A queueing model for the traffic congestion in Bangkok

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Abstract

In this paper we investigate the queueing model for the traffic congestion in Bangkok. We refer the traffic in a crossing as the flow of jobs, the traffic time in the crossing as those of the service time in a queueing system. The traffic consists of two kinds of flows, main and secondary ones. Furthermore we define a traffic controller as a server in the queueing system. At the instant of service completion (cars passed the crossing) in the main flow, the controller is continuously busy as long as there is any job in the main flow. As soon as the controller finds the main flow empty, however, he takes another job in the secondary flow (control of against flows). The service time of job is assumed to be a random variable with exponential distribution. As regards taking services in the secondary flows, two models are considered. In the first model the server returns to the main flow immediately after a single job whether there is a job or not in the main flow, while in the second model he keeps on taking another job until he finds any job present in the main flow upon termination of each job of secondary flow. By taking a job the server utilizes a part or all of his idle time for additional job in the secondary flow.

For each of the models above, the stationary distribution of the system size and that of the waiting time are obtained

1 Introduction

We consider the queueing system where the controller takes another kind of job of secondary flow immediately after he becomes idle at the instant of service completion for the main flow . Upon termination of the secondary flow's service he returns to serve the main flow. If any job is present in the main flow upon termination of main flow's service, the controller keeps for giving his service to each job, that is, the system operates as an ordinary queue. The M/G/1 queue with secondary service system have been studied by Levy and Yechiali [2] and Ōsawa and Doi [3].

To investigate the traffic congestion in Bangkok we consider the GI/M/1 queue with the secondary system's service whose length is assumed to be an exponentially distributed random variable.

With regard to the system above, following two models are considered here.

(Model 1.) Immediately after a single job of secondary flow, the controller returns to the main flow and becomes ready for his service. If there is any job kept waiting during the service of secondary flow, the controller begins to serve at once in the main flow. Otherwise he waits for the first job to arrive at the main flow.

(Model 2.) Unless the controller finds any job present in the main flow upon termination of a job of secondary flow, he keeps on taking another job of secondary flow. When the controller finds any Job in the main flow upon termination of a single job or several jobs in the secondary flow, he begins to serve in the main flow immediately.

While the controller is away from the main flow by taking a secondary flow's service, he provides the secondary flow with the service as an additional work. Thus the server utilizes the idle time. Let the idle time of the controller be the elapsed time during which he sojourns with no main work for him in the main flow waiting for the first job to arrive, then there can be no cases for the controller in Model 2 to possess his idle time.

We can apply this queue to control the traffic congestion in Bangkok. Here, we consider a crossing which have a main flow and secondary one. Arrivals to this crossing are assumed to be the generally distributed. The length of service times for the main flow or secondary one can be assumed to be exponentially distributed random variables. A controller (a signal or a policeman) is controlling the traffic congestion for the crossing.

2 Definition and Notations

We consider the GI/M/1 queue where the inter-arrival times are assumed to be independent and identically distributed (i.i.d.) random variables with distribution function $A(x)$. Further define

$$\frac{1}{\lambda} = \int_0^\infty x dA(x) < +\infty,$$

$$a[s] = \int_0^\infty e^{-sx} dA(x) \qquad for \;\; Re \;\; s > 0.$$

The service times are i.i.d. random variables with common exponential distribution $1 - e^{-\mu x}$. If at the instant of service completion the server finds the system empty he leaves for a secondary flow's service whose duration is a random variable with exponential distribution $1 - e^{-\nu x}$. On the other hand as long as any job is present in the system upon termination of a service, the server continues to give his service to each job as an ordinary queue. Upon termination of a secondary flow's service, in Model 1, the server immediately returns to the system and becomes ready for his service, while in Model 2 the server keeps on taking another secondary flow's service with common distribution until he finds any job kept waiting upon return from the secondary flow's service. The following quantities are introduced;

$$\rho = \frac{\lambda}{\mu} < 1,$$

$$\delta = \frac{\nu}{\mu},$$
$$a_j = \int_0^\infty e^{-\nu x}(\nu x)^j/j!\; dA(x) \quad (j = 0, 1, 2, \cdots),$$
$$b_j = \int_0^\infty e^{-\mu x}(\mu x)^j/j!\; dA(x) \quad (j = 0, 1, 2, \cdots), \tag{1}$$

