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# An Approximation of a Linear Advection Equation using the Meshfree Method

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## Abstract

In this paper, a mesh-free method called "Smoothed Particle Hydrodynamics (SPH)" is applied to the linear advection equation. The cubic spline weight function is used to determine the advection field at the nodes from particles in the support domain. After the nodes have been defined, the derivative of the function is approximated from the derivative of the weight function. This is equivalent to finding the changes in distance between nodes. Results obtained from the SPH method are compared with the corresponding exact solutions. Effects of boundary condition are also investigated.

**Keywords:** Smoothed Particle Hydrodynamics, Advection equation

## 1. Introduction

Numerical method is important for the successful simulation of physical processes as the underlying partial differential equation usually has no analytic solution and has to be approximated. Conventional numerical methods need a priori definition of the connectivity of the nodes, i.e., they rely on a mesh. Finite Element Method (FEM), Finite Difference Method (FDM) and Finite Volume Method (FVM) may be the most well-known members of these thoroughly developed mesh-based methods. The large deformations in highly nonlinear problems that can deteriorate the accuracy because of mesh or element distortion may cause severe loss of accuracy or even complete failure of computations. The mesh-based methods are unsuitable for finding solutions of problems with changing domain shape. A new class of numerical methods has been developed which approximates partial differential equations based on only a set of nodes without the need for an additional mesh. This is called meshfree methods (MMs). Meshfree method is different from FEM because a mesh of element is not used in meshfree method, the field variable $u$ at any point $\mathbf{x} = (x, y, z)$ within the problem domain is interpolated using the displacements at its nodes within the support domain of the point at x, i.e.

$$u(\bar{x}) = \sum_{i=1}^{n} \varphi_i(\bar{x}) u_i = \Phi(\bar{x}) U_s \tag{1}$$

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for $\bar{x} = (x_1, x_2, x_3, ..., x_n)$, where $n$ is the number of nodes include in a "small local domain" of the point at **x**. The local domain means the interpolation area which is represented by point **x**, $u_i$ is the nodal field variable at the *i* th node in the small local domain, $U_s$ is the vector that collects all the field variables at these nodes, and $\varphi_i(\bar{x})$ is the shape function of the *i* th node determined using the nodes that are included in the small domain of **x**. This small local domain is called the support domain of **x**. A support domain of a point **x** determines the number of nodes to be used to support or approximate the function value at **x**. Figure 1 shows the concept of a support domain.

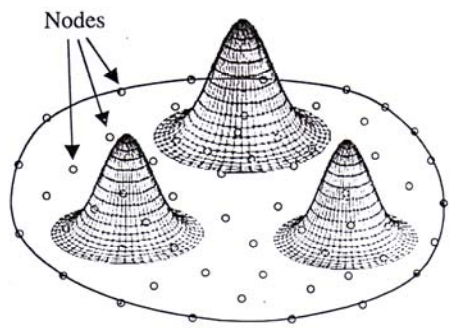


**Figure 1** The support domain of a particle at x [1]

## 2. Materials and Methods

### 2.1 Smoothed Particle Hydrodynamics (SPH)

In the SPH method, the equation to be approximated is represented by a collection of nodes which are under the influence of hydrodynamic. Each node is a representation of surrounding particles in the support domain. The interpolation of support domain is obtained through the summation of the product of multiplying the function with weight function. The weight function is constructed using the information on all particles within the support domain, and the support domain size is defined by the smoothing length. The cubic spline is used to define the weight function in this research. The reason is that, with cubic spline weight function interpolation derivative of a function is transformed into a simple form of derivative of the weight function. That is, for a function $u(x)$ [2-3],

$$u(x) \approx \sum_{j=1}^{N} u(x_j) w(x - x_j, h) \Delta V_j \tag{2}$$

where $w(x\text{-}x_j, h)$ is a weight or smoothing function, $\Delta V_j$ is the volume element carried by particle $j$ and $h$ is the smoothing length. The cubic spline weight function is used in the SPH method for this study. The derivative can be approximated from (1) as

$$u'(x) \approx \sum_{j=1}^{N} u(x_j) \nabla w(x - x_j, h) \Delta V_j \tag{3}$$

A summation representation is valid and converges when e weight function satisfies certain conditions [1].

