A Revision of the Sensory Evaluation by Paired Comparison

Katsumi UJIIE

Research Institute of Education, Tokai University
1117, Kitakaname, Hiratsuka, Kanagawa, 259-1292 Japan
E-mail ; ujiie@keyaki.cc.u-tokai.ac.jp

Abstract

It is difficult to rank the objects which we can't physically observe by our sense. Of course, to rank all of the incentives we receive is almost impossible, but to rank them partially is possible by the paired comparison, a typical method of partial ranking, which is interesting practically as well as theoretically.

This paper shows that the paired comparison can be applied to Böckenholt's partial ranking data, that it is to say, it shows that is possible to use Thurstone's Law of Comparative Judgement for analysis, though there are several techniques in the paired comparison.

We want to rank all of the incentives by psychological measurements which are done by making the numerical value of human incentives. The Thurstone's Law of Comparative Judgement, however, needs a lot of parameters, so it is difficult to make the psychological measurments. We revised the judgement by making the number of parameters fewer. Accordingly, in the case of the fewer parameters we were able to make the measurement value of caseIII and caseV of the Thurstone's Law of Comparative Judgement and to show the presumption and the measurement of the parameters by the revised measurement value before analyzing the practical data.

As a result, we were able to show that the paired comparison can be applied to the ranking of the incentive objects.

Key words : Sensory test, paired comparison, estimation and test.

1. Introduction

The sensory test is carried out about the human sense(taste, hearing, sight and so on). Some objects occur the difference even if the conditions such as temperature, humidity, weather are the same case. And some objects are influenced by fatigue, tension and so on even if the same objects. Also it seems to us that the objects occur the conscious inclination, because objects depend on the sense. Thus we can't judge in many cases easily, because the sensory test tends to be influenced by various things and to change the standard of evaluation. But we can compare with two things and evaluate by the order and the examination easily. We say this evaluation method as a paired comparison method. This method catchs the characteristic of sensory test and is often used actually.

By the way, the paired comparison method shows as a pair of two by two in the object of measurement and obtains the result of comparison conclusion from objects by the evaluation standard(for example, desirability, superiority or inferiority). There is a method expresses the examination or order for the result of the comparison conclusion. The method compared and evaluated by the examination is Scheffé's paired comparison. This method gives the examination by the step for examples 3,5,7 and evaluates the result of comparison by the examination. Recently various methods are studied for the cases of considering the combination effect and order effect, the neglection of the order effect, the small sample as a construction. The method compared and evaluated by the order is Bradley's method and Thurstone's method of comparison conclusion. Bradley's method evaluates by ○ or × whether either of them is suitable for standard of evaluation. In this case, it is not almost considerd the order effect. But there are some methods extended to consider the order effect. Thurstone's method of comparison conclusion orders by the standard of evaluation and carries out the scale constructure based on the evaluation standard by the comparsion result. But it is so difficult to scale because the number of the parameter is so many. Thus Thurstone proposed the five cases for scaling. We describe the rule of comparison decision as follows.

$$R_i - R_j = z_{ij}\sqrt{\sigma_i^2 + \sigma_j^2 - 2\rho_{ij}\sigma_i\sigma_j}$$

where R_i and R_j are psychological scale value which are given for the stimulus O_i and O_j as a characteristic, respectively. z_{ij} is a value on the x-axis for the ratio $p_{i>j}$. σ_i and σ_j are standard deviation of the distribution X_i and X_j, respectively. ρ_{ij} is a correlation coefficient between the distributions X_i and X_j.

Case Ⅰ. We apply the rule for only one subject. The subject carries out the repetition decision in this case.

Case Ⅱ. We apply the rule for many subjects. Each subject carries out the decision once for the pair of each stimulus.

Case Ⅲ. We suppose $\rho_{ij} = 0$.

$$R_i - R_j = z_{ij}\sqrt{\sigma_i^2 + \sigma_j^2}.$$

Case IV. We suppose that there is nearly no difference between standard deviations in addition to $\rho_{ij} = 0$. $\sigma_j = \sigma_i + d(d$ is a small enough value compared with $\sigma_i)$.

$$R_i - R_j = \frac{1}{\sqrt{2}} z_{ij}(\sigma_i + \sigma_j).$$

Case V. We suppose $\sigma_i = \sigma_j$ in addition to $\rho_{ij} = 0$.

$$R_i - R_j = z_{ij}.$$

Above mentioned, we can deal with the rule of comparison decision easily. And we realize that there are various methods even if a paird comparison method is said with a word.

2. The ranking model

Thurstone supposes the random valiable X_i as

$$X_i = \mu_i + \epsilon_i$$

where $\epsilon_i (i = 1, 2, \cdots, t)$ is distributed according to multivariate normal with mean vector $\mathbf{0}$ and covariance matrix $\mathbf{\Sigma}$. Consequently, the probability for observing the rank order $s_l = (1, 2, \cdots, t)$ for a given set of alternatives is

$$\Pr(s_l | O_1, O_2, \cdots, O_t) = \Pr\{(X_1 - X_2 > 0) \cap (X_2 - X_3 > 0) \cap \cdots \cap (X_{t-1} - X_t > 0)\}$$

where $\boldsymbol{X} = (X_1, X_2, \cdots, X_t)'$ follows a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{\Sigma}$ with elements σ_{ij}, that is,

$$\boldsymbol{X} \sim N(\boldsymbol{\mu}, \mathbf{\Sigma}).$$

The distribution of its differences is also multivariate normal,

$$\boldsymbol{C}_l \boldsymbol{X} \sim N(\boldsymbol{C}_l \boldsymbol{\mu}, \boldsymbol{C}_l \mathbf{\Sigma} \boldsymbol{C}_l')$$

where $\boldsymbol{C}_l$ is a matrix of contrasts

$$\boldsymbol{C}_l = \begin{pmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -1 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}$$

Thus, for choice sets with t alternatives, the probability of a certain rank order is given by

$$\Pr(s_l | O_1, O_2, \cdots, O_t) = \Phi_{t-1}(\boldsymbol{D}\boldsymbol{C}_l \boldsymbol{\mu}; \boldsymbol{D}\boldsymbol{C}_l \mathbf{\Sigma} \boldsymbol{C}_l' \boldsymbol{D})$$

where Φ_{t-1} denotes a $(t-1)$-variate normal distribution function and $\boldsymbol{D} = diag\{(\boldsymbol{C}_l\Sigma\boldsymbol{C}_l')\}^{-\frac{1}{2}}$.

