A Perturbation Result for Bounded Solutions of Linear Differential Systems under the Integrable Dichotomy Condition

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Abstract

The aim of this paper is to study the behavior of bounded solutions of linear differential systems under perturbation. We show that if the unperturbed linear differential system has an integrable dichotomy then the perturbed system has a unique bounded solution which converges to the bounded solution of the unperturbed linear differential system when the perturbation is sufficiently small.

Keywords: Bounded solutions, Differential systems, Integrable dichotomy, Perturbation

1. Introduction

The concept of exponential dichotomy has been extensively used when studying bounded solutions of differential equations. Several results on the existence and uniqueness of bounded solutions, periodic solutions and almost periodic solutions of both linear and nonlinear differential equations are obtained under the assumption that the associated homogeneous linear equation

$$x'(t) = A(t)x(t) \tag{1}$$

has an exponential dichotomy (see, for example, [1-3] and references therein). However, there are similar results on the existence and uniqueness of bounded solutions under a more general condition that (1) has only an *integrable dichotomy* [4].

In this paper, we are interested in the behavior of bounded solutions of linear non-homogeneous differential systems under perturbation. Let $A(\cdot)$ and $B_n(\cdot)$ be $N \times N$ continuous matrices. Consider the following unperturbed linear differential systems

$$x'(t) = A(t)x(t) + f(t) \tag{2}$$

in an N-dimensional Euclidean space R^N for $t \in R$. We study the perturbation of the differential system (2) of the form

$$x'(t) = [A(t) + B_n(t)]x(t) + f_n(t) \tag{3}$$

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for $t \in R$. Under the exponential dichotomy condition, it is known that bounded solutions of linear non-homogeneous differential systems persist under a small perturbation of a differential operator due to the robustness of an exponential dichotomy (see, for example, [4, Chapter 3]). The main focus of this paper is to obtain the persistence of bounded solutions under perturbation when we only assume a more general condition, that is, an integrable dichotomy condition for the unperturbed differential system.

An outline of this paper is shown as follows. In Section 2, we review the concept of integrable dichotomy and the existence of bounded solutions of differential systems. In Section 3, we state and prove our perturbation result for bounded solutions.

2. Integrable Dichotomy and Bounded Solutions

We first recall the notion of integrable dichotomy of a linear nonautonomous differential system. Denoted by $\Phi(t)$ the fundamental matrix of system (1), that is, $\Phi(t)$ is a solution matrix of (1) with $\Phi(0) = I$. Let P be a projection matrix. We define a Green matrix $G = G_P$ associated with P by

$$G(t,s) = \begin{cases} \Phi(t)P\Phi^{-1}(s) & \text{for } t \geq s \\ \Phi(t)(I - P)\Phi^{-1}(s) & \text{for } t < s \end{cases} \tag{4}$$

Definition 2.1 We say that the system (1) has an integrable dichotomy if there exist a projection P and a constant $\mu > 0$ such that the associated Green matrix $G = G_P$ satisfies

$$\sup_{t \in R} \int_{-\infty}^{\infty} \|G(t,s)\| \, ds = \mu \tag{5}$$

We denote by $BC(R, R^N)$ the space of bounded and continuous functions from R to R^N. The space $BC(R, R^N)$ is a Banach space with respect to the sup-norm $\|\cdot\|_\infty$ defined by

$$\|f\|_\infty = \sup_{t \in R} \|f(t)\|$$

for $f \in BC(R, R^N)$. Under the integrable dichotomy of the corresponding homogeneous linear system, we have the following existence and uniqueness result for bounded solutions.

Theorem 2.2 (see [4, Proposition 2]) *Suppose that the system (1) has an integrable dichotomy. For any $f \in BC(R, R^N)$, the non-homogeneous system (2) has a unique bounded solution $u \in BC(R, R^N)$ represented by*

$$u(t) = \int_{-\infty}^{\infty} G(t,s) f(s) \, ds \tag{6}$$

for $t \in R$.

3. Perturbation Results

In this section, we state and prove our main result on perturbation of bounded solutions. We assume the following assumption for the unperturbed system.

Assumption 3.1 We assume that

(i) the homogeneous linear system (1) has an integrable dichotomy;

(ii) the non-homogeneous term f is in $BC(R, R^N)$.

Under Assumption 3.1, it follows from Theorem 2.2 that the linear non-homogeneous differential systems (2) has a unique bounded solutions. We shall denote the bounded solution of (2) by $u \in BC(R, R^N)$ throughout the remainder of this paper. We impose the following conditions on the perturbed system (3).

Assumption 3.2 We assume the following conditions.

(i) $\sup_{t\in R}\|B_n(t)\| \to 0$ as $n \to \infty$

(ii) $f_n \in BC(R, R^N)$ for all $n \in N$ and $f_n \to f$ in $BC(R, R^N)$ as $n \to \infty$

We can state our perturbation result as follows.

