(z_i),$$

and so

$$\left| H(b_{j(1)}, b_{j(2)}, \ldots, b_{j(L)})\Delta \right|_p \le \max_{1 \le v \le M} \left| H(\Delta_{ij(1)}, \Delta_{ij(2)}, \ldots, \Delta_{ij(L)}) \right|_p \max_{1 \le v \le M} \left| E^{(s_v)}(z_v) \right|_p .$$

Note that the result remains meaningful though trivial even if the denominator on the right-hand side of the result should vanish, provided we then interpret the lower bound to be zero. Since we shall only be working with p-adic exponential polynomials, we shall first standardize our symbols. Let $\rho(1), \rho(2), \ldots, \rho(m)$ be non-negative integers with sum

$$\sum_{k=1}^{m} \rho(k) = M$$

and let $a_{ks}, 1 \le k \le m, 1 \le s \le \rho(k)$ be M elements of $\mathbb{C}_p$ not all 0. Let α_k $(k = 1, \dots, m)$ be distinct elements of $\mathbb{C}_p$ satisfying

$$|\alpha_k - 1|_p < p^{-1/(p-1)} \qquad (k = 1, \dots, m).$$

Therefore, each α_k^z is an analytic function of z in the domain $\{z \in \mathbb{C}_p; |z|_p \le 1\}$. We shall consider exponential polynomials of the form

$$E(z) = \sum_{k=1}^{m} \sum_{s=1}^{\rho(k)} a_{ks} z^{s-1} \alpha_k^z \qquad (|z|_p \le 1)$$

so that we shall be applying Lemma 1 to the M functions

$$z^{s-1} \alpha_k^z \qquad (k = 1, \dots, m;\ s = 1, \dots, \rho(k)).$$

To avoid any ambiguity, with regard to the terminology in Lemma 1 for $g_j(z)$ and the M functions $z^{s-1}\alpha_k^z$ $(k = 1, \dots, m;\ s = 1, \dots, \rho(k))$, we define

$$\begin{aligned}
&g_1(z) = \alpha_1^z, g_2(z) = z\alpha_1^z, \dots, g_{\rho(1)}(z) = z^{\rho(1)-1}\alpha_1^z,\\
&g_{\rho(1)+1}(z) = \alpha_2^z, g_{\rho(1)+2}(z) = z\alpha_2^z, \dots, g_{\rho(1)+\rho(2)}(z) = z^{\rho(2)-1}\alpha_2^z,\\
&\qquad\vdots\\
&g_{\rho(1)+\dots+\rho(m-1)+1}(z) = \alpha_m^z, g_{\rho(1)+\dots+\rho(m-1)+2}(z) = z\alpha_m^z,\\
&\qquad\vdots\\
&g_{\rho(1)+\dots+\rho(m)}(z) = z^{\rho(m)-1}\alpha_m^z.
\end{aligned}$$

We also find it more convenient to specify the notation in Lemma 1, keeping in mind that the method is also applicable to much more general cases. Thus from Lemma 1, we set as follows:

1. n a fixed non-negative rational integer,
2. $z_j = n + j \quad (j = 1, 2, \dots, M)$,
3. $s_j = 0 \quad (j = 1, 2, \dots, M)$,
4. r = a fixed positive rational integer

Let

$$H(Y_1, Y_2, \dots, Y_M) := \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} \frac{(r-1)!}{(r-t)!} (\log \alpha_h)^{r-t} y_{ht},$$

where

$$\begin{aligned}
&Y_1 = y_{11}, Y_2 = y_{12}, \dots, Y_{\rho(1)} = y_{1,\rho(1)},\\
&Y_{\rho(1)+1} = y_{21}, Y_{\rho(1)+2} = y_{22}, \dots, Y_{\rho(1)+\rho(2)} = y_{2,\rho(2)},\\
&\qquad\vdots\\
&Y_{\rho(1)+\dots+\rho(m-1)+1} = y_{m,1}, Y_{\rho(1)+\dots+\rho(m-1)+2} = y_{m,2}, \dots, Y_{\rho(1)+\dots+\rho(m)} = y_{m,\rho(m)}.
\end{aligned}$$

