Switching Law of a Three Second-order Linear Time Invariant Switched System

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Abstract

In this work, we study a problem of asymptotically stabilizing of a switched system, which consists of second-order linear time invariant subsystems. The subsystems used in this work have complex eigenvalues and two out of three subsystems cannot be stabilized. We then find a third subsystem and a new switching law that makes overall system asymptotically stabilizable. We propose a new sufficient condition on eigenvalues of the 3rd subsystem and a switching law that allows asymptotically stabilizable the overall switched system. A numerical example has been shown to guarantee the effectiveness of the proposed condition.

Keywords: Switched systems; Switching Law; Asymptotic stabilization

1. Introduction

A switched system is a particular kind of hybrid systems. It consists of several subsystems and a switching law, which determine the active subsystem at each time instant. In recent years switched system is a subject of interest in many fields of research. Many switched systems have been successfully applied in real-world processes such as in chemical processes, electrical circuits, computer-controller systems, intelligent-control systems and so on.

Most of the works on switched systems are based on Lyapunov or multiple Lyapunov functions which can be done by using Linear Matrix Inequality (LMI) approach. A switched system can be unstable even if all of its subsystems are stable, or a switched system can be stable even if none of its subsystems is stable; the example can be found in [1-3].

Xu and Antsaklis [4] proposed a method that selected an active subsystem to minimize the distance from the current state to the origin $\|x\|_2$. The authors proposed a selection criteria based on the angle of subsystem vector field and the geometric properties of R^2. A subsequent study [5] used the result of region separation to stabilize switched systems by a static output feedback.

Zhang and colleagues [6] considered a problem on asymptotic stabilization of second-order linear time-invariant (LTI) autonomous switched systems that consist of two subsystems with unstable foci. The authors derived the necessary and sufficient condition for the origin to be asymptotically stabilizable. The method is to find the "most stabilizing" switching law without the use of Lyapunov or multiple Lyapunov functions. The authors studied the locus in which two subsystems's vector fields are parallel.

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In this paper, we consider the asymptotically stabilization problem of second-order LTI autonomous switched system that consists of three subsystems. We study in which two out of subsystems can not be stabilized. We find the third subsystem and derive a new switching law to make the overall system asymptotically stabilizable.

The rest of the paper is organized as follow. In section 2, we present some lemmas and theorems which will be used in the next section. In section 3, we derive a switching law that suffices to make the overall system asymptotically stabilizable. A numerical example is given in this section. Finally in section 4, we summarize our study.

2. Materials and Methods

Consider the linear time-invariant (LTI) system

$$\dot{x}(t) = Ax(t), x(0) = x_0 . \quad (1)$$

It is well known that the solution of (1) is

$$x(t) = e^{At}x_0 \text{ for } t \geq 0$$

where $x = (x_1, x_2)^T$ and $A \in R^{2\times 2}$. The trajectory of the solution can be drawn on R^2 plane. According to [5], we can consider the case when $A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, where $\alpha \pm \beta j$ with $\alpha \neq 0$ and $\beta \neq 0$ are the complex eigenvalues of A. Since for a nonsingular general 2×2 matrices, the matrices could be transformed into the matrix of eigenvalue form. For the detail see [1].

Lemma 2.1 For the LTI autonomous system $\dot{x}(t) = Ax(t)$ with focus, where $A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, the solution with $x(0) = x_0 \neq 0$ has the following properties:

- if $\alpha < 0$ and $\beta > 0$, then the solution $x(t) = e^{At}x_0$ is a logarithmic spiral that converges to the origin clockwise,
- if $\alpha < 0$ and $\beta < 0$, then the solution $x(t) = e^{At}x_0$ is a logarithmic spiral that converges to the origin counterclockwise,
- if $\alpha > 0$ and $\beta > 0$, then the solution $x(t) = e^{At}x_0$ is a logarithmic spiral that diverges to the infinity clockwise,
- if $\alpha < 0$ and $\beta > 0$, then the solution $x(t) = e^{At}x_0$ is a logarithmic spiral that diverges to the infinity clockwise.