3. The estimation and test

3.1 The estimation and test on the ranking

Maximum likelihood methods

The logarithmic likelihood function for the ranking

$$\log L = c + \sum_{l=1}^{t!} r(s_l|O_1, O_2, \cdots, O_t) \log \Pr(s_l|O_1, O_2, \cdots, O_t)$$

where

c is constant

$r(s_l|O_1, O_2, \cdots, O_t)$ is the frequency of observations indicate the ranking pattern s_l

$\Pr(s_l|O_1, O_2, \cdots, O_t)$ is the probability observes the ranking pattern s_l

The test of goodness of fit for the ranking model.
Using the likelihood ratio χ^2 statistic.

$$G^2 = 2\sum_{l=1}^{t!} r(s_l|O_1, O_2, \cdots, O_t) \log\left(\frac{r(s_l|O_1, O_2, \cdots, O_t)}{r^e(s_l|O_1, O_2, \cdots, O_t)}\right)$$

where

$r^e(s_l|O_1, O_2, \cdots, O_t)$ refers to the expected frequencies indicates the ranking pattern s_l

G^2 is distributed according to χ^2-distribution with $(t! - h - 1)$ degrees of freedom

h refers to the number of parameters

3.2 The estimation and test on the partial ranking

When it is difficult to order the overall the ranking and the number of object is so many, we carry out order the ranking and analyze in any choice set contains only m objects from the full choice set $\{O_1, O_2, \cdots, O_t\}$. The estimation of the parameter for the partial raking data uses the maximum likelihood method similarly. Then the logarithmic likelihood function for partial rankings can be written as

$$\log L = c + \sum_{f=1}^{Q}\sum_{l=1}^{m!} r(s_{fl}|W_{O,f}) \log \Pr(s_{fl}|W_{O,f})$$

where c is constant. $W_{O,f}$ be an arbitary choice set containing no more than m objects from full choice set $\{O_1, O_2, \cdots, O_t\}$. s_{fl} is the lth rank order of the choice set $W_{O,f}$. $r(s_{fl}|W_{O,f})$ is the frequency of observations indicate the ranking pattern s_{fl} on $W_{O,f}$. Q is the number of subsets of size m.

The test of goodness of fit for the partial ranking data are available using the likelihood ratio(LR) χ^2 statistic too. For partial ranking data, G^2 can be written as

$$G^2 = 2\sum_{f=1}^{Q}\sum_{l=1}^{m!} r(s_{fl}|W_{O,f}) \log\left(\frac{r(s_{fl}|W_{O,f})}{r^e(s_{fl}|W_{O,f})}\right)$$

where $r^e(s_{fl}|W_{O,f})$ is the expected frequencies indicate the ranking pattern s_{fl} and G^2 will be χ^2-distributed with $[Q(m!-1)-h]$ degrees of freedom and h refers to the number of parameters.

3.3 The special case of partial ranking data <paired comparison method >

The case of $m = 2$ in the method for the partial ranking is the paired comparison method. Here we describe the case of $t = 6$ and the method of estimation and test of parameter for Case V and Case III of the rule of comparison decision of Thurstone.

We suppose Q is the number of the subset the size of $m = 2$. And we suppose s_{fl} is the ranking pattern in the choice set $W_{O,f}$ and $W_{O,f}$ is an any choice set the size of $m = 2$ from full choice set $\{O_1, O_2, O_3, O_4, O_5, O_6\}$, then

$$\begin{aligned}
&W_{O,1} = \{O_1, O_2\},\ \ W_{O,2} = \{O_1, O_3\},\ \ W_{O,3} = \{O_1, O_4\},\\
&W_{O,4} = \{O_1, O_5\},\ \ W_{O,5} = \{O_1, O_6\},\ \ W_{O,6} = \{O_2, O_3\},\\
&W_{O,7} = \{O_2, O_4\},\ \ W_{O,8} = \{O_2, O_5\},\ \ W_{O,9} = \{O_2, O_6\},\\
&W_{O,10} = \{O_3, O_4\},\ \ W_{O,11} = \{O_3, O_5\},\ \ W_{O,12} = \{O_3, O_6\},\\
&W_{O,13} = \{O_4, O_5\},\ \ W_{O,14} = \{O_4, O_6\},\ \ W_{O,15} = \{O_5, O_6\}.
\end{aligned}$$

The logarithmic likelihood function for paired comparison is given by

$$\begin{aligned}
\log L &= c + \sum_{f=1}^{Q}\sum_{l=1}^{m!} r(s_{fl}|W_{O,f}) \log \Pr(s_{fl}|W_{O,f})\\
&= c + \sum_{f=1}^{15}\sum_{l=1}^{2!} r(s_{fl}|W_{O,f}) \log \Pr(s_{fl}|W_{O,f}).
\end{aligned}$$

Then we derive the estimation of probability where O_i is desirable when we compare with the object O_i and O_j.

3.4 The estimation of Case V

The table of $p_{ij} = \Pr(O_i \succ O_j)$.

	O_1	O_2	O_3	O_4	O_5	O_6
O_1	$\Pr(O_1 \succ O_1)$	$\Pr(O_2 \succ O_1)$	$\Pr(O_3 \succ O_1)$	$\Pr(O_4 \succ O_1)$	$\Pr(O_5 \succ O_1)$	$\Pr(O_6 \succ O_1)$
O_2	$\Pr(O_1 \succ O_2)$	$\Pr(O_2 \succ O_2)$	$\Pr(O_3 \succ O_2)$	$\Pr(O_4 \succ O_2)$	$\Pr(O_5 \succ O_2)$	$\Pr(O_6 \succ O_2)$
O_3	$\Pr(O_1 \succ O_3)$	$\Pr(O_2 \succ O_3)$	$\Pr(O_3 \succ O_3)$	$\Pr(O_4 \succ O_3)$	$\Pr(O_5 \succ O_3)$	$\Pr(O_6 \succ O_3)$
O_4	$\Pr(O_1 \succ O_4)$	$\Pr(O_2 \succ O_4)$	$\Pr(O_3 \succ O_4)$	$\Pr(O_4 \succ O_4)$	$\Pr(O_5 \succ O_4)$	$\Pr(O_6 \succ O_4)$
O_5	$\Pr(O_1 \succ O_5)$	$\Pr(O_2 \succ O_5)$	$\Pr(O_3 \succ O_5)$	$\Pr(O_4 \succ O_5)$	$\Pr(O_5 \succ O_5)$	$\Pr(O_6 \succ O_5)$
O_6	$\Pr(O_1 \succ O_6)$	$\Pr(O_2 \succ O_6)$	$\Pr(O_3 \succ O_6)$	$\Pr(O_4 \succ O_6)$	$\Pr(O_5 \succ O_6)$	$\Pr(O_6 \succ O_6)$

The table of z_{ij} as $\Pr(O_i \succ O_j) = \Phi(z_{ij})$.