We define the following random variables for the instant of the n-th arrival, say t_n, in our queueing system ;

$$Z_n = \begin{cases} 0\ ; & \text{if the server is on secondary service at time } t_n, \\ 1\ ; & \text{if the server is not on secondary service at time } t_n, \end{cases}$$
$$\xi_n = \text{the number of jobs in the system just before the instant } t_n.$$

Then the processes $\{Z, \xi\} = \{(Z_n, \xi_n); n = 1, 2, \cdots\}$ are homogeneous Markov chains. In Model 1, the process, say $\{Z, \xi\}_1$, processes the state space S_1;

$$S_1 = \{(v, j); v = 0, 1, j = 0, 1, 2, \cdots\},$$

however, in Model 2, the state (1,0) is unable to exist because of successive secondary flow's services, and therefore the state space of the process $\{Z, \xi\}_2$ is

$$S_2 = S_1 - \{(1, 0)\}.$$

For these processes we use common notations, besides $\{Z, \xi\}$ and S, with respect to the stationary transition probabilities and their generating functions;

$$p_{u,v}(i, j) = P\{Z_{n+1} = v, \xi_{n+1} = j | Z_n = u, \xi_n = i\}$$
$$((u, i), (v, j) \in S; n = 1, 2, \cdots),$$
$$Q_{u,v}(z; j) = \sum_{i=0}^{\infty} z^i P_{u,v}(i, j) \qquad ((v, j) \in S; u = 0, 1)$$

for $|z| < 1$, where, in Model 2, we assume that $p_{1v}(0, j) = 0$ for convenience. From the expressions of the transition probabilities given in the following sections, it follows easily that both chains $\{Z, \xi\}$ are irreducible and aperiodic. On the assumption that the steady state in our queue exists, we consider the system of equations for the stationary probabilities;

$$\pi_{vj} = \sum_{i=0}^{\infty} \pi_{0i} p_{0v}(i, j) + \sum_{i=0}^{\infty} \pi_{1i} p_{1v}(i, j) \quad ((v, j) \in S), \tag{2}$$

where $\pi_{ui} = \lim_{n \to \infty} P\{Z_n = u, \xi_n = i\}$ for $(u, i) \in S$ and we take, in Model 2, $\pi_{10} = 0$ for convenience.

Define the waiting time, say W_q, as the elapsed time between the instant the job arrives at the system and the instant the server begins to serve him. Using the stationary distribution $\{\pi_{vj}; (v,j) \in S\}$, we deduce that the distribution function $F(x)$ of the waiting time are given by

$$dF(x) = \begin{cases} \pi_{10} & (x = 0). \\ \sum_{i=1}^{\infty} \pi_{1i} e^{-\mu x}(\mu x)^{i-1}/(i-1)!\mu dx + \pi_{00} e^{-\nu x}\nu dx \\ \quad + \sum_{i=1}^{\infty} \pi_{0i}\nu e^{-\nu x} * \{e^{-\mu x}(\mu x)^{i-1}/(i-1)!\}\mu dx & (x > 0), \end{cases} \tag{3}$$

where $f * g$ is convolution of two functions f and g.
Further we introduce the following notations;

$\pi_j = \pi_{0j} + \pi_{1j} \quad (j = 0, 1, 2, \cdots)$,
$L =$ the mean system size just before the instant of ajob's arrival,
$W =$ the duration during which a job sojourns in the system.

