Approximation of Amplitude Equalizer Based on q-Bessel Polynomials

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Abstract

In this paper, a method to equalize the amplitude distortion of linear slope signal in waveform transmission is presented. In order to design the linear gain slope amplitude equalizer with maximally flat group delay, a non-minimum phase network technique is utilized. Thus, the transmission zeros possess quadrant symmetry and the denominator of the transfer function is approximated by using q-Bessel polynomials. As it is known, the q-Bessel polynomial has more parameter to adjust than does the simple Bessel polynomials. In order to investigate the performance of the design equalizer. Simulation of the proposed gain slope response is carried out. Moreover, the stability of the approximated transfer function is investigated by using a Mihailov's criterion. It is shown that the simulation results agree with the theoretical ones.

Keywords: q-Bessel Polynomials, non-minimum phase, amplitude equalizer, Mihailov's criterion.

1. Introduction

As it is known, in signal transmission when the signal is transmitted to receiver via transmission media. The quality of receiving signal depends on many factors such as amplitude and phase distortions. These factors will cause degradation of signal and signal quality.

In this paper, we investigated the signal caused by linear amplitude distortion only. In the previous literature the design of amplitude equalizers has been published in many aspect [1-3]. Unfortunately, the group delay error is still incomplete. It maintains the effect of group delay.

In this paper, a non-minimum phase network is utilized. In order to design the approximated magnitude response of the proposed gain equalizer without the effect of delay error. We introduced the q-Bessel polynomials as the denominator of the transfer function. As it is known, Bessel polynomials gives a strictly Hurwitz polynomial, and the q-Bessel polynomials parameter can vary between $0 \le q \le 1$. With appropriate choice of the q-parameter, we can achieve the specified requirement of the gain equalizer.

The paper is organized as follows. In section 2, gain equalizer based on q-Bessel polynomials is presented. In section 3, we describe the non-minimum phase transfer function. Section 4 shows the simulation results of the proposed gain equalizer. Finally, the conclusion is given in section 5.

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2. The q-bessel Polynomials

The Bessel polynomials were introduced by Krall and Frink [4]. The classical generalized Bessel polynomials solution is given as

$$y_n(s,\alpha,\beta) = \sum_{k=0}^{n} \left(\frac{(-n)_k (\alpha+n-1)_k}{k!} \right) \left(-\frac{S}{\beta} \right)^k . \tag{1}$$

$$\text{Where}\ldots \frac{(-n)_k}{k!} = \frac{(-1)^k n!}{(n-k)!k!}$$

Herein, we apply q-special functions known as q-Bessel polynomials. The application of q-Bessel polynomials was studied by Ernst [5] who proposed a new method and its application to q-Bessel polynomials.

In this paper, a classical q-Bessel polynomial which are mathematical operators,have been applied to gain slope equalizer.

The generalized q-Bessel polynomials is given by

$$y_n(s,\alpha,\beta,q) = \sum_{k=0}^{n} \left(\frac{(-n)_k (\alpha+n-1)_k}{k!\beta^k} \right)_q \cdot \left(\frac{-s}{\beta} \right)^k . \tag{2}$$

From (2) by setting $\alpha = \beta = 2$, then (2) becomes classical q-Bessel polynomials.

$$y_n(s,2,2,q) = \sum_{k=0}^{n} \left(\frac{(-n)_k (n+1)_k}{k!} \right)_q \cdot \left(-\frac{s}{2} \right)^k . \tag{3}$$

$$\text{where } (n+1)_k = \frac{(n+k)!}{n!}$$

For simplicity, (3) can be rewritten in term of fractional notation as

$$y_n(s,2,2,q) = \sum_{k=0}^{n} \frac{(n+k)_q!}{(n-k)_q!} \cdot \frac{1}{(k)_q!} \cdot \frac{1}{2^k} s^k . \tag{4}$$

For each non-negative integer n the q-integer $[n]_q$ can be defined as

$$[n]_q = \frac{(1-q^n)}{1-q} \ ; \quad 0 \langle q \langle 1 \tag{5}$$

and the q factorial $[n]_q!$ as

$$[n]_q! = \begin{cases} [n]_q [n-1]_q \ldots [1]_q & if \quad n \ge 1 \\ 1 & if \quad n = 0 \end{cases} \tag{6}$$

For the integer n and k with $0 \le k \le n$, the q-binomial coefficients are then defined as follows