We consider such linear form H because of the following identity

$$H(a_{11}, \dots, a_{m,\rho(m)}) = \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} \frac{(r-1)!}{(r-t)!} (\log \alpha_h)^{r-t} a_{ht} = E^{(r-1)}(0).$$

3. Main Results

By Lemma 1, we must then consider the determinant

$$\Delta = \left| (n+j)^{t-1} \alpha_h^{n+j} \right|_{ht,j} \quad (1 \le h \le m, 1 \le t \le \rho(h), 1 \le j \le M)$$

(the numbering of row and column indexes is clear from the display of functions $g_j(z)$'s, and the linear form *H* in its cofactors).

The crux of van der Poorten's method, and indeed the most difficult part, is to handle these determinants effectively. Consequently, we first compute these determinants. Since the arguments used in van der Poorten's sledge-hammer approach to evaluate these determinants are algebraic in nature, then they are also applicable in the *p*-adic case. We shall be brief here and refer to a more detailed discussion in van der Poorten [3]. We first let

$$\alpha_{ht} \ (1 \le h \le m; 1 \le t \le \rho(h))$$

be formal quantities. The next lemma, again due to van der Poorten [2], enables us to obtain nice identities later. Its proof is elementary, and can be found in van der Poorten [2].

Lemma 2. *Denote by P the product*

$$P = \prod_{k=1}^{m} \prod_{s=1}^{\rho(k)} \prod_{l=1}^{s-1} (\alpha_{ks} - \alpha_{kl}),$$

and let R_1 *and* R_2 *be functions in the* α_{ht} $(i \le h \le m, 1 \le t \le \rho(h))$, *which are divisible by P. Then*

$$\lim_{\substack{\alpha_{ks} \to \alpha_k \\ \text{all } ks}} \left(\prod_{k=1}^{m} \prod_{s=1}^{\rho(k)} \frac{\left((\alpha_{ks} \frac{\partial}{\partial \alpha_{ks}})^{s-1} \right) R_1}{\left((\alpha_{ks} \frac{\partial}{\partial \alpha_{ks}})^{s-1} \right) R_2} \right) = \lim_{\substack{\alpha_{ks} \to \alpha_k \\ \text{all } ks}} \left(\frac{R_1 / P}{R_2 / P} \right).$$

(*The limiting processes in Lemma 2 and in what follows can be thought of as those with respect to the p-adic topology.*)

Next, let *D*be the Vandermonde determinant

$$D = \left| \alpha_{ht}^{n+i} \right|_{ht,i} = \prod_{k=1}^{m} \sum_{s=1}^{\rho(k)} \Big(\alpha_{ks}^{n+1} \prod_{jr<ks} (\alpha_{ks} - \alpha_{jr}) \Big),$$

where $jr < ks$ means either $j < k$ or if $j = k$, then $r < s$ .A more explicit form is

$$D = \prod_{k=1}^{m} \prod_{s=1}^{\rho(k)} \Big(\alpha_{ks}^{n+1} \prod_{l=1}^{s-1} (\alpha_{ks} - \alpha_{kl}) \prod_{j=1}^{k-1} \prod_{r=1}^{\rho(k)} (\alpha_{ks} - \alpha_{jr}) \Big),$$

By direct differentiation (see also [6] or [7]), we get

$$\begin{aligned} \Delta &:= \left| (n+i)^{t-1} \alpha_h^{n+i} \right|_{ht,i} \quad (1 \le h \le m, 1 \le t \le \rho(h), 1 \le i \le M) \\ &= \lim_{\substack{\alpha_{ks} \to \alpha_k \\ 1 \le k \le m; 1 \le s \le \rho(k)}} \left(\prod_{k=1}^{m} \prod_{s=1}^{\rho(k)} \Big(\alpha_{ks} \frac{\partial}{\partial \alpha_{ks}} \Big)^{s-1} \right) D \\ &= \prod_{k=1}^{m} \prod_{j=1}^{\rho(k)} \Big(\alpha_k^{n+s} (s-1)! \prod_{j=1}^{k-1} (\alpha_k - \alpha_j)^{\rho(j)} \Big). \qquad (3.1) \end{aligned}$$