Note that

$$E(W) = E(W_q) + 1/\mu.$$

Let ζ denote the solution of the equation

$$\zeta = a[\mu(1-\zeta)], \quad 0 < \zeta < 1. \tag{4}$$

It should be noted that such a unique solution ζ exists if and only if $\rho < 1$. Evaluate $\alpha(z) = a[\mu(1-z)]$ at $z = 1 - \delta$ and $z = 0$,

$$\alpha(1-\delta) = a_0,$$
$$\alpha(0) = b_0.$$

3 The Analysis of Model 1

As already noted the process $\{Z, \xi\}_1$ is the Markov chain with state space S_1. In a usual manner its transition probabilities are given as follows;

$$p_{11}(i,j) = \begin{cases} \int_0^\infty e^{-\mu y}(\mu y)^{i+1-j}/(i+1-j)!dA(y) & (i+1 \geq j \neq 0), \\ \int_0^\infty \int_0^y e^{-\mu t}(\mu t)^i/i!\mu dt(1 - e^{-\nu(y-t)})dA(y) & (j = 0), \end{cases} \tag{5}$$

$$p_{01}(i,j) = \begin{cases} \int_0^\infty \int_0^y \nu e^{-\nu t}dte^{-(y-t)} \\ \quad [\mu(y-t)]^{i+1-j}/(i+1-j)!dA(y) & (i+1 \geq j \neq 0), \\ \int_0^\infty \int_0^y \nu e^{-\nu t}dt \int_0^{y-t} e^{-\mu\tau}(\mu\tau)^i/i! \\ \quad \mu d\tau(1 - e^{-\nu(y-t-\tau)})dA(y) & (j = 0), \end{cases} \tag{6}$$

$$p_{10}(i,0) = \int_0^\infty \int_0^y e^{-\mu t}(\mu t)^i/i!\mu dte^{-\nu(y-t)}dA(y), \tag{7}$$

$$p_{00}(i,j) = \begin{cases} \int_0^\infty e^{-\nu y} dA(y) & (j = i+1), \\ \int_0^\infty \int_0^y \nu e^{-\nu t} dt \int_0^{y-t} e^{-\mu\tau} (\mu\tau)^i / i! \\ \qquad \mu d\tau e^{-\nu(y-t-\tau)} dA(y) & (j = 0), \end{cases} \tag{8}$$

$$p_{uv}(i,j) = 0 \qquad \text{(otherwise)} \tag{9}$$

for non-negative integers i. From (5)~(9) we derive the generating functions $Q_{uv}(z;j)$ for $|z| < 1$, $u = 0, 1$ and $(v,j) \in S$;

$$Q_{11}(z;j) = \begin{cases} z^{j-1} a[\mu(1-z)] & (j = 1, 2, \cdots), \\ (1 - a[\mu(1-z)])/(1-z) - f(z) & (j = 0), \end{cases} \tag{10}$$

$$Q_{01}(z;j) = \begin{cases} \delta f(z) z^{j-1} & (j = 1, 2, \cdots), \\ (1-a_0)/(1-z) - a_1/(1-z-\delta) \\ \qquad +\delta^2 f(z)/\{(1-z)(1-z-\delta)\} & (j = 0), \end{cases} \tag{11}$$

$$Q_{10}(z;0) = f(z), \tag{12}$$

$$Q_{00}(z;0) = (a_1 - \delta f(z))(1 - z - \delta), \tag{13}$$

where $f(z) = \{a_0 - a[\mu(1-z)]\}/(1-z-\delta)$. Note that the generating functions $Q_{uv}(z;j)$ given as above are analytic inside the unit circle. If $|1-\delta| < 1$, using L'Hospital's rule, we find that $f(1-\delta) = a_1/\delta$ and that

$$Q_{11}(1-\delta;0) = (1 - a_0 - a_1)/\delta, \tag{14}$$

$$Q_{01}(1-\delta;j) = \begin{cases} a_1(1-\delta)^{j-1} & (j = 1, 2, \cdots), \\ (1 - a_0 - a_1 - a_2)/\delta & (j = 0), \end{cases} \tag{15}$$

$$Q_{10}(1-\delta;0) = a_1/\delta, \tag{16}$$

$$Q_{00}(1-\delta;0) = a_2/\delta. \tag{17}$$

These equations $Q_{uv}(1-\delta;j)$ are utilized for obtaining the stationary distribution of the process $\{Z,\xi\}_1$ in the case of $a_0 + \delta = 1$.