	O_1	O_2	O_3	O_4	O_5	O_6
O_1	z_{11}	z_{21}	z_{31}	z_{41}	z_{51}	z_{61}
O_2	z_{12}	z_{22}	z_{32}	z_{42}	z_{52}	z_{62}
O_3	z_{13}	z_{23}	z_{33}	z_{43}	z_{53}	z_{63}
O_4	z_{14}	z_{24}	z_{34}	z_{44}	z_{54}	z_{64}
O_5	z_{15}	z_{25}	z_{35}	z_{45}	z_{55}	z_{65}
O_6	z_{16}	z_{26}	z_{36}	z_{46}	z_{56}	z_{66}
total	$\sum z_{1j}$	$\sum z_{2j}$	$\sum z_{3j}$	$\sum z_{4j}$	$\sum z_{5j}$	$\sum z_{6j}$
average	$\bar{z}_1$	$\bar{z}_2$	$\bar{z}_3$	$\bar{z}_4$	$\bar{z}_5$	$\bar{z}_6$
R_i	$\bar{z}_1 - \bar{z}_1$	$\bar{z}_2 - \bar{z}_1$	$\bar{z}_3 - \bar{z}_1$	$\bar{z}_4 - \bar{z}_1$	$\bar{z}_5 - \bar{z}_1$	$\bar{z}_6 - \bar{z}_1$

We derive $z'_{ij} = R_i - R_j$ by this table and derive the estimation of the probability by the table of normal distribution. Thus the estimation of probability where O_i is desirable when we compare with the object O_i and O_j is given by

$$\hat{\Pr}(O_i \succ O_j) = \Phi(z'_{ij}) = \Phi(R_i - R_j)$$

where R_i and R_j are psychological scale value which are given for O_i and O_j as a characteristic respectively. z_{ij} is a value on the x-axis for the ratio $p_{i>j}$.

3.5 The estimation of Case Ⅲ

If we carry out the scaling by Case Ⅲ, we need to estimate the standard deviation for the stimulus. Burros[4] proposed the following several equation.

$$\sigma_i = \frac{c}{V_{z_i}}$$

where σ_i is the standard deviation of X_i. V_{z_i} is the variance of each O_i for the table of z_{ij}.

$$c = \frac{t}{\sum \frac{1}{V_{z_i}}}.$$

We derive V_{z_i} and σ_i by the table of Case V and make the table of the value of $z_{ij}\sqrt{\sigma_i^2+\sigma_j^2}$.

The table of $z_{ij}\sqrt{\sigma_i^2+\sigma_j^2}$.

	O_1	O_2	O_3	O_4	O_5	O_6
O_1	$z_{11}\sqrt{\sigma_1^2+\sigma_1^2}$	$z_{21}\sqrt{\sigma_2^2+\sigma_1^2}$	$z_{31}\sqrt{\sigma_3^2+\sigma_1^2}$	$z_{41}\sqrt{\sigma_4^2+\sigma_1^2}$	$z_{51}\sqrt{\sigma_5^2+\sigma_1^2}$	$z_{61}\sqrt{\sigma_6^2+\sigma_1^2}$
O_2	$z_{12}\sqrt{\sigma_1^2+\sigma_2^2}$	$z_{22}\sqrt{\sigma_2^2+\sigma_2^2}$	$z_{32}\sqrt{\sigma_3^2+\sigma_2^2}$	$z_{42}\sqrt{\sigma_4^2+\sigma_2^2}$	$z_{52}\sqrt{\sigma_5^2+\sigma_2^2}$	$z_{62}\sqrt{\sigma_6^2+\sigma_2^2}$
O_3	$z_{13}\sqrt{\sigma_1^2+\sigma_3^2}$	$z_{23}\sqrt{\sigma_2^2+\sigma_3^2}$	$z_{33}\sqrt{\sigma_3^2+\sigma_3^2}$	$z_{43}\sqrt{\sigma_4^2+\sigma_3^2}$	$z_{53}\sqrt{\sigma_5^2+\sigma_3^2}$	$z_{63}\sqrt{\sigma_6^2+\sigma_3^2}$
O_4	$z_{14}\sqrt{\sigma_1^2+\sigma_4^2}$	$z_{24}\sqrt{\sigma_2^2+\sigma_4^2}$	$z_{34}\sqrt{\sigma_3^2+\sigma_4^2}$	$z_{44}\sqrt{\sigma_4^2+\sigma_4^2}$	$z_{54}\sqrt{\sigma_5^2+\sigma_4^2}$	$z_{64}\sqrt{\sigma_6^2+\sigma_4^2}$
O_5	$z_{15}\sqrt{\sigma_1^2+\sigma_5^2}$	$z_{25}\sqrt{\sigma_2^2+\sigma_5^2}$	$z_{35}\sqrt{\sigma_3^2+\sigma_5^2}$	$z_{45}\sqrt{\sigma_4^2+\sigma_5^2}$	$z_{55}\sqrt{\sigma_5^2+\sigma_5^2}$	$z_{65}\sqrt{\sigma_6^2+\sigma_5^2}$
O_6	$z_{16}\sqrt{\sigma_1^2+\sigma_6^2}$	$z_{26}\sqrt{\sigma_2^2+\sigma_6^2}$	$z_{36}\sqrt{\sigma_3^2+\sigma_6^2}$	$z_{46}\sqrt{\sigma_4^2+\sigma_6^2}$	$z_{56}\sqrt{\sigma_5^2+\sigma_6^2}$	$z_{66}\sqrt{\sigma_6^2+\sigma_6^2}$
total	$\sum z_{1j}\sqrt{\sigma_1^2+\sigma_j^2}$	$\sum z_{2j}\sqrt{\sigma_2^2+\sigma_j^2}$	$\sum z_{3j}\sqrt{\sigma_3^2+\sigma_j^2}$	$\sum z_{4j}\sqrt{\sigma_4^2+\sigma_j^2}$	$\sum z_{5j}\sqrt{\sigma_5^2+\sigma_j^2}$	$\sum z_{6j}\sqrt{\sigma_6^2+\sigma_j^2}$
average	$\bar{z}_1$	$\bar{z}_2$	$\bar{z}_3$	$\bar{z}_4$	$\bar{z}_5$	$\bar{z}_6$
R_i	$\bar{z}_1-\bar{z}_1$	$\bar{z}_2-\bar{z}_1$	$\bar{z}_3-\bar{z}_1$	$\bar{z}_4-\bar{z}_1$	$\bar{z}_5-\bar{z}_1$	$\bar{z}_6-\bar{z}_1$

We derive $z'_{ij} = (R_i - R_j)/\sqrt{\sigma_i^2+\sigma_j^2}$ similar to Case V. Thus the estimation of probability where O_i is desirable when we compare with the object O_i and O_j is given by

$$\hat{\Pr}(O_i \succ O_j) = \Phi(z'_{ij}) = \Phi\left(\frac{R_i - R_j}{\sqrt{\sigma_i^2+\sigma_j^2}}\right)$$

3.6 The test

We test by using the likelihood ratio χ^2 statistic

$$G^2 = 2\sum_{f=1}^{Q}\sum_{l=1}^{m!} r(s_{fl}|W_{O,f})\log\left(\frac{r(s_{fl}|W_{O,f})}{r^e(s_{fl}|W_{O,f})}\right)$$

for paired comparison is distributed according to χ^2-distribution with $[Q(m!-1)-h]$ degrees of freedom where $r(s_{fl}|W_{O,f})$ is the frequency of observation and $r^e(s_{fl}|W_{O,f})$ is the expecterd frequencies. When $Q=15, t=6, m=2$, the test statistic

$$G^2 = 2\sum_{f=1}^{15}\sum_{l=1}^{2!} r(s_{fl}|W_{O,f})\log\left(\frac{r(s_{fl}|W_{O,f})}{r^e(s_{fl}|W_{O,f})}\right)$$

is distributed according to χ^2-distribution with 9 degrees of freedom.