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} \tag{7}$$

Let the q-Pochammers symbol $\{a\}_{n,q}$ be defined by

$$\{a\}_{n,q} = \prod_{m=0}^{n-1}(a+m)_q \tag{8}$$

The inverse q-Bessel polynomials used in this technical paper can be written as

$$\theta_{n,q}(s) = s^n y_{n,q}\left(\frac{1}{s}\right) \tag{9}$$

The inverse q-Bessel polynomials

$$\theta_{n,q}(s) = \sum_{k=0}^{n} \frac{(n+k)_q!}{(n-k)_q!}\frac{1}{(k)_q!}\frac{1}{2^k}s^{n=k} \tag{10}$$

Herein,let $\theta_{2,q}(s)$ be a second order of q-Bessel polynomials. By substituting $n=2$ onto (10) yields

$$\theta_{2,q}(s) = s^2 + \frac{(1+q)(1+q+q^2)}{2}s + \frac{(1+q)(1+q^2)(1+q+q^2)}{4} \tag{11}$$

Let the denominator of the proposed transfer function $D_n(s)$ be equal to $\theta_{2,q}(s)$

3. Non-Minimum Phase

The proposed example of linear chrominance gain slope equalizer for amplitude enhance of +1dB at a normalized angled angular velocity $\omega = 1$ is given as

$$F_S(\omega) = 1 + 0.258\omega \qquad 0 \le \omega \le 1 \tag{12}$$

Let $H(s)$ be a non-minimize transfer function given by

$$H(s) = \frac{N_m(s)N_m(-s)}{{D_n}^2(s)} \tag{13}$$

where $D_n(s)$ is the q-Bessel polynomials of n^{th} order and $N_m(s)$ is the polynomials of m^{th} order in which $m \le n$ To approach the ideal function, the least mean square error is given in the following equation.

$$E = \int_0^{w_0}\{F_S(\omega) - |H(\omega)|\}^2 d\omega \tag{14}$$

where E is the mean squared error. $F_S(\omega)$ is the desired function. $H(\omega)$ is the approximated function. The desired function is shown in Figure 1.

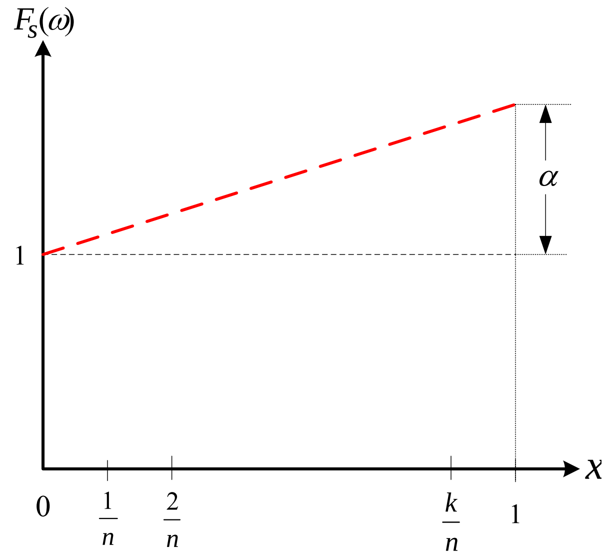


Figure 1 The desired amplitude characteristic.

For simplicity, the magnitude squared characteristic for $n = 2$ is given in the following equation.

$$H(\omega) = \frac{a_2\omega^2 + a_0}{\omega^4 + b_2\omega^2 + b_0} \tag{15}$$

The coefficient of numerator a_2 is obtained by using (15). Thus

$$a_2 = \frac{A_1^f - a_0 A_2}{A_4} \tag{16}$$

where

$$A_1^f = \int_0^{\omega_0} \frac{F_s(\omega)\omega^2}{\omega^4 + b_2\omega^2 + b_0} d\omega \tag{17}$$

$$A_2 = \int_0^{\omega_0} \left\{ \frac{\omega}{\omega^4 + b_2\omega^2 + b_0} \right\}^2 d\omega \tag{18}$$

$$A_4 = \int_0^{\omega_0} \left\{ \frac{\omega^2}{\omega^4 + b_2\omega^2 + b_0} \right\}^2 d\omega \tag{19}$$