See the details in [5]. In 2005, Liguo Zhang, Yangzhou Chen and Pingyuan Cui [7] studied the second-order linear time-invariant (LTI) autonomous switched system is described by

$$\dot{x}(t) = A_{\sigma(t)}x(t)$$
$$\sigma(t) : [0,\infty) \to \{1, 2\} \quad (2)$$

where $x \in R^2$ and $\sigma(t)$ is the switching law indicating active subsystem at each instant. The necessary and sufficient condition is deserved by the following theorem.

Theorem 2.1 [7] The autonomous switched system (2) that consists of two subsystem with unstable focus equilibrium is asymptotically stabilizable if and only if $D > 0$, and $\rho < 1$, where

$$\rho = \exp\left[\rho_1 \arctan\left(\frac{\rho_1 R - \rho_2}{\sqrt{D}}\right) + \rho_2 \arctan\left(\frac{\rho_2 R - \rho_1}{\sqrt{D}}\right) - \frac{\pi}{2}(\rho_1 + \rho_2)\right] \times \sqrt{\frac{\rho_1\rho_2 + R + \sqrt{D}}{\rho_1\rho_2 + R - \sqrt{D}}},$$

$D = R^2 + 4\rho_1\rho_2 R - 4(1+\rho_1^{\ 2} + \rho_2^{\ 2})$, $R = (E + 1/E)$, $\rho_1 = \alpha_1/\beta_1$, $\rho_2 = \alpha_2/\beta_2$.
For the switching law, see [7].

3. Results and Discussion

In this section, we study the case in which the two subsystems have unstable foci and cannot be stabilized. To stabilize the overall system, we introduce a third subsystem which is a stable system that has two complex eigenvalues with negative real parts. This subsystem will be used to stabilize the overall system according to concept illustrated in Figure 1.

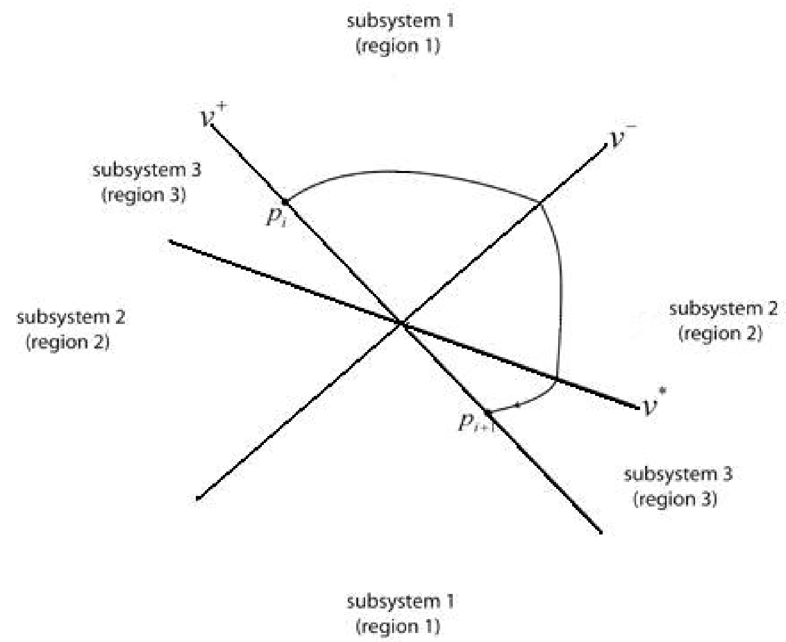

Figure 1 The idea of the concept

According to figure 1, given the line v^+ and v^-, the plane is divided into four regions. We are looking for the line v^* that makes the system asymptotically stabilizable. Without loss of generality, we let the point p_i lying on the line v^+. At this point, the switched system will use subsystem 1.

The trajectory will traverse region 1 in the clockwise direction until it intersects the line v^-. At this point, the switched system will switch to subsystem 2.

Then, we let the trajectory traverse region 2 in the clockwise direction until it intersect the v^* line, where the switched system switches to subsystem 3. The trajectory will then traverse region 3 until it intersect the v^+ line again.