Denote by $D_{i,ht}$ and, respectively, $\Delta_{i,ht}$ the cofactor of α_{ht}^{n+i} and, respectively, of $(n+i)^{t-1} \alpha_h^{n+i}$ in *D* and respectively in Δ. In a similar manner as we derived Δ from *D*, we also have

$$\Delta_{i,ht} = \lim_{\substack{\alpha_{ks}\to\alpha_k \\ \text{all } ks}} \left(\prod_{k=1}^{m} \prod_{\substack{s=1 \\ ks\neq ht}}^{\rho(k)} \left(\alpha_{ks} \frac{\partial}{\partial \alpha_{ks}}\right)^{s-1} \right) D_{i,ht}.$$

By expanding the determinant D through its cofactors, we see that

$$\frac{D}{\alpha_{ht}^{n+1}} \delta_{ks,ht} = \sum_{i=1}^{M} \alpha_{ks}^{i-1} D_{i,ht} \qquad (1 \le k \le m, 1 \le s \le \rho(k)),$$

(where $\delta_{ks,ht}$ is the usual Kronecker δ), which asserts that $D_{i,ht}$ is exactly the coefficient of z^{i-1} in the polynomial

$$\frac{D}{\alpha_{ht}^{n+1}} \prod_{k=1}^{m} \prod_{\substack{s=1 \\ ks\neq ht}}^{\rho(k)} \left(\frac{z-\alpha_{ks}}{\alpha_{ht}-\alpha_{ks}}\right).$$

Thus for any u we have by the above expressions

$$\alpha_h^u \sum_{t=1}^{\rho(h)} u^{t-1} \Delta_{i,ht} = \lim_{\substack{\alpha_{ks}\to\alpha_k \\ \text{all } ks}} \left(\prod_{k=1}^{m} \prod_{s=1}^{\rho(k)} \left(\alpha_{ks} \frac{\partial}{\partial \alpha_{ks}}\right)^{s-1} \right) \sum_{t=1}^{\rho(h)} \alpha_{ht}^u D_{i,ht},$$

and by Lemma 2, we see that it is the coefficient of z^{i-1} in the polynomial

$$\lim_{\substack{\alpha_{ks}\to\alpha_k \\ \text{all } ks}} \Delta \sum_{t=1}^{\rho(h)} \frac{1}{\alpha_{ht}^{n+1-u}} \prod_{k=1}^{m} \prod_{\substack{s=1 \\ ks\neq ht}}^{\rho(k)} \left(\frac{z-\alpha_{ks}}{\alpha_{ht}-\alpha_{ks}}\right). \qquad (3.2)$$

Similarly for r a positive integer, we get

$$\sum_{t=1}^{\rho(h)} \frac{(r-1)!}{(r-t)!} (\log \alpha_h)^{r-t} \Delta_{i,ht} = \lim_{\substack{\alpha_{ks}\to\alpha_k \\ \text{all } ks}} \left(\prod_{k=1}^{m} \prod_{s=1}^{\rho(k)} \left(\alpha_{ks} \frac{\partial}{\partial \alpha_{ks}}\right)^{s-1} \right) \sum_{t=1}^{\rho(h)} (\log \alpha_{ht})^{r-1} D_{i,ht},$$

which by Lemma 2 becomes the coefficient of z^{i-1} in the polynomial

$$\lim_{\substack{\alpha_{ks}\to\alpha_k \\ \text{all } ks}} \Delta \sum_{t=1}^{\rho(h)} \frac{(\log \alpha_{ht})^{r-1}}{\alpha_{ht}^{n+1}} \prod_{k=1}^{m} \prod_{\substack{s=1 \\ ks\neq ht}}^{\rho(k)} \left(\frac{z-\alpha_{ks}}{\alpha_{ht}-\alpha_{ks}}\right). \qquad (3.3)$$