Substituting (12) on to (14), yields

$$E = \int_0^1 \left\{ F_s(\omega) - \frac{a_2\omega^2 + a_0}{D^2(\omega)} \right\}^2 d\omega \tag{20}$$

where $D^2(\omega)$ is an all pole magnitude square function that has maximally flat group delay characteristic. Let $D(s)$ be a second order q-Bessel polynomials. Such that its magnitude squared characteristic is given as

$$D^2(\omega) = \omega^4 + \frac{(1+q)(1+q+q^2)(q^3+2q-1)\omega^2}{4} + \left[\frac{(1+q)(1+q^2)(1+q+q^2)}{4}\right]^2 \quad (21)$$

The first consideration of the parameter of $q = 0.2$ is to examine the magnitude characteristic. Thus the transfer function is given by

$$H(\omega)\Big|_{q=0.2} = \frac{1.6529\omega^2 + 0.15}{\omega^4 + 0.7757\omega^2 + 0.15} \quad (22)$$

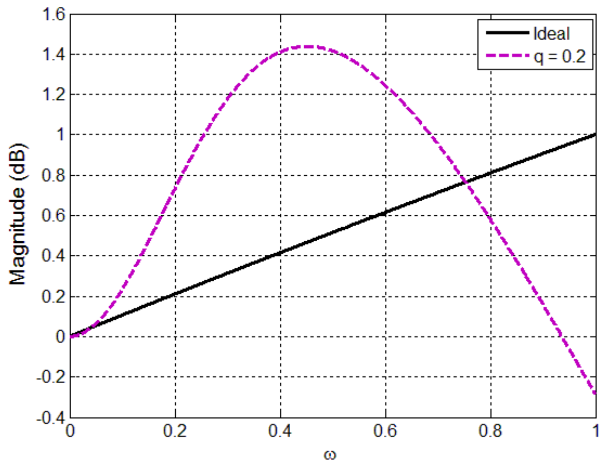


Figure 2 The magnitude characteristic with $q = 0.2$.

Figure 2 shows the magnitude response of the transfer function with $q = 0.2$. It can be seen that the purple line is different from ideal line or it has many errors as shown in Figure 3.

The closed form of the magnitude squared function for q-Bessel polynomials is given Appendix.

The next consideration, setting q = 0.8, 0.85, 0.9 and 1 respectively, the resulting of approximated transfer function can be written as

$$H(\omega)\Big|_{q=0.8} = \frac{4.8126\omega^2 + 3.24}{\omega^4 + 2.3536\omega^2 + 3.24} \quad (23)$$

$$H(\omega)\Big|_{q=0.85} = \frac{7.33\omega^2 + 4.2}{\omega^4 + 4.09\omega^2 + 4.2} \quad (24)$$

$$H(\omega)\Big|_{q=0.9} = \frac{4.5418\omega^2 + 7.433}{\omega^4 + 3.25\omega^2 + 7.433} \quad (25)$$

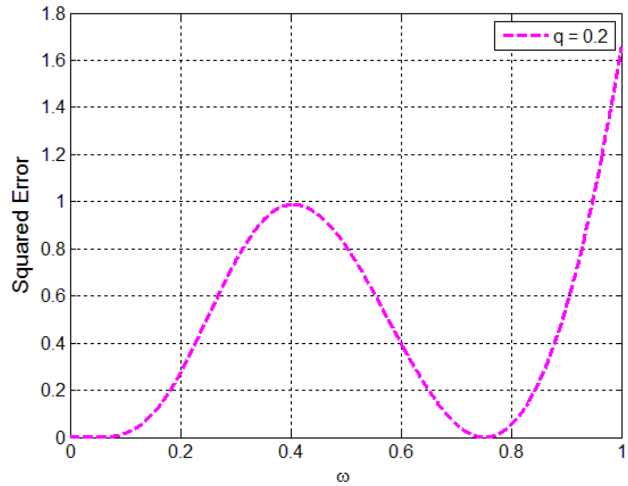


Figure 3 The squared error with $q = 0.2$.

$$H(\omega)\big|_{q=1} = \frac{5.268\omega^2 + 9}{\omega^4 + 3\omega^2 + 9} \tag{26}$$

Note that for $q = 1$ corresponding to classical Bessel polynomials.The plot of the approximated transfer functions for the magnitude, phase and delay responsesare shown in Figure 4, Figure 5 and Figure 6 respectively.