4. The numerical example

4.1 The case of the number of objects are 6

We carried out the paired comparison 120 times about the desirability of juice for 15 subsets from 6 kinds of juice $\{O_1, O_2, O_3, O_4, O_5, O_6\}$ respectively. As a result, we gained the frequency of observation $r(s_{fl}|W_{O,f})$ as follows.

$$r((1,2)|\{O_1,O_2\}) = 65, \quad r((2,1)|\{O_1,O_2\}) = 55,$$

$$r((1,3)|\{O_1,O_2\}) = 72, \quad r((3,1)|\{O_1,O_2\}) = 48,$$

$$r((1,4)|\{O_1,O_2\}) = 74, \quad r((4,1)|\{O_1,O_2\}) = 46,$$

$$r((1,5)|\{O_1,O_2\}) = 71, \quad r((5,1)|\{O_1,O_2\}) = 49,$$

$$r((1,6)|\{O_1,O_2\}) = 53, \quad r((6,1)|\{O_1,O_2\}) = 67,$$

$$r((2,3)|\{O_1,O_2\}) = 54, \quad r((3,2)|\{O_1,O_2\}) = 66,$$

$$r((2,4)|\{O_1,O_2\}) = 53, \quad r((4,2)|\{O_1,O_2\}) = 67,$$

$$r((2,5)|\{O_1,O_2\}) = 82, \quad r((5,2)|\{O_1,O_2\}) = 38,$$

$$r((2,6)|\{O_1,O_2\}) = 59, \quad r((6,2)|\{O_1,O_2\}) = 61,$$

$$r((3,4)|\{O_1,O_2\}) = 48, \quad r((4,3)|\{O_1,O_2\}) = 72,$$

$$r((3,5)|\{O_1,O_2\}) = 70, \quad r((5,3)|\{O_1,O_2\}) = 50,$$

$$r((3,6)|\{O_1,O_2\}) = 46, \quad r((6,3)|\{O_1,O_2\}) = 74,$$

$$r((4,5)|\{O_1,O_2\}) = 83, \quad r((5,4)|\{O_1,O_2\}) = 37,$$

$$r((4,6)|\{O_1,O_2\}) = 59, \quad r((6,4)|\{O_1,O_2\}) = 61,$$

$$r((5,6)|\{O_1,O_2\}) = 34, \quad r((6,5)|\{O_1,O_2\}) = 86.$$

4.2 The estimation of Case V

We represent the following table for the probability where the object O_i is more desirable than O_j by the frequency of observation.

The table of $\Pr(O_i \succ O_j)$.

$\Pr(O_i \succ O_j)$	O_1	O_2	O_3	O_4	O_5	O_6
O_1	0.500	0.458	0.400	0.383	0.408	0.558
O_2	0.542	0.500	0.550	0.558	0.317	0.508
O_3	0.600	0.450	0.500	0.600	0.417	0.617
O_4	0.617	0.442	0.400	0.500	0.308	0.508
O_5	0.592	0.683	0.583	0.692	0.500	0.717
O_6	0.442	0.492	0.383	0.492	0.283	0.500

By the table of $\Pr(O_i \succ O_j)$, we represent the following table for z_{ij} where $\Pr(O_i \succ O_j) = \Phi(z_{ij})$ by the table of normal distribution.

The table of z_{ij} of $\Pr(O_i \succ O_j) = \Phi(z_{ij})$.

z_{ij}	O_1	O_2	O_3	O_4	O_5	O_6
O_1	0.000	-0.106	-0.253	-0.298	-0.233	0.146
O_2	0.106	0.000	0.126	0.146	-0.476	0.020
O_3	0.253	-0.126	0.000	0.253	-0.210	0.298
O_4	0.298	-0.146	-0.253	0.000	-0.502	0.020
O_5	0.233	0.476	0.210	0.502	0.000	0.574
O_6	-0.146	-0.020	-0.298	-0.020	-0.574	0.000
total	0.744	0.078	-0.468	0.583	-1.995	1.058
average	0.124	0.013	-0.078	0.097	-0.333	0.176
R_i	0.457	0.346	0.255	0.430	0.000	0.509

We derive $z'_{ij} = R_i - R_j$ by this table and the estimation of probability by the table of normal distribution. When Case V, the estimation of probability where the object O_i is more desirable than O_j is given by

$$\hat{\Pr}(O_i \succ O_j) = \hat{\Pr}((i,j)|\{O_i, O_j\}) = \Phi(z'_{ij}) = \Phi(R_i - R_j).$$

Thus the estimations of probability in Case V are given by

$$\hat{\Pr}(O_1 \succ O_2) = \Phi(0.111) = 0.5442, \quad \hat{\Pr}(O_2 \succ O_1) = \Phi(-0.111) = 0.4558,$$

$$\hat{\Pr}(O_1 \succ O_3) = \Phi(0.202) = 0.5801, \quad \hat{\Pr}(O_3 \succ O_1) = \Phi(-0.202) = 0.4199,$$

$$\hat{\Pr}(O_1 \succ O_4) = \Phi(0.027) = 0.5108, \quad \hat{\Pr}(O_4 \succ O_1) = \Phi(-0.027) = 0.4892,$$

$$\hat{\Pr}(O_1 \succ O_5) = \Phi(0.457) = 0.6761, \quad \hat{\Pr}(O_5 \succ O_1) = \Phi(-0.457) = 0.3239,$$

$$\hat{\Pr}(O_1 \succ O_6) = \Phi(-0.052) = 0.4793, \quad \hat{\Pr}(O_6 \succ O_1) = \Phi(0.052) = 0.5207,$$

$$\hat{\Pr}(O_2 \succ O_3) = \Phi(0.091) = 0.5363, \quad \hat{\Pr}(O_3 \succ O_2) = \Phi(-0.091) = 0.4637,$$

$$\hat{\Pr}(O_2 \succ O_4) = \Phi(-0.084) = 0.4665, \quad \hat{\Pr}(O_4 \succ O_2) = \Phi(0.084) = 0.5335,$$

$$\hat{\Pr}(O_2 \succ O_5) = \Phi(0.346) = 0.6353, \quad \hat{\Pr}(O_5 \succ O_2) = \Phi(-0.346) = 0.3647,$$

$$\hat{\Pr}(O_2 \succ O_6) = \Phi(-0.163) = 0.4352, \quad \hat{\Pr}(O_6 \succ O_2) = \Phi(0.163) = 0.5648,$$

$$\hat{\Pr}(O_3 \succ O_4) = \Phi(-0.175) = 0.4305, \quad \hat{\Pr}(O_4 \succ O_3) = \Phi(0.175) = 0.5695,$$

$$\hat{\Pr}(O_3 \succ O_5) = \Phi(0.255) = 0.6007, \quad \hat{\Pr}(O_5 \succ O_3) = \Phi(-0.255) = 0.3993,$$

$$\hat{\Pr}(O_3 \succ O_6) = \Phi(-0.254) = 0.3997, \quad \hat{\Pr}(O_6 \succ O_3) = \Phi(0.254) = 0.6003,$$

$$\hat{\Pr}(O_4 \succ O_5) = \Phi(0.430) = 0.6664, \quad \hat{\Pr}(O_5 \succ O_4) = \Phi(-0.430) = 0.3336,$$

$$\hat{\Pr}(O_4 \succ O_6) = \Phi(-0.079) = 0.4685, \quad \hat{\Pr}(O_6 \succ O_4) = \Phi(0.079) = 0.5315,$$

$$\hat{\Pr}(O_5 \succ O_6) = \Phi(-0.509) = 0.3053, \quad \hat{\Pr}(O_6 \succ O_5) = \Phi(0.509) = 0.6947.$$

