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A Survey of Extremal Rays for Fano Manifolds

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Abstract

In this expository note, we give an introduction to algebraic geometry with the emphasis on Fanomanifolds which forms one of the building blocks of algebraic varieties. After a brief review on the theory of intersection numbers, we introduce numerical invariants on Fano manifolds and discuss related problems.

Keywords: algebraicvarieties, Fano manifolds, extremal ray

1. Introduction

Algebraic geometry is a branch of mathematics studying the structures of algebraic varieties i.e. the set of zeros of a system of polynomial equations. It is natural to consider such varieties not in an affine space (=euclidian n-space) but in a projective space.

Let k be a field. Recall that the n-dimensional projective space $P^n(k)$ is defined as the quotient of the set

$$k^{n+1} - \{(0, 0, \cdots, 0)\}$$

by the equivalence relation:

$$(x0, x1,\cdots ,xn) = \lambda\, (y0, y1,\cdots ,yn)$$

where λ is some non-zero element of k. The ground field k is usually assumed to be algebraically closed. A simple reason is briefly explained in the book [9] Chapter I. Furthermore, if we consider the field of complex numbers C, then the study of algebraic varieties is naturally linked to other branches of mathematics e.g. complex geometry, differential geometry or topology. In this note, we put P^n: $=P^n(C)$. The points in P^n are written in the form of homogeneous coordinates: $(x0 : x1 : \cdots : xn)$ with $(x0,x1,\cdots ,xn) \in C^{n+1} - \{(0,0,\cdots 0)\}$. For example, in P^2we have $(1:i : -1) = (-i : 1 : i)$ since $(1, i, -1) = i \cdot (-i, 1, i)$ in C^3. Consider the algebraic variety $X:=\{(x:y:z:w)\in P^3|xz-yw=0\}$.The dimension of X is two, because it is defined by one equation in the three dimensional projective space. To investigate the structure of X, we consider the curve$C:=\{(x:y:z:w)\in P^3|x=y=0\}$which is contained in X. We have the self-intersection number $(C^2) = 0$ (see [1] Chapter I for the intersection theory on algebraic surfaces). This does not happen for any curve on P^2 (any curve on P^2has strictly positive self-intersection number). We conclude that X is a different algebraic variety from P^2. However, X and $P^2$are birational. Indeed, if we consider the projectionmap: φ: P3→P2 given by

$$(x: y : z : w) \to (x : y : z),$$

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then the restriction to X is a birational map. This X is called quadric surface because it is defined by an equation of degree two. The main problem of algebraic geometry is the birational classification of algebraic varieties (see [6] Chapter 1.8). For example, P2 and the X defined above are in a same equivalence class in this point of view. It is known that the (smooth) cubic surface is also birationalto P2. However, a surface defined by equations of degree 4 or more 12 are not birational to P2 and hence are in different classes (this can be verified by using numerical invariants e.g. ”Kodaira dimension” (see [7]). In higher dimensions, the birational classification is very compli- cated. However, there is a method to get a ”simple” model of algebraic varieties in each birational class (see [4]).

2. Fano manifolds

A Fano manifold is a projective variety whose anticanonical bundle is ample (see [5] Chapter 1 for a definition of ”ample”). The anticanonical bundle of a projective space is expressed as

$$-K= (n + 1) H$$

where H is a hyperplane in Pn. Since H is (very) ample, we conclude that Pnis a Fano manifold. Let X be a hypersurface of degree d in P^{n+1}, then, by ”adjunction formula” we have

$$-K_X= (-K-X)|_X= ((n+2)H -dH)|_X= (n+2-d)H|_X.$$

Hence X is a Fano manifold if and only if $n+2-d > 0$, i.e. $d \leq n+1$. From this example, we can say that (the degree of) Fano manifolds is bounded in some sense. This observation is correct in general: the anticanonical degree of the Fano manifolds of dimension n is bounded by a function of n. This bound implies that there are only finitely many types of Fano manifolds in each dimension. This very deep result has been shown using the theory of rational connectedness (see [3] p.251 for a brief history)

The classification of Fano manifolds up to dimension three is following (see [8] and references therein):

- (dimX = 1) X is isomorphic to P1
- (dim X = 2) these are ”Del Pezzo surfaces” : P2, P1 × P1, theblow-ups of P2 at most 8 general points
- (dim X = 3) there are 109 deformation types of smooth Fano3-folds.