1. $w(x - x', h) > 0$ over $\Omega$ (Positivity)
2. $w(x - x', h) = 0$ outside $\Omega$ (Compact)
3. $\int_\Omega w(x - x', h) dx' = 1$ (Unity)

4. $w$ is monotonically decreasing function (Decay)
5. $w(s,h) \to \delta(s) \quad as \quad h \to 0$ (Delta function)

## 2.2 The Linear Advection Equation

The linear advection equation is an unsteady state flow problem. This is a simple atmospheric model. The one-dimensional linear advection equation is given by the expressions [4].

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} \tag{4}$$

where $u = u\ (x,\ t)$ is the advection quantity. The initial conditions for (4) are $u = \sin\ (kx)$, $k = 2\pi/Lx$, $Lx = 1000$m, and the advection speed is $c = 10$m/s. Equation (4) is approximated from $t = 0$s to $t = 10$s with the time step of 0.01s. The exact solution of (4) is $u = \sin\ (k(x-ct))$.

## 2.3 The Linear Advection Equation in SPH form

From (4), the derivative of $u$ can be written as

$$\frac{\partial u}{\partial x} = \sum_{j=1}^{N} u_j \nabla_i w_{ij}\left(x_i - x_j, h\right)\Delta V_j \tag{4.1}$$

Such that $\frac{\partial w_{ij}}{\partial x_i} = \nabla_i w_{ij} = \frac{dx}{x_{ij}}\frac{\partial w_{ij}}{\partial x_{ij}}$

where $dx = x_j - x_i$ and $x_{ij} = |dx|$

The weight function which will be used for approximation is the cubic spline weight function [1].

$$\hat{W}(S) = \begin{cases} \frac{2}{3} - 4S^2 + 4S^3 & , \ S \le \frac{1}{2} \\ \frac{4}{3} - 4S + 4S^2 - \frac{4}{3}S^3 & , \ \frac{1}{2} < S \le 1 \\ 0 & , \ S > 1 \end{cases} \tag{4.2}$$

where $S = \frac{d_c}{d_s}$

$d_c$ is the nodal spacing and $d_s$ is the size of the support domain for the weight function.

The forward time difference is used for time derivative [5],

$$\frac{\partial u}{\partial t} \approx \frac{u_m^{n+1} - u_m^n}{\Delta t} \tag{4.3}$$

The time step is controlled in the computation to satisfy the following Courant-Friedrichs-Lewy (CFL) condition: $\Delta t \le \frac{\Delta x}{c}$

From (4.1) and (4.3), the advection equation (4) can be written in the form of SPH method as,

$$u^{n+1} = u^n - c\Delta t\left[\sum_{j=1}^{N} u_j \nabla_i w_{ij}\left(x_i - x_j, h\right)\Delta V_j\right] \tag{4.4}$$

This straightforward approximation is usually not accurate, and often destroys the conservation property of the associated continuous system. Nonetheless, when the approximation is combined with an additional term which contains a null expression, it may produce better results [2]. A Null expression term is the term that approximate the derivative at the node by using weight function,

$$\begin{aligned}\sum_{j=1}^{N} u_i \nabla_i w_{ij}\left(x_i - x_j, h\right)\Delta V_j &= u_i \sum_{j=1}^{N} \nabla_i w_{ij}\left(x_i - x_j, h\right)\Delta V_j \\ &= u_i \nabla_i \sum_{j=1}^{N} w_{ij}\left(x_i - x_j, h\right)\Delta V_j \\ &= u_i \nabla_i (1) = 0\end{aligned}$$

Since $\left\{w_{ij}\right\}$ is a partition of unity, the better results are obtained. Thus, approximation of the derivative function using function as in (4.1) with a null expression term is then,

$$\begin{aligned}\frac{\partial u}{\partial x} &= \sum_{j=1}^{N} u_j \nabla_i w_{ij}\left(x_i - x_j, h\right)\Delta V_j - \sum_{j=1}^{N} u_i \nabla_i w_{ij}\left(x_i - x_j, h\right)\Delta V_j \\ &= \sum_{j=1}^{N} u_j \nabla_i w_{ij}\left(x_i - x_j, h\right)\Delta V_j - u_i \sum_{j=1}^{N} \nabla_i w_{ij}\left(x_i - x_j, h\right)\Delta V_j \\ &= \sum_{j=1}^{N} \left(u_j - u_i\right) \nabla_i w_{ij}\left(x_i - x_j, h\right)\Delta V_j\end{aligned}$$