4.3 The test of Case V

We carry out the likelihood ratio test

$$\text{null hypothesis } H_0 : \Pr(O_i \succ O_j) = \hat{\Pr}((i,j)|\{O_i, O_j\}).$$

Here we test by using the likelihood ratio χ^2 statistic

$$G^2 = 2\sum_{f=1}^{Q}\sum_{l=1}^{m!} r(s_{fl}|W_{O,f}) \log\left(\frac{r(s_{fl}|W_{O,f})}{r^e(s_{fl}|W_{O,f})}\right)$$

is distributed according to χ^2-distribution with $[Q(m!-1)-h]$ degrees of freedom for the partial ranking. When $Q = 15, t = 6, m = 2$, the test statistic

$$G^2 = 2\sum_{f=1}^{15}\sum_{l=1}^{2!} r(s_{fl}|W_{O,f}) \log\left(\frac{r(s_{fl}|W_{O,f})}{r^e(s_{fl}|W_{O,f})}\right)$$

is distributed according to χ^2-distribution with 9 degrees of freedom.

If we calculate the frequency of expectation, we gain the following results.

$r^e((1,2)|\{O_1,O_2\}) = 120\times0.5442 = 65.304,\ r^e((2,1)|\{O_1,O_2\}) = 120\times0.4558 = 54.696,$

$r^e((1,3)|\{O_1,O_2\}) = 120\times0.5801 = 69.612,\ r^e((3,1)|\{O_1,O_2\}) = 120\times0.4199 = 50.388,$

$r^e((1,4)|\{O_1,O_2\}) = 120\times0.5108 = 61.296,\ r^e((4,1)|\{O_1,O_2\}) = 120\times0.4892 = 58.704,$

$r^e((1,5)|\{O_1,O_2\}) = 120\times0.6761 = 81.132,\ r^e((5,1)|\{O_1,O_2\}) = 120\times0.3239 = 38.868,$

$r^e((1,6)|\{O_1,O_2\}) = 120\times0.4793 = 57.516,\ r^e((6,1)|\{O_1,O_2\}) = 120\times0.5207 = 62.484,$

$r^e((2,3)|\{O_1,O_2\}) = 120\times0.5363 = 64.356,\ r^e((3,2)|\{O_1,O_2\}) = 120\times0.4637 = 55.644,$

$r^e((2,4)|\{O_1,O_2\}) = 120\times0.4665 = 55.980,\ r^e((4,2)|\{O_1,O_2\}) = 120\times0.5335 = 64.020,$

$r^e((2,5)|\{O_1,O_2\}) = 120\times0.6353 = 76.236,\ r^e((5,2)|\{O_1,O_2\}) = 120\times0.3647 = 43.764,$

$r^e((2,6)|\{O_1,O_2\}) = 120\times0.4352 = 52.224,\ r^e((6,2)|\{O_1,O_2\}) = 120\times0.5648 = 67.776,$

$r^e((3,4)|\{O_1,O_2\}) = 120\times0.4305 = 51.660,\ r^e((4,3)|\{O_1,O_2\}) = 120\times0.5695 = 68.340,$

$r^e((3,5)|\{O_1,O_2\}) = 120\times0.6007 = 72.084,\ r^e((5,3)|\{O_1,O_2\}) = 120\times0.3993 = 47.916,$

$r^e((3,6)|\{O_1,O_2\}) = 120\times0.3997 = 47.964,\ r^e((6,3)|\{O_1,O_2\}) = 120\times0.6003 = 72.036,$

$r^e((4,5)|\{O_1,O_2\}) = 120\times0.6664 = 79.968,\ r^e((5,4)|\{O_1,O_2\}) = 120\times0.3336 = 40.032,$

$r^e((4,6)|\{O_1,O_2\}) = 120\times0.4685 = 56.220,\ r^e((6,4)|\{O_1,O_2\}) = 120\times0.5315 = 63.780,$

$r^e((5,6)|\{O_1,O_2\}) = 120\times 0.3053 = 36.636,\ \ r^e((6,5)|\{O_1,O_2\}) = 120\times 0.6947 = 83.364.$

If we calculate $\log \frac{r(s_{fl}|W_{O,f})}{r^e(s_{fl}|W_{O,f})}$, we gain the following results.

$$\log\left(\frac{r((1,2)|\{O_1,O_2\})}{r^e((1,2)|\{O_1,O_2\})}\right) = -0.0047,\quad \log\left(\frac{r((2,1)|\{O_1,O_2\})}{r^e((2,1)|\{O_1,O_2\})}\right) = 0.0055,$$

$$\log\left(\frac{r((1,3)|\{O_1,O_2\})}{r^e((1,3)|\{O_1,O_2\})}\right) = 0.0337,\quad \log\left(\frac{r((3,1)|\{O_1,O_2\})}{r^e((3,1)|\{O_1,O_2\})}\right) = -0.0486,$$

$$\log\left(\frac{r((1,4)|\{O_1,O_2\})}{r^e((1,4)|\{O_1,O_2\})}\right) = 0.1884,\quad \log\left(\frac{r((4,1)|\{O_1,O_2\})}{r^e((4,1)|\{O_1,O_2\})}\right) = -0.2439,$$

$$\log\left(\frac{r((1,5)|\{O_1,O_2\})}{r^e((1,5)|\{O_1,O_2\})}\right) = -0.1334,\quad \log\left(\frac{r((5,1)|\{O_1,O_2\})}{r^e((5,1)|\{O_1,O_2\})}\right) = 0.2316,$$

$$\log\left(\frac{r((1,6)|\{O_1,O_2\})}{r^e((1,6)|\{O_1,O_2\})}\right) = -0.0818,\quad \log\left(\frac{r((6,1)|\{O_1,O_2\})}{r^e((6,1)|\{O_1,O_2\})}\right) = 0.0698,$$

$$\log\left(\frac{r((2,3)|\{O_1,O_2\})}{r^e((2,3)|\{O_1,O_2\})}\right) = -0.1754,\quad \log\left(\frac{r((3,2)|\{O_1,O_2\})}{r^e((3,2)|\{O_1,O_2\})}\right) = 0.1707,$$