We let the point at which the trajectory intersect v^+ line again be p_{i+1}. From p_i to p_{i+1}, The trajectory has traversed the total angle of π. The overall switched system which consists of three subsystems is asymptotically stabilizable if $\|p_{i+1}\| < c\|p_i\|$ where $0 < c < 1$.
In this work, we consider a switched system

$$\dot{x}(t) = A_{\sigma(t)}x(t)$$
$$\sigma(t) : [0,\infty) \to \{1, 2, 3\} \tag{3}$$

where $x \in R^2$ and $\sigma(t)$ is a switching law indicating active subsystem at each instant.
Let $\lambda_i = \alpha_i \pm j\beta_i$ be the eigenvalues of A_i, respectively. $\beta_i > 0$ (for $i = 1,2,3$), ($\alpha_i > 0$, $i = 1$, 2, $\alpha_i < 0$, $i = 3$).

According the figure 1, the switching law would be: switching to subsystem 1 whenever the system trajectory enters the region 1. Switching to subsystem 2 whenever the system trajectory enters the region 2. And switching to subsystem 3 whenever the system trajectory enters the region 3.

Lemma 3.1 A switched system which consists of three subsystems is asymptotically stabilizable if

$$\|p_{i+1}\| \le \frac{1}{c}\|p_i\| \text{ where } c > 1.$$

Proof. $\|p_2\| \le c\|p_1\|$,

$$\|p_3\| \le c\|p_2\| = c^2\|p_1\|.$$

By Mathematics Induction, we obtain

$$\|p_n\| \le c^{n-1}\|p_1\|.$$

As $n \to \infty$, we obtain

$$\lim_{n\to\infty}\|p_n\| \le \lim_{n\to\infty} c^{n-1}\|p_1\| = 0.$$

The trajectory of system converges to the origin. Thus, the switched system is asymptotically stabilizable.

Definition 3.1 [8] An $n \times n$ matrix A is said to be diagonalizable if there exists a nonsingular matrix P and a diagonal matrix D such that

$$P^{-1}A\,P = D.$$

We say that P diagonalizes A.

Lemma 3.2 For the system (1) with $A_1 = \begin{bmatrix} \alpha_1 & \beta_1/E \\ -E\beta_1 & \alpha_1 \end{bmatrix}$ and complex eigenvalues $\alpha_1 \pm \beta_1 j$ with $\alpha_1 \ne 0$ and $\beta_1 \ne 0$, the solution is

$$x(t) = e^{A_1 t}x_0 \text{ For } t \ge 0$$

$$\text{where } e^{A_1 t} = e^{\alpha t}\begin{bmatrix} \cos\beta\,t & \frac{-1}{E}\sin\beta\,t \\ E\sin\beta t & \cos\beta t \end{bmatrix}.$$

Proof. We want to find a nonsingular matrix P such that $PA_1P^{-1} = D$ where $D = \begin{bmatrix} \alpha_1 - j\beta_1 & 0 \\ 0 & \alpha_1 + j\beta_1 \end{bmatrix}$. Let $A_1 = \begin{bmatrix} \alpha_1 & \beta_1 / E \\ -E\beta_1 & \alpha_1 \end{bmatrix}$ with the eigenvalues $\lambda_1 = \alpha_1 - \beta_1 j$ and $\lambda_2 = \alpha_1 + \beta_1 j$. It is easy to see that $v_1 = [1 \quad E/j]^T$ and $v_2 = [1/E \quad j]^T$ are the eigenvectors, corresponding to the eigenvalues λ_1 and λ_2, respectively.

Thus we have $P = \begin{bmatrix} 1 & 1/E \\ E/j & j \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 1 & -1/2jE \\ E/2 & 1/2j \end{bmatrix}$.

Since $P^{-1}A\,P = D$, then we have $PDP^{-1} = A$.