In order to derive the stationary distributions in Model 1, we now consider the system of equations (2). When $v = 0$, from (8), equations (2) are written by

$$\pi_{0j} = \pi_{0j-1} a_0 \qquad (j = 1, 2, \cdots),$$

and thus we have

$$\pi_{0j} = c a_0^j \qquad (j = 1, 2, \cdots), \tag{18}$$

where $c = \pi_{00}$. Therefore the first term of the right hand side of equations (2) is given by

$$\sum_{i=0}^{\infty} \pi_{0i} p_{0v}(i,j) = cQ_{0v}(a_0; j) \qquad ((v,j) \in S_1).$$

By using the equation (1), we rewrite (5) as follows;

$$p_{11}(i,j) = \begin{cases} b_{i+1-j} & (i+1 \geq j \neq 0), \\ \sum_{k=t+1}^{\infty} b_k - r_i & (j = 0), \end{cases}$$

where $r_i = p_{10}(i, 0)$. Then the system of equations (2) for π_{1j} and π_{00} become

$$\pi_{10} = \gamma_0 + \sum_{i=0}^{\infty} \pi_{1i} (\sum_{k=i+1}^{\infty} b_k - r_i)$$
$$= \gamma_0 - \sum_{i=0}^{\infty} \pi_{1i} r_i + \sum_{k=1}^{\infty} b_k \sum_{i=0}^{k-1} \pi_{1i}, \tag{19}$$
$$\pi_{1j} = \gamma_j + \sum_{i=j-1}^{\infty} \pi_{1i} b_{i+1-j} \qquad (j = 1, 2, \cdots), \tag{20}$$
$$c = cQ_{00}(a_0; 0) + \sum_{i=0}^{\infty} \pi_{1i} r_i, \tag{21}$$

where $\gamma_j = cQ_{01}(a_0; j)$. If we define

$$p_j = \sum_{i=0}^{j} \pi_{1i} \qquad (j = 0, 1, 2, \cdots),$$

then we have, from (19) and (21),

$$p_0 = \gamma_0 - c(1 - Q_{00}(a_0; 0)) + \sum_{k=0}^{\infty} p_k b_{k+1}. \tag{22}$$

Summing the equations for π_{1i} from $i = 0$ to $i = j$ yields

$$p_j = q_j + \sum_{k=j-1}^{\infty} p_k b_{k+1-j} \quad , \tag{23}$$

where we use the equation

$$q_j = \sum_{k=0}^{j} \gamma_k - c(1 - Q_{00}(a_0; 0)).$$

Let the vectors $\boldsymbol{p}$, $\boldsymbol{q}$ and a matrix $\boldsymbol{B}$ be defined by

$$\boldsymbol{p} = (p_0, p_1, p_2, \cdots),$$
$$\boldsymbol{q} = (q_0, q_1, q_2, \cdots),$$
$$\boldsymbol{B} = \begin{pmatrix} b_1 & b_0 & & & 0 \\ b_2 & b_1 & b_0 & & \\ b_3 & b_2 & b_1 & b_0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{24}$$

then we deduce from (22) and (23) that these vectors satisfy

$$\boldsymbol{p} = \boldsymbol{q} + \boldsymbol{p}\boldsymbol{B}. \tag{25}$$

Now we use the method of *Matrix-Geometric Solution* [1]. Then we have the following stationary distributions. for the Model 1.

Theorem 1.