Theorem 3.3 *Suppose that Assumptions 3.1 and 3.2 are satisfied. For all n sufficiently large, the perturbed system (3) has a unique bounded solution $u_n \in BC(R, R^N)$. In addition, u_n converges to the bounded solution u of (2) in $BC(R, R^N)$ as $n \to \infty$.*

The main tool to establish our perturbation result is the contraction mapping principle. Define the operator $L : BC(R, R^N) \to BC(R, R^N)$ to be the constant map

$$Lv := u \tag{7}$$

for $v \in BC(R, R^N)$, where u is the bounded solution of (2) given by (6). For $n \in N$, we consider the operator $L_n : BC(R, R^N) \to BC(R, R^N)$ given by

$$(L_n v)(t) := \int_{-\infty}^{\infty} G(t,s)B_n(s)v(s)ds + \int_{-\infty}^{\infty} G(t,s)f_n(s)ds \tag{8}$$

for $t \in R$ if $v \in BC(R, R^N)$. It follows from Assumption 3.2 and the integrable dichotomy assumption that the operator L_n given by (8) is well-defined for all n sufficiently large. Indeed, for any given $v \in BC(R, R^N)$, we have that $L_n v$ is the bounded solution of

$$x'(t) = A(t)x(t) + B_n(t)v(t) + f_n(t)$$

for all n sufficiently large. Moreover, it can be easily seen that u_n is a bounded solution of the system (3) if and only if u_n is a fixed point of L_n.

To prove our convergence result, we consider the following lemmas.

Lemma 3.4 *Suppose that Assumptions 3.1 and 3.2 are satisfied. The maps L_n given by (8) are uniform contractions for all n sufficiently large.*

Proof Let v, w be functions in $BC(R, R^N)$. Then, we have

$$\left\|(L_n v)(t) - (L_n w)(t)\right\| = \left\| \int_{-\infty}^{\infty} G(t,s) B_n(s)[v(s) - w(s)] ds \right\|$$

$$\leq \int_{-\infty}^{\infty} \left\|G(t,s)\right\| \left\|B_n(s)\right\| \left\|v(s) - w(s)\right\| ds \tag{9}$$

$$\leq \sup_{s\in R} \left\|B_n(s)\right\| \left\|v\text{-}w\right\|_\infty \int_{-\infty}^{\infty} \left\|G(t,s)\right\| ds \ ,$$

for all $t \in R$. By Assumption 3.2 (i), we have that $\sup_{t\in R} \left\|B_n(t)\right\| < \frac{1}{\mu}$ for all n sufficiently large. Hence, we get from (5) and (9) that

$$\left\|(L_n v)(t) - (L_n w)(t)\right\| < \left\|v - w\right\|_\infty ,$$

for all $t \in R$. This implies that L_n is a contraction for all n sufficiently large. Since the estimate above is independent on n (when n is sufficiently large), the maps L_n are uniform contractions.

Lemma 3.5 *Suppose that Assumptions 3.1 and 3.2 are satisfied. For any* $v \in BC(R, R^N)$ *, we have that* $L_n v \to Lv$ *in* $BC(R, R^N)$ *as* $n \to \infty$.

Proof Let $v \in BC(R, R^N)$ be arbitrary. We have from (6) – (8) that

$$\left\|(L_n v)(t) - (Lv)(t)\right\| \leq \int_{-\infty}^{\infty} \left\|G(t,s) B_n(s) v(s)\right\| ds + \int_{-\infty}^{\infty} \left\|G(t,s)[f_n(s) - f(s)]\right\| ds$$

$$\leq \int_{-\infty}^{\infty} \left\|G(t,s)\right\| \left\|B_n(s)\right\| \left\|v(s)\right\| ds + \int_{-\infty}^{\infty} \left\|G(t,s)\right\| \left\|f_n(s) - f(s)\right\| ds$$

$$\leq \sup_{s\in R} \left\|B_n(s)\right\| \left\|v\right\|_\infty \int_{-\infty}^{\infty} \left\|G(t,s)\right\| ds + \left\|f_n - f\right\|_\infty \int_{-\infty}^{\infty} \left\|G(t,s)\right\| ds$$

$$\leq \sup_{s\in R} \left\|B_n(s)\right\| \left\|v\right\|_\infty \mu + \left\|f_n - f\right\|_\infty \mu ,$$

for all $t \in R$. It follows from Assumption 3.2 that $L_n v \to Lv$ in $BC(R, R^N)$ as $n \to \infty$

We are now in a position to prove our main result.

Proof of Theorem 3.3 We first show the existence and uniqueness of bounded solution of the perturbed system (3) for large n. By Lemma 4.1 and the contraction mapping principle, L_n has a unique fixed point for all n sufficiently large. Since u_n is a bounded solution of the system (3) if and only if u_n is a fixed point of L_n, this proves the existence and uniqueness of bounded solution of the perturbed system (3).

Next, we show the convergence of solutions under perturbation. We notice that the bounded solution u of (2) is the fixed point of the constant map L given by (7). Hence, to show that $u_n \to u$ in $BC(R, R^N)$, we need to show that the fixed point of L_n converges to the fixed point of L in $BC(R, R^N)$. Since L_n and L are uniform contractions, we can apply Lemma 3.5 and a parameter dependent version of contraction mapping principle (see [2, Section 1.2.6]) to conclude that the fixed point converges as required.

Remark 3.6 Similar perturbation results could be obtained for bounded solutions of differential systems of semilinear types (possibly with delays) under suitable assumptions on the nonlinear terms such as in [4].

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