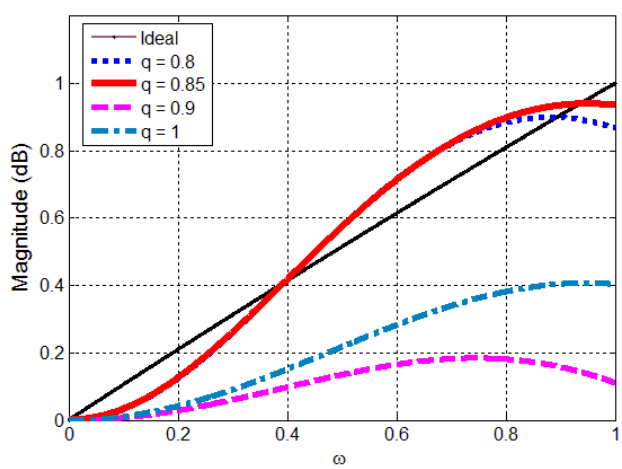


Figure 4 The comparison magnitude responses with $q = 0.8$, 0.85, 0.9 and 1.

These plots clearly indicate that the optimal approximation $H(\omega)$ can be achieved with appropriate q-Bessel parameters. It is shown that $q = 0.85$ satisfied the desired magnitude characteristic, the linear phase and the constant delay. Moreover, it is less than error as shown in Figure 7.

4. Stability Test

4.1 Stability Consideration

The stability of the given transfer function can be proved by using a Mihailov's criterion [6]. The Mihailov's criterion states that an n^{th} order polynomials be defined as

$$D(j\omega) = u(\omega) + jv(\omega) \tag{27}$$

where $u(\omega) = a_0 - a_2\omega^2 + ...a_{2n}\omega^{2n}$, and $v(\omega) = a_1\omega \;\; - a_3\omega^3 + ...a_{2n-1}\omega^{2n-1}$

The curve is obtained as ω varies from $-\infty$ to 0 and ∞ to 0 with $u(\omega)$ as abscises and $v(\omega)$ are ordinated. For the contour is encircle around the origin in anticlockwise direction. The system has proven to be stable. This method is used widely in the analog control system.

In this paper, the designed amplitude equalize is assumed $+1\,\text{dB}$ and the denominator of 2^{th} order and $q = 0.85$ is given as

$$D_2(s) = s^2 + 2.385s + 2.05 \tag{28}$$

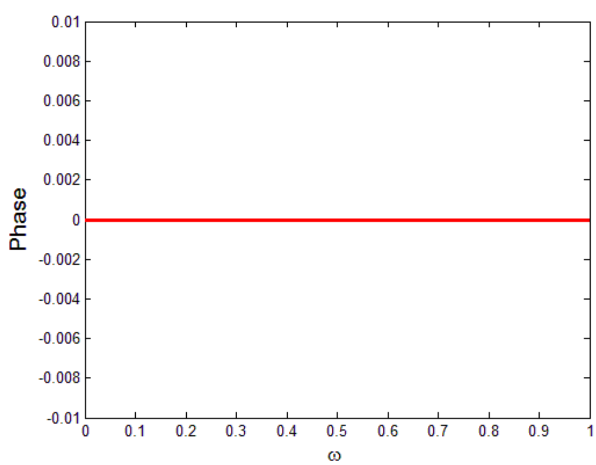


Figure 5 The comparison phaseresponses with $q = 0.8$, 0.85, 0.9 and 1.

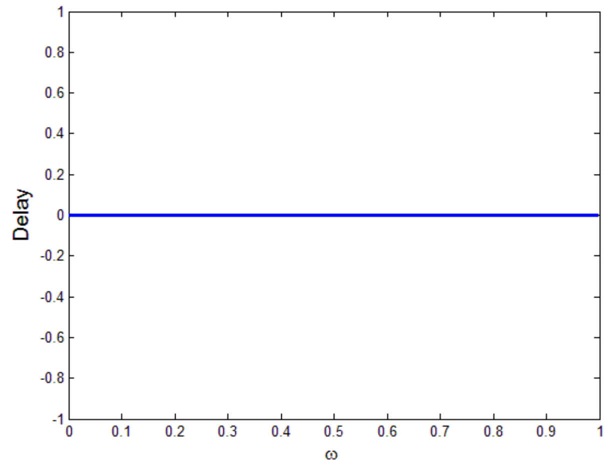


Figure 6 The comparison delayresponses with $q = 0.8, 0.85, 0.9$ and 1.