If $\rho < 1$ and $a_0 + \delta \neq 1$ then we obtain the stationary distributions as follows:

$$\pi_j = (1-\zeta)\zeta^j + (1-\zeta)\{(1-a_0)a_0^j - (1-\zeta)\zeta^j\}/(1-\zeta+\kappa),$$
$$L = \zeta/(1-\zeta) + (a_0-\zeta)/\{(1-a_0)(1-\zeta+\kappa)\}$$
$$\text{where } \kappa = (1-a_0-a_1-\delta)/f(\zeta).$$

4 The Analysis of Model 2

In this section we deal with Model 2. We have the following transition probabilities:

$$p_{11}(i,j) = \int_0^\infty e^{-\mu y}(\mu y)^{i+1-j}!dA(y) \qquad (i+1 \geq j \neq 0), \tag{26}$$

$$p_{01}(i,j) = \int_0^\infty \int_0^y \nu e^{-\nu t}dte^{-\mu(y-t)}\{\mu(y-t)\}^{i+1-j}/(i+1-j)!dA(y) \qquad (i+1 \geq j \neq 0) \tag{27}$$

$$p_{10}(i,0) = \int_0^\infty \int_0^y e^{-\mu t}(\mu t)^i/i!\mu dtdA(y) \tag{28}$$

$$p_{00}(i.j) = \begin{cases} \int_0^\infty e^{-\nu y}dA(y) & (j = i+1) \\ \int_0^\infty \int_0^y \nu e^{-\nu t}dt \int_0^{y-t} e^{\mu\tau}(\mu\tau)^i/i!\mu d\tau dA(y) & (j = 0) \end{cases} \tag{29}$$

$$p_{uv}(i,j) = 0 \qquad \text{(otherwise)}. \tag{30}$$

In the same way of Model 1 we have the following theorem.

Theorem 2.

If $\rho < 1$ and $a_0 + \delta \neq 1$ then we obtain the stationary distributions as follows:

$$\pi_j = (1-\zeta)\zeta^j + (1-\zeta)\{(1-a_0)a_0^j - (1-\zeta)\zeta^j\}/(1-\zeta-\delta),$$
$$L = \zeta/(1-\zeta) + (a_0-\zeta)/\{(1-a_0)(1-\zeta-\delta)\}.$$

5 Numerical Example

We consider the case that the inter-arrival time has E_2 or M. Let the mean inter-arrival rate be equal to 1 and the mean service rate be equal to $1.1 - 2.0$. Fig.1 and Fig.2 shows the behavior of L for Model 1. From Theorem 1 and Theorem 2 we make out the fact that the mean queue length of Model 2 is greater than that of Model 1. Hence we see in the crossing the controller may pass only one car from the secondary flow if the main flow is idle.

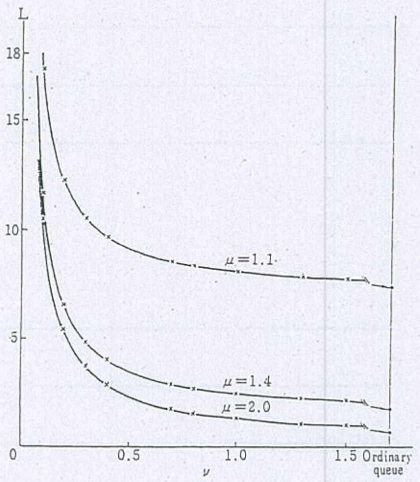

Fig. 1. Mean queue size in Model 1; $E_2/M/1$ queue with $\lambda=1.0$.

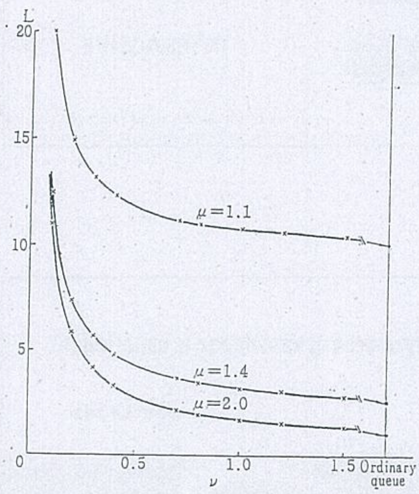

Fig. 2. Mean queue size in Model 1; M/M/1 queue with $\lambda=1.0$.

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