From Lemma 1 and the shape of the linear form H, we must get a p-adic upper bound for the expression

$$q_i := \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} \frac{(r-1)!}{(r-t)!} (\log \alpha_h)^{r-t} \frac{\Delta_{i,ht}}{\Delta}, \quad (1 \le i \le M).$$

We have observed that q_i is exactly the coefficient of z^{i-1} in the polynomial

$$Q(z) = \lim_{\substack{\alpha_{ks}\to\alpha_k \\ \text{all } ks}} \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} \frac{(\log \alpha_{ht})^{r-1}}{\alpha_{ht}^{n+1}} \prod_{k=1}^{m} \prod_{\substack{s=1 \\ ks\neq ht}}^{\rho(k)} \left(\frac{z-\alpha_{ks}}{\alpha_{ht}-\alpha_{ks}}\right).$$

At this point, we could of course derive a p-adic upper bound for the coefficients q_i directly from the shape of Q. However, a bound obtained in this way is weak and untidy. To obtain a clean and strong bound, we resort to the use of interpolation. Neglecting the limit for a moment, $Q(z)$ is exactly the polynomial of degree $M-1$ such that

$$Q(\alpha_{ht}) = \frac{(\log \alpha_{ht})^{r-1}}{\alpha_{ht}^{n+1}}, 1 \le h \le m; 1 \le t \le \rho(h). \qquad (3.4)$$

Taking the quantities α_{ht} as formally distinct, these conditions determine Q. We suppose that the distinct quantities α_{ht} lie (*p*-adically) arbitrarily close to the α_h $(1 \le h \le m)$. Still neglecting the limit in $Q(z)$, we write it as an interpolation series

$$Q(z) = b_{11} + b_{12}(z-\alpha_{11}) + b_{13}(z-\alpha_{11})(z-\alpha_{12}) + \ldots + b_{m\rho(m)}(z-\alpha_{11})\ldots(z-\alpha_{m,\rho(m)-1}) \quad (3.5)$$

By the conditions (3.4) defining *Q*, and by similar derivation as in Laohakosol and Pitman [1], we can represent the interpolation coefficients b_{ht} as Schnirelman integrals

$$b_{ht} = \int_{1,R} \frac{(\log z)^{r-1}(z-1)dz}{z^{n+1}(z-\alpha_{11})\cdots(z-\alpha_{ht})} \quad (1 \le h \le m, 1 \le t \le \rho(h)),$$

where the integrals are taken along the circle with center 1 and radius *R* with $p^{-1/(p-1)} < R < 1$. Here, the lower bound for *R* guarantees that all α_{ij} 's lie inside the circle, and the upper bound ensures that the point *O* lies outside. By uniform convergence (see [4]) of the series defining the integrals, the integrals all remain well-defined when the relevant limit is taken and indeed we can drop any implicit assumption that the α_h be distinct. We require by Lemma 1 to find an upper bound for the quantities

$$\left|q_i\right|_p \quad (1 \le i \le M).$$

Using the fact that $|\alpha_j|_p = 1$, from the equation (3.5), we have for $1 \le i \le M$

$$|q_i|_p = \max_{ht} |b_{ht}|_p = \max_{ht} \left| \int_{1,R} \frac{(\log z)^{r-1}(z-1)dz}{z^{n+1}(z-\alpha_{11})\cdots(z-\alpha_{ht})} \right|_p$$

$$\le \frac{R^{r-1}R}{1 \cdot R^{\rho(1)+\ldots+\rho(m)}} = R^{r-M},$$

because $|z-\alpha|_p = |(z-1)+(1-\alpha)|_p = R$. To get optimal bounds, we consider two separate cases.*Case* $r > M$. Since the inequality