It follows that

$$\begin{bmatrix} 1 & 1/E \\ E/j & j \end{bmatrix} \begin{bmatrix} \alpha_1 - j\beta_1 & 0 \\ 0 & \alpha_1 + j\beta_1 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2jE \\ E/2 & 1/2j \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 / E \\ -E\beta_1 & \alpha_1 \end{bmatrix}.$$

Therefore,

$$e^{A_1 t} = P \begin{bmatrix} e^{(\alpha_1 - j\beta_1)t} & 0 \\ 0 & e^{(\alpha_1 + j\beta_1)t} \end{bmatrix} P^{-1} = e^{\alpha_1 t} \begin{bmatrix} e^{-j\beta t} & \frac{1}{E} e^{j\beta t} \\ \frac{E}{j} e^{-j\beta t} & j e^{j\beta t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{-1}{2jE} \\ \frac{E}{2} & \frac{1}{2j} \end{bmatrix}$$

$$= e^{\alpha_1 t} \begin{bmatrix} \frac{1}{2} e^{-j\beta t} + \frac{1}{2} e^{j\beta t} & \frac{-1}{2jE} e^{-j\beta t} + \frac{1}{2Ej} e^{j\beta t} \\ \frac{-1}{2} Eje^{-j\beta t} + \frac{1}{2} Eje^{j\beta t} & \frac{1}{2} e^{-j\beta t} + \frac{1}{2} e^{j\beta t} \end{bmatrix}$$

$$= e^{\alpha_1 t} \begin{bmatrix} \cos\beta_1 t & \frac{-1}{E}\sin\beta_1 t \\ E\sin\beta_1 t & \cos\beta_1 t \end{bmatrix}$$

Since, the solution of the system is

$$x(t) = e^{At} x_0$$

and if $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we have $x(t) = e^{\alpha_1 t} \begin{bmatrix} \cos\beta_1 t \\ E\sin\beta_1 t \end{bmatrix}$. Thus the proof is completed.

In the next theorem, we will derive the switching law of switched system which have three subsystems. That is the main result of our study.

Theorem 3.2 The autonomous switched system (3) consisting of three subsystems is asymptotical-ly stabilizable if

$$\frac{\alpha_3(\theta_2 - \theta_3)}{\beta_3} < -\frac{\alpha_1\theta_1}{\beta_1} - \frac{\alpha_2(\theta_1 - \theta_2)}{\beta_2} - \ln\sqrt{\cos^2(\theta_1) + E^2\sin^2(\theta_1)} - \ln c\,.$$

θ_1 is the angle between v^+ and v^- line, θ_2 is the angle between v^- and v^* line and θ_3 is the angle between v^* and v^+ line. $\theta_1 + \theta_2 + \theta_3 = \pi$ and $c > 1$.

Proof. By using the transformation and rotation matrix, the switched system which has three subsystems is described by

$$QA_1Q^{-1} = \begin{bmatrix} \alpha_1 & \beta_1 / E \\ -E\beta_1 & \alpha_1 \end{bmatrix},\ QA_2Q^{-1} = \begin{bmatrix} \alpha_2 & \beta_2 \\ -\beta_2 & \alpha_2 \end{bmatrix},\ A_3 = \begin{bmatrix} \alpha_3 & \beta_3 \\ -\beta_3 & \alpha_3 \end{bmatrix}$$

For a general 2×2 nonsingular matrices, the matrices could be transformed into the eigenvalue form. So we let A_3 is the form as above for ease in the analysis.

Let $(\rho(t), \theta(t))$ be the solution of the switched system in polar coordinate with initial condition $\rho(0) = 1$ and $\theta(0) = 0$.

The system will use subsystem 1 at which the trajectory intersect the v^-, thus $t \geq 0$ and $t < t_1$. By Lemma 3.2, the system obeys the equation

$$x(t) = \rho_0 e^{\alpha_1 t} \begin{bmatrix} \cos(-\beta_1 t) \\ E\sin(-\beta_1 t) \end{bmatrix}$$

At $t = t_1$, we define $\theta_1 = \beta_1 t_1$; i.e. $t_1 = \theta_1 / \beta_1$, thus we have

$$x(t_1) = e^{\alpha_1 \theta_1 / \beta_1} \begin{bmatrix} \cos(-\theta_1) \\ E\sin(-\theta_1) \end{bmatrix}$$

Therefore,

$$\rho_1(t_1) = e^{\alpha_1 \theta_1 / \beta_1} \sqrt{\cos^2(\theta_1) + E^2 \sin^2(\theta_1)}$$

After instant t_1, the system will use the subsystem 2 at which the trajectory intersect v^* line, thus $t_2 > t_1$. The system obeys the equation

$$x(t) = \rho_1(t_1) e^{\alpha_2 t_2} \begin{bmatrix} \cos(-\beta_2 t_2 + \theta_1) \\ \sin(-\beta_2 t_2 + \theta_1) \end{bmatrix}$$

At t_2 we define $\theta_2 = -\beta_2 t_2 + \theta_1$, then $t_2 = \dfrac{\theta_1 - \theta_2}{\beta_2}$.