$$\log\left(\frac{r((2,4)|\{O_1,O_2\})}{r^e((2,4)|\{O_1,O_2\})}\right) = -0.0547,\quad \log\left(\frac{r((4,2)|\{O_1,O_2\})}{r^e((4,2)|\{O_1,O_2\})}\right) = 0.0455,$$

$$\log\left(\frac{r((2,5)|\{O_1,O_2\})}{r^e((2,5)|\{O_1,O_2\})}\right) = 0.0729,\quad \log\left(\frac{r((5,2)|\{O_1,O_2\})}{r^e((5,2)|\{O_1,O_2\})}\right) = -0.1412,$$

$$\log\left(\frac{r((2,6)|\{O_1,O_2\})}{r^e((2,6)|\{O_1,O_2\})}\right) = 0.1220,\quad \log\left(\frac{r((6,2)|\{O_1,O_2\})}{r^e((6,2)|\{O_1,O_2\})}\right) = -0.1053,$$

$$\log\left(\frac{r((3,4)|\{O_1,O_2\})}{r^e((3,4)|\{O_1,O_2\})}\right) = -0.0735,\quad \log\left(\frac{r((4,3)|\{O_1,O_2\})}{r^e((4,3)|\{O_1,O_2\})}\right) = 0.0522,$$

$$\log\left(\frac{r((3,5)|\{O_1,O_2\})}{r^e((3,5)|\{O_1,O_2\})}\right) = -0.0293,\quad \log\left(\frac{r((5,3)|\{O_1,O_2\})}{r^e((5,3)|\{O_1,O_2\})}\right) = 0.0426,$$

$$\log\left(\frac{r((3,6)|\{O_1,O_2\})}{r^e((3,6)|\{O_1,O_2\})}\right) = -0.0418,\quad \log\left(\frac{r((6,3)|\{O_1,O_2\})}{r^e((6,3)|\{O_1,O_2\})}\right) = 0.0269,$$

$$\log\left(\frac{r((4,5)|\{O_1,O_2\})}{r^e((4,5)|\{O_1,O_2\})}\right) = 0.0372,\quad \log\left(\frac{r((5,4)|\{O_1,O_2\})}{r^e((5,4)|\{O_1,O_2\})}\right) = -0.0788,$$

$$\log\left(\frac{r((4,6)|\{O_1,O_2\})}{r^e((4,6)|\{O_1,O_2\})}\right) = 0.0483,\quad \log\left(\frac{r((6,4)|\{O_1,O_2\})}{r^e((6,4)|\{O_1,O_2\})}\right) = -0.0446,$$

$$\log\left(\frac{r((5,6)|\{O_1,O_2\})}{r^e((5,6)|\{O_1,O_2\})}\right) = -0.0747,\quad \log\left(\frac{r((6,5)|\{O_1,O_2\})}{r^e((6,5)|\{O_1,O_2\})}\right) = 0.0311.$$

Thus the realization value of test statistic is

$$G^{2*} = 2\sum_{f=1}^{15}\sum_{l=1}^{2!} r(s_{fl}|W_{O,f}) \log\left(\frac{r(s_{fl}|W_{O,f})}{r^e(s_{fl}|W_{O,f})}\right) = 18.35$$

and G^2 is distributed according to χ^2-distribution with $[Q(m!-1)-h] = 9$ degrees of freedom. Then the rejections region for each significance level are given by

When the significance level $\alpha = 0.01$, the rejection region $R = (21.70, \infty)$,
When the significance level $\alpha = 0.05$, the rejection region $R = (16.92, \infty)$.

Thus $G^{2*} \notin R$ when the significance level $\alpha = 0.01$. We don't reject the hypothesis H_0. By this, the order of desirability of 6 kinds of juice is to be $(O_6, O_1, O_4, O_2, O_3, O_5)$.

Also when $G^{2*} \in R$ when the significance level $\alpha = 0.05$. We reject the hypothesis H_0.

4.4 The estimation of Case III

We must to derive V_{z_i} and σ_i in Case III. We represent the following table for V_{z_i} and σ_i by the table of z_{ij} in Case V.

The table.

	O_1	O_2	O_3	O_4	O_5	O_6	total
V_{zi}	0.0246	0.0458	0.0400	0.0616	0.0405	0.0423	
$1/V_{zi}$	40.650	21.834	25.000	16.234	24.691	23.641	152.050
σ_i	1.6057	0.8624	0.9875	0.6412	0.9753	0.9338	6.0059
σ_i^2	2.5783	0.7437	0.9752	0.4111	0.9512	0.8720	

In order to derive $R_i - R_j$, we make the following table.

The table of $z_{ij}\sqrt{\sigma_i^2 + \sigma_j^2}$.

z_{ij}	O_1	O_2	O_3	O_4	O_5	O_6
O_1	0.000	-0.193	-0.477	-0.515	-0.438	0.271
O_2	0.193	0.000	0.165	0.157	-0.620	0.025
O_3	0.477	-0.165	0.000	0.298	-0.291	0.405
O_4	0.515	-0.157	-0.298	0.000	-0.586	0.023
O_5	0.438	0.620	0.291	0.586	0.000	0.775
O_6	-0.271	-0.025	-0.405	-0.023	-0.775	0.000
total	1.352	0.080	-0.724	0.503	-2.710	1.499
average	0.225	0.013	-0.121	0.084	-0.452	0.250
R_i	0.677	0.465	0.331	0.536	0.000	0.702

We derive $z'_{ij} = (R_i - R_j)/\sqrt{\sigma_i^2 + \sigma_j^2}$ similar to Case V. In Case III, the estimation of probability where the object O_i is more desirable than O_j is given by

$$\hat{\Pr}(O_i \succ O_j) = \hat{\Pr}((i,j)|\{O_i, O_j\}) = \Phi(z'_{ij}) = \Phi\left(\frac{R_i - R_j}{\sqrt{\sigma_i^2 + \sigma_j^2}}\right)$$

Thus the estimations of probability in Case III are given by

$$\hat{\Pr}(O_1 \succ O_2) = \Phi(0.116) = 0.5462, \quad \hat{\Pr}(O_2 \succ O_1) = \Phi(-0.116) = 0.4538,$$

$$\hat{\Pr}(O_1 \succ O_3) = \Phi(0.184) = 0.5730, \quad \hat{\Pr}(O_3 \succ O_1) = \Phi(-0.184) = 0.4270,$$

$$\hat{\Pr}(O_1 \succ O_4) = \Phi(0.082) = 0.5327, \quad \hat{\Pr}(O_4 \succ O_1) = \Phi(-0.082) = 0.4673,$$

$$\hat{\Pr}(O_1 \succ O_5) = \Phi(0.360) = 0.6406, \quad \hat{\Pr}(O_5 \succ O_1) = \Phi(-0.360) = 0.3594,$$

$$\hat{\Pr}(O_1 \succ O_6) = \Phi(-0.013) = 0.4948, \quad \hat{\Pr}(O_6 \succ O_1) = \Phi(0.013) = 0.5052,$$