Hence, (4.4) can be written as

$$u^{n+1} = u^n - c\Delta t \sum_{j=1}^{N} \left(u_j^n - u_i^n\right) \nabla_i w_{ij}\left(x_i - x_j, h\right)\Delta V_j \tag{4.5}$$

## 3. Results and Discussion

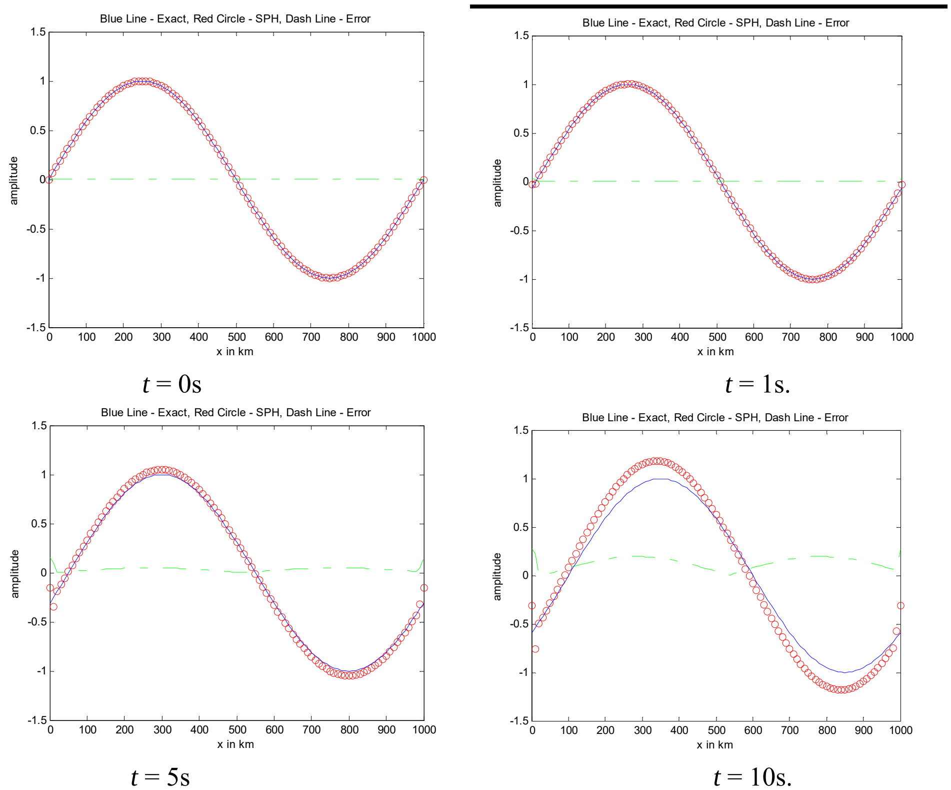


**Figure 2** Results of advection equation with cyclic boundary for time $t = 0$s and $t = 10$s.

From Figure 2, the advection of sine wave moves from left hand side to right hand side for time step $t$ = 0s to $t$ = 10s. The result from SPH method is compared with the exact solution where the blue line is the exact solution, the red circle line is the SPH result, and the green dash line represents the approximation error. In the figure, the error occurred in SPH method when time step $t$ = 5s. At the later time, when $t$ = 10s, the results have large errors. The boundary of sine wave had a complex pattern, large error occurred near the boundary due to asymmetry of weight function at the boundary (shown in Figure 3). So, the main cause of error in this approximation is the accumulate error on the boundary in every time step.