4.2 Plotting Mikhailov's Stability

From (13), the denominator of the transfer function is

$$D^2(s)\Big|_{q=0.85} = \left(s^2 + 2.385s + 2.05\right)^2 \tag{29}$$

$$D^2(s)\Big|_{q=0.85} = s^4 + 4.76s^3 + 9.76s^2 + 9.76s + 4.2 \tag{30}$$

Separating into real and imaginary part yield

$$D^2(j\omega) = u(\omega) + jv(\omega) \tag{31}$$

By

$$u(\omega) = 4.2 - 9.76\omega^2 + \omega^4 \tag{32}$$

$$v(\omega) = 9.76\omega - 4.76\omega^3 \tag{33}$$

From (32) to (33), Mikhailov's hodograph is plotted in Figure 8. It is seen that the contour is encircle around the original in the anticlockwise direction. Thus the proposed equalizer is stable.

In view of system stability criteria, the stability of electrical system can be represented by using delay function's properties. The system criteria states that the group delay function $\tau(\omega)$ which is the corresponding denominator of transfer function based on q-Bessel polynomials is positive where the poles are lie on the left half of s-plane. Therefore, the system is stable.

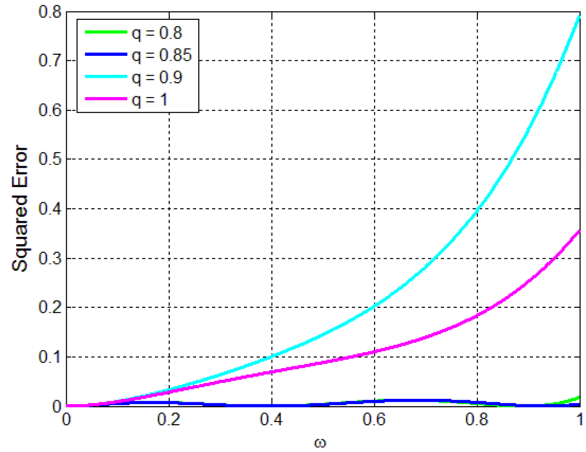
From (31), plotting of group delay function can be obtained as follows

$$\tau(\omega) = \frac{uv' - vu'}{u^2 + v^2} \tag{34}$$

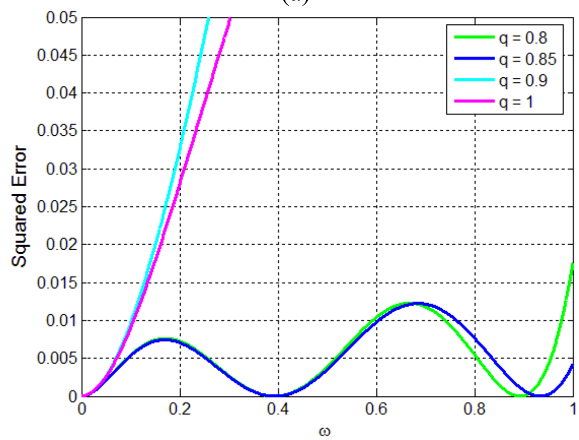
where u' and v' are its first derivative of $u(\omega)$ and $v(\omega)$, respectively. Substituting (32) and (33) onto (34) yields

$$\tau(\omega) = \frac{40.98 + 39.15\omega^2 + 15.27\omega^4 + 5.12\omega^6}{17.64 + 13.27\omega^2 + 10.74\omega^4 + 3.137\omega^6 + \omega^8} \tag{35}$$

The group delay function is shown in Figure 9. It is seen that the group delay function of q-Bessel polynomials is positive on the basis of the stability's property.



(a)



(b)

Figure 7 The squared error for q = 0.8, 0.85, 0.9 and 1.

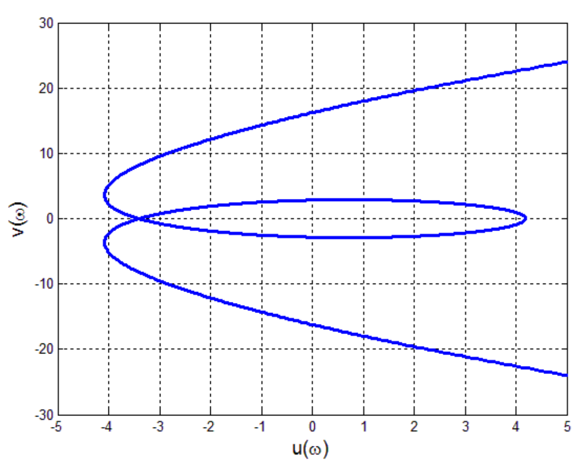


Figure 8 The stability using Mihailov hodograph of q = 0.85.