$$\left|q_i\right|_p \le R^{r-M} \quad (i = 1,\ldots,M)$$

holds for all R satisfying $p^{-1/(p-1)} < R < 1$, taking here the limit as $R \to p^{-1/(p-1)}$, we have in this case

$$|q_i|_p \le R^{(M-r)/(p-1)} \quad (i = 1,\ldots,M).$$

Case $r \le M$. In this case we take the limit as $R \to 1$ to get

$$\left|q_i\right|_p \le 1 \quad (i = 1,\ldots,M).$$

Hence, by Lemma 1, there exists a rational integer *v* such that $1 \le v \le M$ and

$$\left|E(n+v)\right|_p \ge \begin{cases} p^{(r-M)/(p-1)} \left|E^{(r-1)}(0)\right|_p & \text{if } r > M, \\ \left|E^{(r-1)}(0)\right|_p & \text{if } r \le M. \end{cases}$$

We observe, moreover that by taking $r = 1$ (and so $r \le M$), and replacing n by $n-u$ in (3.3), we obtain the expression (3.2). Consequently, the corresponding linear forms are

$$H' = \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} \alpha_h^u u^{t-1} a_{ht} = E(u),$$

$$q_i' = \sum_{h=1}^{m} \sum_{t=1}^{\rho(h)} \alpha_h^u u^{t-1} \frac{\Delta_{i,ht}}{\Delta} \quad (i = 1,\ldots,M),$$

and the interpolation coefficients b_{ht}' take the form

$$b'_{ht} = \int_{1,R} \frac{(z-1)dz}{z^{n+1}(z-\alpha_{11})\cdots(z-\alpha_{ht})} \quad (1 \le h \le m, 1 \le t \le \rho(h))$$

with the same *R* as above. By the same arguments as for the case of q_i, we see that

$$\left|q'_i\right|_p \le 1 \quad (i = 1, \ldots, M).$$

Hence, we have proved

Theorem 1. *Let* $\alpha_1, \ldots, \alpha_m$ *be distinct elements of* $\mathbb{C}_p$ *satisfying*

$$|\alpha_j - 1|_p < p^{-1/(p-1)} \quad (j = 1, \ldots, m).$$

Denote by E the exponential polynomial

$$E(z) = \sum_{k=1}^{m} \sum_{s=1}^{\rho(k)} a_{ks} z^{s-1} \alpha_k^z \quad (|z|_p \le 1),$$

where $\rho(1), \ldots, \rho(m)$ *are non-negative integers with sum M, and* a_{ks} *are constants in* $\mathbb{C}_p$ *not all zero. Then for integers* $n \ge 0$ *and* $r \ge 1$, *we have*

$$\max_{n+1 \le v \le n+M} \left|E(v)\right|_p \ge \begin{cases} p^{(r-M)/(p-1)} \left|E^{(r-1)}(0)\right|_p & \text{if } r > M, \\ \left|E^{(r-1)}(0)\right|_p & \text{if } r \le M, \end{cases}$$

and for any rational integer $u \le n$, *we have*

$$\max_{n+1 \le v \le n+M} |E(v)|_p \ge |E(u)|_p .$$

An immediate consequence is the following

Corollary 1. *Let the notation be as in Theorem 1. Then*

$$\max_{n+1 \le v \le n+M} \left| \sum_{k=1}^{m} a_k \alpha_k^v \right|_p \ge \left| a_1 + \ldots + a_m \right|_p .$$

Before proceeding to our next result, we make the following remarks.

Remarks.