$$\hat{\Pr}(O_2 \succ O_3) = \Phi(0.102) = 0.5406, \quad \hat{\Pr}(O_3 \succ O_2) = \Phi(-0.102) = 0.4594,$$

$$\hat{\Pr}(O_2 \succ O_4) = \Phi(-0.066) = 0.4737, \quad \hat{\Pr}(O_4 \succ O_2) = \Phi(0.066) = 0.5263,$$

$$\hat{\Pr}(O_2 \succ O_5) = \Phi(0.357) = 0.6395, \quad \hat{\Pr}(O_5 \succ O_2) = \Phi(-0.357) = 0.3605,$$

$$\hat{\Pr}(O_2 \succ O_6) = \Phi(-0.186) = 0.4263, \quad \hat{\Pr}(O_6 \succ O_2) = \Phi(0.186) = 0.5737,$$

$$\hat{\Pr}(O_3 \succ O_4) = \Phi(-0.174) = 0.4309, \quad \hat{\Pr}(O_4 \succ O_3) = \Phi(0.174) = 0.5691,$$

$$\hat{\Pr}(O_3 \succ O_5) = \Phi(0.238) = 0.5940, \quad \hat{\Pr}(O_5 \succ O_3) = \Phi(-0.238) = 0.4060,$$

$$\hat{\Pr}(O_3 \succ O_6) = \Phi(-0.273) = 0.3924, \quad \hat{\Pr}(O_6 \succ O_3) = \Phi(0.273) = 0.6076,$$

$$\hat{\Pr}(O_4 \succ O_5) = \Phi(0.459) = 0.6768, \quad \hat{\Pr}(O_5 \succ O_4) = \Phi(-0.459) = 0.3232,$$

$$\hat{\Pr}(O_4 \succ O_6) = \Phi(-0.147) = 0.4416, \quad \hat{\Pr}(O_6 \succ O_4) = \Phi(0.147) = 0.5584,$$

$$\hat{\Pr}(O_5 \succ O_6) = \Phi(-0.520) = 0.3015, \quad \hat{\Pr}(O_6 \succ O_5) = \Phi(0.520) = 0.6985.$$

4.5 The test of Case III

We carry out the likelihood ratio test

$$\text{null hypothesis } H_0 : \Pr(O_i \succ O_j) = \hat{\Pr}((i,j)|\{O_i, O_j\}).$$

Here we test by using the likelihood ratio χ^2 statistic

$$G^2 = 2\sum_{f=1}^{Q}\sum_{l=1}^{m!} r(s_{fl}|W_{O,f}) \log\left(\frac{r(s_{fl}|W_{O,f})}{r^e(s_{fl}|W_{O,f})}\right)$$

is distributed according to χ^2-distribution with $[Q(m!-1)-h]$ degrees of freedom for the partial ranking. When $Q = 15, t = 6, m = 2$, the test statistic

$$G^2 = 2\sum_{f=1}^{15}\sum_{l=1}^{2!} r(s_{fl}|W_{O,f}) \log\left(\frac{r(s_{fl}|W_{O,f})}{r^e(s_{fl}|W_{O,f})}\right)$$

is distributed according to χ^2-distribution with 9 degrees of freedom.

If we calculate the frequency of expectation, we gain the following results.

$r^e((1,2)|\{O_1,O_2\}) = 120\times0.5462 = 65.544,\ r^e((2,1)|\{O_1,O_2\}) = 120\times0.4538 = 54.456,$

$r^e((1,3)|\{O_1,O_2\}) = 120\times0.5462 = 68.760,\ r^e((3,1)|\{O_1,O_2\}) = 120\times0.4538 = 51.240,$

$r^e((1,4)|\{O_1,O_2\}) = 120\times0.5462 = 63.924,\ r^e((4,1)|\{O_1,O_2\}) = 120\times0.4538 = 56.076,$

$r^e((1,5)|\{O_1,O_2\}) = 120\times0.5462 = 76.872,\ r^e((5,1)|\{O_1,O_2\}) = 120\times0.4538 = 43.128,$

$r^e((1,6)|\{O_1,O_2\}) = 120\times0.5462 = 59.376,\ r^e((6,1)|\{O_1,O_2\}) = 120\times0.4538 = 60.624,$

$r^e((2,3)|\{O_1,O_2\}) = 120\times0.5462 = 64.872,\ r^e((3,2)|\{O_1,O_2\}) = 120\times0.4538 = 55.128,$

$r^e((2,4)|\{O_1,O_2\}) = 120\times0.5462 = 56.844,\ r^e((4,2)|\{O_1,O_2\}) = 120\times0.4538 = 63.156,$

$r^e((2,5)|\{O_1,O_2\}) = 120\times0.5462 = 76.740,\ r^e((5,2)|\{O_1,O_2\}) = 120\times0.4538 = 43.260,$

$r^e((2,6)|\{O_1,O_2\}) = 120\times0.5462 = 51.156,\ r^e((6,2)|\{O_1,O_2\}) = 120\times0.4538 = 68.844,$

$r^e((3,4)|\{O_1,O_2\}) = 120\times0.5462 = 51.708,\ r^e((4,3)|\{O_1,O_2\}) = 120\times0.4538 = 68.292,$

$r^e((3,5)|\{O_1,O_2\}) = 120\times0.5462 = 71.280,\ r^e((5,3)|\{O_1,O_2\}) = 120\times0.4538 = 48.720,$

$r^e((3,6)|\{O_1,O_2\}) = 120\times0.5462 = 47.088,\ r^e((6,3)|\{O_1,O_2\}) = 120\times0.4538 = 72.912,$

$r^e((4,5)|\{O_1,O_2\}) = 120\times0.5462 = 81.216,\ r^e((5,4)|\{O_1,O_2\}) = 120\times0.4538 = 38.784,$

$r^e((4,6)|\{O_1,O_2\}) = 120\times0.5462 = 52.992,\ r^e((6,4)|\{O_1,O_2\}) = 120\times0.4538 = 67.008,$

$r^e((5,6)|\{O_1,O_2\}) = 120\times0.5462 = 36.180,\ r^e((6,5)|\{O_1,O_2\}) = 120\times0.4538 = 83.820.$

If we calculate $\log \frac{r(s_{fl}|W_{O,f})}{r^e(s_{fl}|W_{O,f})}$, we gain the following results.