In the dimension greater than or equal to four, there are only partial results on explicit classifications of Fano manifolds.The study of Fano manifolds is very important in algebraic geometry. According to the minimal model theory (MMP), every uniruled varietyis birationally equivalent to a Fanofibration i.e. a fiber space whose fibers are Fano manifolds (e.g. the projection Pn× Y → Y is a Fanofibration since the fiber Pnis a Fano manifold). Fano manifolds itself have rich geometry: they are of great interest concerning the rationality problem of algebraic varieties, the theory of algebraic groups and the Einstein metric in differential geometry etc. (see [10]).

3. Cone of curves and extremal rays for Fano manifolds

Let X be a (smooth) projective variety of dimension n. A divisor Don X is a formal sum of codimension onesubvarietieswith integral (rational or real) coefficients. A 1-cycle C on X is a formal sum of 1-dimensional subvarieties with integral (rational or real) coefficients. Divisors and 1-cycles arerelated by the intersection pairing:

$$(D, C) \mapsto D \cdot C$$

which defines a bilinear map. Remark: the intersection number $D \cdot C$ is a number of points in the intersection $D \cap C$ with multiplicity. For example, let X be the complex 3-space C3, D a hyperplane say $D:= \{(x,y,z) \mid z = 0\}$ and C the curve (1-cycle) defined by $\{(x, y, z) \mid x = 0, z = y^2\}$.

Then, the intersection $D \cap C$ consists of one point (0, 0, 0). However, we have $D \cdot C = 2$ because D has multiplicity two at the point (0, 0, 0).
Two 1-cycles Aand Bare said to be numerically equivalent and denoted by $A \equiv B$if we have

$$D \cdot A = D \cdot B$$

for any divisor D on X.We define the (real) vector space N1(X) := { 1-cycles on X}/ ≡whose dimension is known to be finite. Its dimension denoted by $\rho(X)$ is called Picard number. Note that for two classes [A] and [B] in N1(X), we have [A] = [B]ifandonlyif $A \equiv B$.There is a subcone called Kleiman-Mori cone or cone of curves:

$$NE(X) := \{[C] \in N1(X) \mid C\text{: 1-cycle with positive coefficients}\}$$

which plays an important role in the minimal model theory. We often consider its closure NE(X) in euclidian topology of N1(X).
An extremal ray R of NE(X) is a half line satisfying the following:

$$u, v \in NE(X) \text{ and } u + v \in R \implies u, v \in R.$$

It is known that the Kleiman-Mori cone of a Fano manifold is polyhedral and each extremal ray is generated by a rational curve (i.e. a curve birational to P1). Let X be a Fano manifold. We consider the following two natural numbers:

- a(X) := the number of extremal rays of NE(X)
- l(X) := the minimum of $-K \cdot C$ where C is a rational curvein some extremal ray

By definition, we have $a(X) \geq \rho(X)$. Using a classification result from [11], we get the following:
Theorem Let X be a smooth Fano 4-fold having a birational con- traction. Assume $l(X) \geq 2$. Then we have $a(X) = \rho(X)$.

4. Conclusions

In general, a(X) is greater than the Picard number $\rho(X)$. If X is a cubic surface in P3, there are 27 lines contained in X. Let C1, C2, C27 be these rational curves. By an elementary observation, we see that each [Ci] generates an extremal ray of NE(X). Therefore, we obtain a(X) = 27. On the other hand, the cubic surface is obtained by blowing up 6 points in P2. Hence we have $\rho(X) = 7$.
It is natural to ask the following:

Problem Find an explicit bound for the number of extremal rays for Fano manifolds.

Remark: The explicit upper bound for the Picard number of Fano manifolds of dimension n is conjectured to be 9n/2 (for n even), 9(n − 1)/2 + 1 (for n odd). However, it is only verified up to dimension three (see [2]).

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