The solution at $t = t_2$ is

$$x(t_2) = \rho_1(t_1) e^{\alpha_2 \frac{(\theta_1 - \vartheta_2)}{\beta_2}} \begin{bmatrix} \cos(\beta_2 t_2 - \theta_1) \\ \sin(\beta_2 t_2 - \theta_1) \end{bmatrix}$$

Therefore,

$$\rho_2(t_2) = e^{\frac{\alpha_1 \theta_1}{\beta_1}} e^{\alpha_2 \frac{(\theta_1 - \theta_2)}{\beta_2}} \sqrt{\cos^2(\theta_1) + E^2 \sin^2(\theta_1)}$$

After instant t_2, the system will use subsystem 3 at which the trajectory intersect v_+ line. Thus $t_3 > t_2$. The system obeys the equation

$$x(t) = \rho_2(t_2) e^{\alpha_3 t_3} \begin{bmatrix} \cos(-\beta_3 t_3 + \theta_2) \\ \sin(-\beta_3 t_3 + \theta_2 \end{bmatrix}$$

At t_3 we define $\theta_3 = -\beta_3 t_3 + \theta_2$, thus $t_3 = \dfrac{\theta_2 - \theta_3}{\beta_3}$.

The solution at $t = t_3$ is

$$x(t_3) = \rho_2(t_2) e^{\alpha_3 \frac{(\theta_2 - \vartheta_3)}{\beta_3}} \begin{bmatrix} \cos(\beta_3 t_3 - \theta_2) \\ \sin(\beta_3 t_3 - \theta_2) \end{bmatrix}$$

Therefore,

$$\rho_3(t_3) = e^{\frac{\alpha_1\theta_1}{\beta_1}} e^{\alpha_2 \frac{(\theta_1 - \theta_2)}{\beta_2}} e^{\alpha_3 \frac{\theta_2 - \theta_3}{\beta_3}} \sqrt{\cos^2(\theta_1) + E^2 \sin^2(\theta_1)}$$

By Lemma 3.1, the system is asymptotically stabilized if

$$\rho_3(t_3) < \frac{1}{c}.$$

This is equivalent to the following

$$e^{\frac{\alpha_1\theta_1}{\beta_1}} e^{\alpha_2 \frac{(\theta_1 - \theta_2)}{\beta_2}} e^{\alpha_3 \frac{(\theta_2 - \theta_3)}{\beta_3}} \sqrt{\cos^2(\theta_1) + E^2 \sin^2(\theta_1)} < \frac{1}{c},$$

$$\ln\left[e^{\frac{\alpha_1\theta_1}{\beta_1}} e^{\alpha_2 \frac{(\theta_1 - \theta_2)}{\beta_2}} e^{\alpha_3 \frac{(\theta_2 - \theta_3)}{\beta_3}} \sqrt{\cos^2(\theta_1) + E^2 \sin^2(\theta_1)} \right] < \ln c^{-1},$$

$$\frac{\alpha_1\theta_1}{\beta_1} + \frac{\alpha_2(\theta_1 - \theta_2)}{\beta_2} + \frac{\alpha_3(\theta_2 - \theta_3)}{\beta_3} + \ln\sqrt{\cos^2(\theta_1) + E^2 \sin^2(\theta_1)} < -\ln c.$$

Therefore,

$$\frac{\alpha_3(\theta_2 - \theta_3)}{\beta_3} < -\frac{\alpha_1\theta_1}{\beta_1} - \frac{\alpha_2(\theta_1 - \theta_2)}{\beta_2} - \ln\sqrt{\cos^2(\theta_1) + E^2 \sin^2(\theta_1)} - \ln c$$

where $\theta_1 + \theta_2 + \theta_3 = \pi$ and $c > 1$.