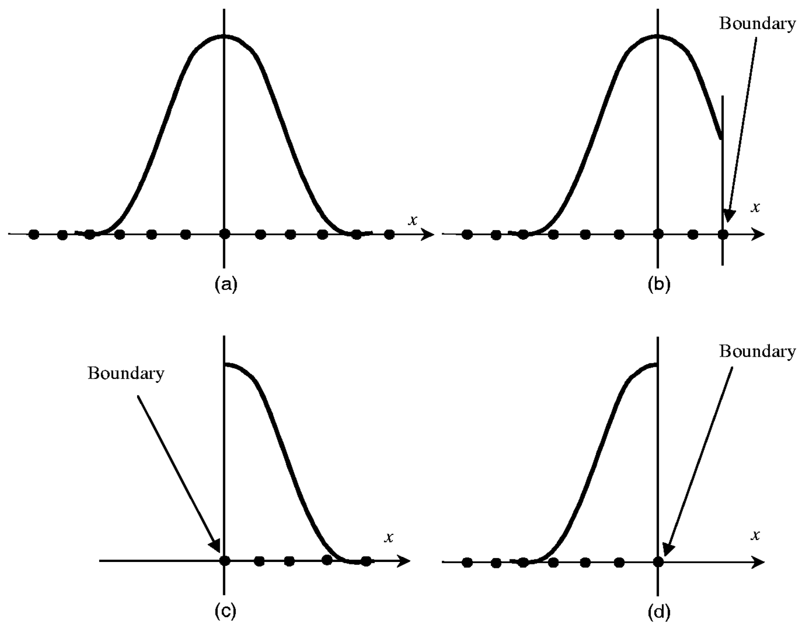


**Figure 3** The weight function on the boundary [3]

There have been very few formal studies of the accuracy of SPH carried out. Even most code papers on SPH report only circumstantial evidence for SPH's accuracy. What is especially missing are rigorous studies of the convergence rate of SPH towards known analytic solutions, which is ultimately one of the most sensitive tests of the accuracy of a numerical method. For example, this may involve measuring the error in the final result of an SPH calculation at time $t$ = 1$s$ in terms of an L1 error norm that can be defined as

$$\mathrm{L1} = \frac{1}{N}\sum_i |u_{SPH} - u_{exact}| \tag{5}$$

where N is the number of SPH nodes, $u_{SPH}$ is the numerical solution and $u_{exact}$ is the analytic solution.

To study the effect of increasing number of nodes in the SPH approximation, an experiment is performed using 10, 20, 50, 100, 500 and 1,000 nodes. It is found that increasing number of nodes results in better approximation, as shown in Figure 4. The accuracy of the SPH approximation of the advection equation is also depends on the number of node used in computation. If the number of node is increased, the accuracy will increase but the time for computing will increase too.

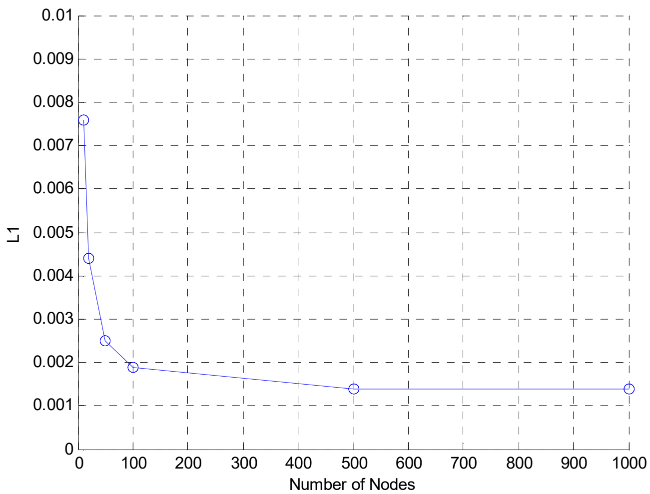


**Figure 4** Convergence rate for different numbers of nodes.

## 4. Conclusions

Smooth particle hydrodynamics (SPH) is a meshfree method, the concept is differential equations approximation on nodes without mesh. With SPH method, the equation to be approximated is represented by a collection of nodes which are under the influence of hydrodynamic. Each node is a representation of surrounding particles in the support domain. The interpolation of support domain is obtained from the summation of multiplying the function with weight function. The weight function is constructed using the information on all particles within the support domain, and the support domain size is defined by the smoothing length. Application of the SPH method to the linear advection equation, by using the cubic spline weight function, shows that the error of approximation is mainly due to accumulate error of boundary approximation in each time step. The weak point of approximation which is asymmetric weight function on boundary.

## 5. Acknowledgements

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