1. Corollary 1 is another version of *p*-adic Turán's theorem. In this *p*-adic case, since $|\alpha_j|_p = 1$ for all *j*, the distinction between the two main theorems of Turánin the classical case (see [5]) disappears.
2. The condition that $|\alpha_j - 1|_p < p^{-1/(p-1)} = 1$ for all *j* is necessary to make α_j^z a well-defined analytic function of *z* with $|z|_p \le 1$. It seems restrictive if one only aims at proving Corollary 1; indeed in Theorem 3 of Laohakosol and Pitman [1], there is no such restriction. However, by a result of Cassels [6] there are infinitely many *p*-adic fields $\mathbb{Q}_p$ for which all α_j's $(j = 1, \ldots, m)$ can be embedded as *p*-adic units, i.e. $|\alpha_j|_p = 1$ for all *j*. Moreover being *p*-adic units, by a well-known result (see [7]), there exists a positive integer *d* such that

$$|\alpha_j^d - 1|_p < p^{-1/(p-1)} \quad (j = 1, \ldots, m).$$

Consequently, by considering α_j^d instead of α_j, we have an abundant supply of *p*-adic fields to work with.
3. It should be observed that the estimates in the *p*-adic case here are much easier to compute than the corresponding ones in the classical case; as a by-product we do not have to bound the values of *r* from above to get an optimal bound as in the classical case (see [2]). As to the values of *n* in Theorem 1, the condition $n \ge 0$ is mainly imposed so that the determinants involved are non-zero.

4. The bounds obtained in Theorem 1 and Corollary 1 cannot in general be improved as the following examples show.

(i) Let $p \neq 2$, and let

$$E_1(z) = 1-(1+p)^z.$$

Then

$$\max_{0+1\le v\le 0+2} \left| E_1(v) \right|_p = \max\left(| p |_p, | p^2 + 2p |_p\right) = | p |_p$$
$$= \left| \log(1+p) \right|_p = \left| E_1'(0) \right|_p = \left| E_1^{(2-1)}(0) \right|_p .$$

(ii) Let $p \neq 2$, and let

$$E_2(z) = (1+p)^2 + (1-p)^z.$$

Then

$$\max_{0+1\le v\le 0+2} \left| E_2(v) \right|_p = 1 = | 1+1 |_p .$$

Next, we prove

Theorem 2. *Let $\alpha_1,\ldots,\alpha_m$ be distinct elements in $\mathbb{C}_p$ satisfying*

$$\left|\alpha_j - 1\right|_p < p^{-1/(p-1)} \quad (j = 1,\ldots,m).$$

Let E be the exponential polynomial

$$E(z) = \sum_{k=1}^{m}\sum_{s=1}^{\rho(k)} a_{ks} z^{s-1}\alpha_k^z \quad (|z|_p \le 1),$$

where $\rho(1),\ldots,\rho(m)$ are non-negative integers with sum M, and a_{ks} $(1\le k\le m, 1\le s\le \rho(k))$ are constants in $\mathbb{C}_p$ not all zero. Further, let

$$\delta = \min_{\substack{1\le j,h\le m\\ j\ne h}} \left|\alpha_h - \alpha_j\right|_p ,$$

$$\rho = \max_{1\le k\le m} \rho(k).$$

Then if n is a nonnegative integer, we have, for each ht $(1\le h\le m, 1\le t\le \rho(h))$,

$$\max_{n+1\le v\le n+M} \left|E(v)\right|_p \ge \left|a_{ht}\right|_p \delta^{3m^2\rho^2/2-5m\rho/2+2} p^{-m\rho(\rho-1)/2(p-1)}.$$

Remark The value of δ, though non-zero, is usually very small because

$$0 < \delta \le | \alpha_h - \alpha_j |_p = | (\alpha_h - 1) - (\alpha_j - 1) |_p < p^{-1/(p-1)} , h \ne j.$$

Proof Here, we consider the linear form H in Lemma 1 to be a linear polynomial in one variable

$$H(y_{ht}) = y_{ht}$$

for a certain index ht. By Lemma 1, we need an upper bound for $\left| \Delta_{i,ht} / \Delta \right|_p$ because

$$\max_{1\le v\le M} \left| E(n+v) \right|_p \ge \left| a_{ht} \right|_p \cdot \max_{1\le v\le M} \left|\Delta / \Delta_{i,ht}\right|_p .$$

As pointed out in [2], the shape of this last polynomial is quite complicated, and we have no hope of using interpolation to derive a neat bound. Therefore, we instead compute directly a *p*-adic lower bound for Δ and an upper bound for $\Delta_{i,ht}$. To obtain a *p*-adic upper bound for $\Delta_{i,ht}$, we note the following:

- first, a p-adic upper bound for the coefficients of z^j $(1 \le j \le M)$ in $\prod_{k=1}^{m}\prod_{\substack{s=1\\ ks\neq ht}}^{\rho(k)}\left(\frac{z-\alpha_{ks}}{\alpha_{ht}-\alpha_{ks}}\right)$ is $\delta^{-m\rho+1}$;
- second, by the shape of D, and $\left|\alpha_j\right|_p = 1$ for all j, we get $\left|D/\alpha_{ht}^{n+1}\right|_p \le 1$.

Thus, a p-adic upper bound for the coefficients (of z^j) in $D\alpha_{ht}^{-n-1}\prod_{k=1}^{m}\prod_{\substack{s=1\\ ks\neq ht}}^{\rho(k)}\left(\frac{z-\alpha_{ks}}{\alpha_{ht}-\alpha_{ks}}\right)$ is $\delta^{-m\rho+1}$.

Now each partial derivative $\alpha\frac{\partial}{\partial\alpha}$ increases the p-adic upper bound for the coefficients at most by a factor of $1/\delta$. Therefore, a p-adic upper bound for the coefficients is

$$\prod_{k=1}^{m}\prod_{\substack{s=1\\ ks\neq ht}}^{\rho(k)}\delta^{-m\rho+1-s+1} \le \delta^{(-m\rho+2)(m\rho-1)}\prod_{k=1}^{m}\delta^{-\rho(k)(\rho(k)+1)/2}$$
$$\le \delta^{(-m\rho+2)(m\rho-1)}\delta^{-\rho(\rho+1)m/2} = \delta^{-\rho^2(m^2+m/2)+5m\rho/2-2}.$$

Now from the shape of Δ in equation (3.1), we have

$$\left|\Delta\right|_p \ge \prod_{k=1}^{m}\prod_{s=1}^{\rho(k)}\left(p^{(-s+1)/(p-1)}\delta_k^{\rho(1)+\cdots+\rho(k-1)}\right)$$
$$\ge \prod_{k=1}^{m}\left(p^{-(\rho(k)-1)/2(p-1)}\delta^{\rho(1)+\cdots+\rho(k-1)}\right)^{\rho(k)} \ge \delta^{m(m-1)\rho^2/2}p^{-m\rho(\rho-1)/2(p-1)}.$$

Combining these estimates, the required result follows.

An immediate consequence of Theorem 2 is the following:

Corollary 2. *Let the notation be as in Theorem 2. Then for some positive integer* h $(1 \le h \le m)$, *we have*

$$\max_{n+1\le v\le n+m}\left|\sum_{k=1}^{m}a_k\alpha_k^v\right|_p \ge \left|a_h\right|_p\delta^{(3m^2-5m+4)/2}.$$

References

[1] Laohakosol, V. and Pitman, J., 1981. Applications of p-adic interpolation to exponentialpolynomials and sum of powers. *Journal of Australian Mathematical Society*, (Ser. A) 30,419-437.

[2] van der Poorten, A.J., 1970. Generalisations of Turán's main theorems on lower bounds for sums of powers. *Bulletin of the Australian Mathematical Society*, 2, 15-37.

[3] van der Poorten, A.J., 1976. Some determinants that should be better known. *Journal of Australian Mathematical Society*, (Ser. A) 21, 278-288.

[4] Adams, W.W., 1966. Transcendental numbers in the p-adic domain, *American Journal of Mathematics*, 88, 279-308.

[5] Turán's, P., 1953. *Eine neue Methode in der Analysis und deren Anwendungen* Akademiai Kiado: Budapest.

[6] Cassels, J.W.S., 1976. An embedding theorem for fields. *Bulletin of the Australian Mathematical Society*, 14, 147-168.

[7] Amice, Y., 1959. Analyse p-adiques. *Seminaire Delange-Pisot*, 1, 63.