In the next section, we present an example to show the effectiveness of the result in Theorem 3.2.

Example 1 Consider a switched system that consists of three subsystems where

$$A_1 = \begin{pmatrix} -1 & 2 \\ -10 & 3 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 13 \\ -2 & 3 \end{pmatrix}.$$

The subsystem's state matrices can be transformed into

$$A_1' = \begin{pmatrix} 1 & 4/E \\ -4E & 1 \end{pmatrix},\ A_2' = \begin{pmatrix} 2 & 5 \\ -5 & 2 \end{pmatrix},\ E = 0.15$$

From the Theorem 2.1 [7], the switched systems that consists of subsystem 1 and subsystem 2 can not be stabilized when $D > 0$ and $\rho > 1$. For subsystem 1 and subsystem 2, we have $D = 10.7945$, $\rho = 2.3457$.

From our study, when the switched system has two subsystems consisting of unstable foci trajectories can not be stabilized whenever $\rho > 1$. We will be looking for the condition of subsystem 3 that makes the overall system asymptotically stabilizable.
By the Theorem 3.2

$$\frac{\alpha_3(\theta_2 - \theta_3)}{\beta_3} < -\frac{\alpha_1\theta_1}{\beta_1} - \frac{\alpha_2(\theta_1 - \theta_2)}{\beta_2} - \ln\sqrt{\cos^2(\theta_1) + E^2\sin^2(\theta_1)} - \ln c$$

where $\theta_1 + \theta_2 + \theta_3 = \pi$ and $c > 1$.

We let $\theta_1 = 0.640$, $\theta_2 = 2.027$, $\theta_3 = \pi - 0.640 - 2.027$, $\frac{\alpha_1}{\beta_1} = \frac{1}{4}$, $\frac{\alpha_2}{\beta_2} = \frac{2}{5}$, $E = 0.15$, and $c = 2$.
Then we have

$$\frac{\alpha_3}{\beta_3} < -0.05.$$

Therefore, we have a matrix of subsystem 3 that corresponds with the above condition
$A_3 = \begin{pmatrix} -5 & 10 \\ -10 & -5 \end{pmatrix}$ where $\lambda_3 = -5 \pm 10j$ are the eigenvalues.

The Figure 2 shows the trajectory of the system with initial condition $x_0 = (-10,-5)^T$.

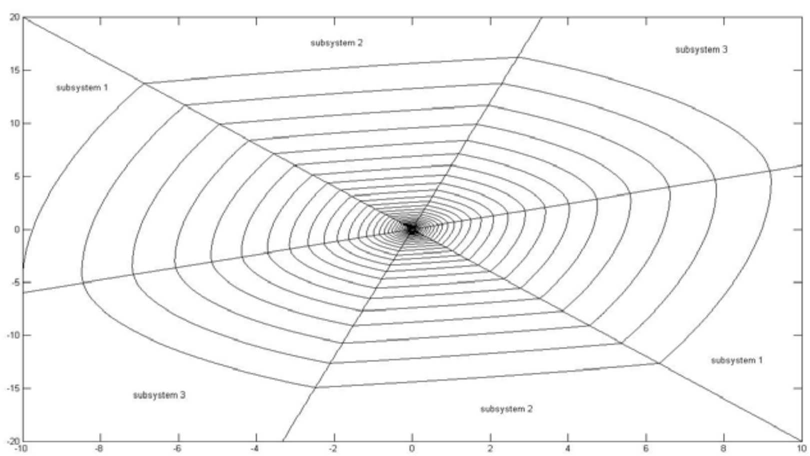

Figure 2 Trajectory of example 1

4. Conclusions

In this work, we study a problem of asymptotically stabilizing a switched system which consists of three second-order linear time invariant (LTI) subsystem. The subsystem used in this work have complex eigenvalues where two subsystem have positive real part and one subsystem has negative real part. From the previous with two unstable foci subsystems cannot be stabilized by a switching law [7]. Therefore, we propose a sufficient condition to stabilize the switched system by finding another subsystem along with a new switching law to guarantee overall switched system to be asymptotically stabilizable.

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