$$\log\left(\frac{r((1,2)|\{O_1,O_2\})}{r^e((1,2)|\{O_1,O_2\})}\right) = -0.0083,\quad \log\left(\frac{r((2,1)|\{O_1,O_2\})}{r^e((2,1)|\{O_1,O_2\})}\right) = 0.0099,$$

$$\log\left(\frac{r((1,3)|\{O_1,O_2\})}{r^e((1,3)|\{O_1,O_2\})}\right) = 0.0460,\quad \log\left(\frac{r((3,1)|\{O_1,O_2\})}{r^e((3,1)|\{O_1,O_2\})}\right) = -0.0653,$$

$$\log\left(\frac{r((1,4)|\{O_1,O_2\})}{r^e((1,4)|\{O_1,O_2\})}\right) = 0.1464,\quad \log\left(\frac{r((4,1)|\{O_1,O_2\})}{r^e((4,1)|\{O_1,O_2\})}\right) = -0.1981,$$

$$\log\left(\frac{r((1,5)|\{O_1,O_2\})}{r^e((1,5)|\{O_1,O_2\})}\right) = -0.0795,\quad \log\left(\frac{r((5,1)|\{O_1,O_2\})}{r^e((5,1)|\{O_1,O_2\})}\right) = 0.1276,$$

$$\log\left(\frac{r((1,6)|\{O_1,O_2\})}{r^e((1,6)|\{O_1,O_2\})}\right) = -0.1136,\quad \log\left(\frac{r((6,1)|\{O_1,O_2\})}{r^e((6,1)|\{O_1,O_2\})}\right) = 0.1000,$$

$$\log\left(\frac{r((2,3)|\{O_1,O_2\})}{r^e((2,3)|\{O_1,O_2\})}\right) = -0.1834,\quad \log\left(\frac{r((3,2)|\{O_1,O_2\})}{r^e((3,2)|\{O_1,O_2\})}\right) = 0.1800,$$

$$\log\left(\frac{r((2,4)|\{O_1,O_2\})}{r^e((2,4)|\{O_1,O_2\})}\right) = -0.0700,\quad \log\left(\frac{r((4,2)|\{O_1,O_2\})}{r^e((4,2)|\{O_1,O_2\})}\right) = 0.0591,$$

$$\log\left(\frac{r((2,5)|\{O_1,O_2\})}{r^e((2,5)|\{O_1,O_2\})}\right)=0.0663,\quad \log\left(\frac{r((5,2)|\{O_1,O_2\})}{r^e((5,2)|\{O_1,O_2\})}\right)=-0.1296,$$
$$\log\left(\frac{r((2,6)|\{O_1,O_2\})}{r^e((2,6)|\{O_1,O_2\})}\right)=0.1427,\quad \log\left(\frac{r((6,2)|\{O_1,O_2\})}{r^e((6,2)|\{O_1,O_2\})}\right)=-0.1210,$$
$$\log\left(\frac{r((3,4)|\{O_1,O_2\})}{r^e((3,4)|\{O_1,O_2\})}\right)=-0.0744,\quad \log\left(\frac{r((4,3)|\{O_1,O_2\})}{r^e((4,3)|\{O_1,O_2\})}\right)=0.0529,$$
$$\log\left(\frac{r((3,5)|\{O_1,O_2\})}{r^e((3,5)|\{O_1,O_2\})}\right)=-0.0181,\quad \log\left(\frac{r((5,3)|\{O_1,O_2\})}{r^e((5,3)|\{O_1,O_2\})}\right)=0.0259,$$
$$\log\left(\frac{r((3,6)|\{O_1,O_2\})}{r^e((3,6)|\{O_1,O_2\})}\right)=-0.0234,\quad \log\left(\frac{r((6,3)|\{O_1,O_2\})}{r^e((6,3)|\{O_1,O_2\})}\right)=0.0148,$$
$$\log\left(\frac{r((4,5)|\{O_1,O_2\})}{r^e((4,5)|\{O_1,O_2\})}\right)=0.0217,\quad \log\left(\frac{r((5,4)|\{O_1,O_2\})}{r^e((5,4)|\{O_1,O_2\})}\right)=-0.0471,$$
$$\log\left(\frac{r((4,6)|\{O_1,O_2\})}{r^e((4,6)|\{O_1,O_2\})}\right)=0.1074,\quad \log\left(\frac{r((6,4)|\{O_1,O_2\})}{r^e((6,4)|\{O_1,O_2\})}\right)=-0.0939,$$
$$\log\left(\frac{r((5,6)|\{O_1,O_2\})}{r^e((5,6)|\{O_1,O_2\})}\right)=-0.0621,\quad \log\left(\frac{r((6,5)|\{O_1,O_2\})}{r^e((6,5)|\{O_1,O_2\})}\right)=0.0257.$$

Thus the realization value of test statistic is

$$G^{2*}=2\sum_{f=1}^{15}\sum_{l=1}^{2!} r(s_{fl}|W_{O,f})\log\left(\frac{r(s_{fl}|W_{O,f})}{r^e(s_{fl}|W_{O,f})}\right)=16.0344$$

and G^2 is distributed according to χ^2-distribution with $[Q(m!-1)-h]=9$ degrees of freedom. Then the rejections region for each significance level are given by

When the significance level $\alpha=0.01$, the rejection region $R=(21.70,\infty)$,
When the significance level $\alpha=0.05$, the rejection region $R=(16.92,\infty)$.

Thus $G^{2*}\notin R$ for both cases when the significance level $\alpha=0.01$ and $\alpha=0.05$. We don't reject the hypothesis H_0 for both cases. By this, the order of desirability of 6 kinds of juice is to be $(O_6,O_1,O_4,O_2,O_3,O_5)$.

5. Considerations

1. In this paper, we carried out the test of goodness of fit by using likelihood ratio χ^2-distribution. But we had to deal with the large sample as a sample. If the sample is small sample, it is desirable to use another paired comparison method. Thus we must to investigate what the paired comparison methods are suitable before the experiments for the contents of the comparison experiments.

If we carry out the paired comparison, we must to test the comparison experiments of $\frac{t(t-1)}{2}$ pairs at least for the number of object t. Thus the number of combination becomes much larger as t increases. Then the experiments become difficult, because of it needs many subjects and becomes lots of frequency of experiments for objects in case the objects are small. It is certain that it expends long time for the experience even if it is easy to calculate by developing the personal computer. We would like to deal with the case to reduce the number of pair of stimulus in order to carry out the experiences some extent in future.

2. As a numerical example, we carried out the test and estimation in Case V and Case III for the case of 6 objects. Then both cases didn't change the order. This case, we could judge that the ranking model catchs the nature of each object.

References

[1] Ulf Böckenholt(1992). Thurstonian representation for partial ranking data. British Journal of Mathematical and Statistical Psychology, 45, 31-49.

[2] L.L.Thurstone(1927). A law of comparative judgment. Psychological Review, 15, 284-297.

[3] Frederick Mosteller(1951). Remarks on the method of paired comparisons:I. The least squares solution assuming equal standard deviations and equal correlations. Psychometrika, 16, No.1,3-9.

[4] Raymond H.Burros and W.A.Gibson(1954). A solution for Case III of the Law of comparative judgment. Psychometrika, 19, No.1, 57-64.

[5] Iizuka.M and Ujiie.K(2002). A Generalized Modification of Scheffé's Paired Comparisons -A theoretical approach to decrease the numbers of experiments-. Measurement and Multivariate Analysis, the Institute of Statistical Mathematics. Springer Verlag. 77-86.

[6] Ujiie.K and Inoue.Y(1997). A Modification of Scheffé's Paired Comparisons -A theoretical approach to decrease the number of experiments-. Proceeding of the Faculty of Science of Tokai University 32, 43-53.