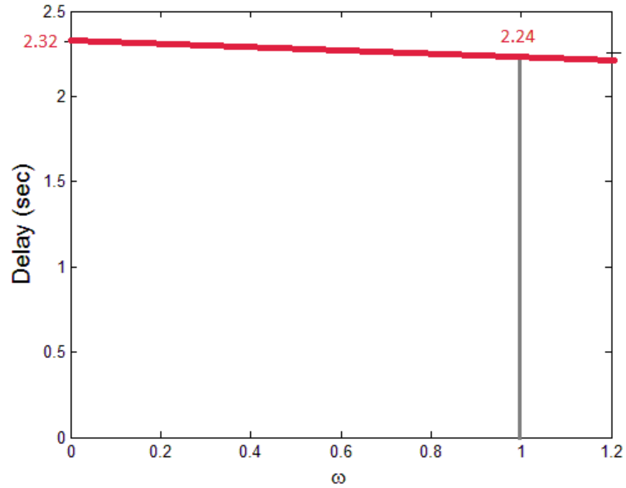


Figure 9 Maximally flat group delay response for $q = 0.85$.

The refore, the proposed transfer function is applicable to any electrical networks whose stability depends on the poles of the characteristic equation, located in the left half of s-plane.

5. Conclusions

A method for the design of amplitude equalizer based on q-Bessel polynomials has been presented. The designed linear gain slope equalizer for +1 dB at specified frequency is given as an example. In general the design of magnitude response will cause an inequality of phase characteristic. Thus, a non-minimum phase method together with q-Bessel polynomials is utilized. In section 4,comparison was made by varying q-Bessel polynomials parameters. Finally, the stability of the proposed equalizer was made by using a Mihailov's criterion.

Appendix

We can derive the closed form of the magnitude response for q-Bessel polynomials in the following equations. Brafman [7] was first to introduce the two product of Generalized Bessel polynomials. We extend this method for two product of q-Bessel polynomials. The algebraic manipulation is derived as follows:

$$y_{n,a}(x)y_{n,a}(y) = \sum_{k=0}^{n} \frac{(-n)_k (a+n-1)_k}{k!} \left(\frac{-x+y}{2}\right)^k y_k\left(\frac{xy}{x+y}\right) \tag{A.1}$$

Its reversed Generalized Bessel polynomials of (A.1) is obtained

$$\theta_{n,a}(x)\theta_{n,a}(y) = \sum_{k=0}^{n} \frac{(-1)^k (-n)_k}{k!} \frac{(a+n-1)_k}{2^k} (xy)^{n-k} \theta_k(x+y) \tag{A.2}$$

Setting $x = -y = s$ then

$$\theta_k(x+y) = \theta_k(0) = \frac{(a+k-1)}{2^k} \tag{A.3}$$

Substituting (A.3) onto (A.2) yields

$$A_{n,2}(s) = \theta_{n,a}(s)\theta_{n,a}(-s) = \left|\theta_n(s)\right|^2\Big|_{s=j\omega} \tag{A.4}$$

Equation (A.4) is obtained in Factorial form.

$$A_{n,2}(s) = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \frac{(a+n+k-2)!}{(a+n-2)!} \frac{(a+2k-2)!}{(a+k-2)!} \frac{s^{2(n-k)}}{2^{2k}} \quad \text{(A.5)}$$

Substituting $a = 2$ in (A.5), its becomes two product of simple Bessel polynomials

$$A_{n,2}(s) = \sum_{k=0}^{n} \frac{(n+k)!(2k)!}{(n-k)!(k!)^2} \frac{s^{2(n-k)}}{2^{2k}} \quad \text{(A.6)}$$

Equation (A.6) can be written in two product of q-Bessel polynomials in the following equation.

$$A_{n,2,q}(s) = \sum_{k=0}^{n} \frac{(n+k)_q!(2k)_q!}{(n-k)_q!(k!)_q{}^2} \frac{s^{2(n-k)}}{2^{2k}} \quad \text{(A.7)}$$

Therefore, (A.7) is the closed form of the magnitude squared unction of q-Bessel polynomials. It is applicable in calculating and graph's plotting for electrical network.

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