On Synchronization Queues

Hideo ŌSAWA

College of Economics, Nihon University, Japan

Chikayoshi Shima

College of Science and Engineering, Nihon University, Japan

Makoto Doi

Faculty of Science, Tokai University, Japan

Abstract

This paper considers synchronization queues (or synchronization nodes) with two input flows and finite or infinite buffers. There is one flow of tokens for each buffer, called a stream. Each stream is assumed to be a point process with finite intensity. Tokens are held in the buffer until one is available from each flow and a group-token is instantaneously released as a synchronized departure. In this paper, we review the system state and the output processes of synchronization queues.

Keywords : Synchronization queue, output process, Markov renewal process, matrix analytical method, phase-type distribution, point process, rate conservation law.

1 Introduction

This paper concerns some synchronization queues (or synchronization nodes) consisting of two buffers with finite or infinite capacities. At a synchronization queue, tokens arrive on distinct input flows and are stored. There is one flow for each buffer. The flow of tokens to each buffer is called a stream, and is assumed to be a point process with finite intensity. Tokens are held in the buffers until one is available from each flow. As soon as this happens, one token from each buffer is taken to form a group-token which is instantaneously released as a synchronized departure. Therefore, at least one buffer has no tokens at any instant and tokens in the other buffers are held in each buffer until one is available from each stream.

A synchronization queue having two input flows forms a so-called double-ended queue (see Srivastava and Kashyap, 1982). A good example of this case is the taxi cab problem where taxis or passengers form two different queues to wait for each other. The another is seen in the assembly-like queue as a kitting problem (see for example Bhat, 1986; Hopp and Simon, 1989; Latouche, 1981; Lipper and Sengupta, 1986; Som, Wilhelm and Disney, 1994; Takahashi, Ōsawa and Fujisawa, 1998, 2000). The almost of works on such problems have focused on the system state process. In addition, such queueing models appear in various areas, for instance, parallel processing, database concurrency control, flexible manufacturing systems, communication protocols and on.

Considering queueing networks with synchronization nodes, the output flow from such a node forms the input flow to the other node. Hence, studying output processes of synchronization queues or nodes seem to be very important. Som, Wilhelm and Disney (1994) and Takahashi, Ōsawa and Fujisawa (1998, 2000) investigated the distribution of time intervals during consecutive synchronized outputs from a synchronization node with two input flows. Prabhakar, Bambos and Mountford (2000) and Ōsawa (2001) studied the output process of general synchronization queues with some input flows. In this paper, we introduce their results for the system state and output processes of the synchronization queues.

The rest of paper is organized as follows. In the next section, the mathematical model and notation used in the paper is introduced to describe the system state of synchronization queues. Further, the system state processes are considered for synchronization queues with Poisson, PH-renewal and general streams. Moreover, the output processes are discussed for synchronization queues with two finite or infinite buffers. By applying conservation law, relations between the stationary distributions of the system state at an arbitrary point in time, just prior and after arrival time and an output point in time are discussed.

2 Model description and notation

We consider a synchronization queue with two buffers labeled as 1, 2. The ith buffer, $i = 1, 2$, has a random flow called stream i and the capacity K_i where $K_i \leq \infty$, that is, the capacity of buffer may be finite or infinite. Arriving tokens are held in the buffers, until one is available from each stream. Therefore, at least one buffer is always empty. By arriving to stream i when only buffer i is empty, exactly one token is taken from each buffer and forms a group to be released as a synchronized departure.

Let $Q_i(t)$ be the number of tokens held in buffer i at time t. Since exactly one buffer is empty at any instant by definition of the synchronization operation, the state space of $\{\boldsymbol{Q}(t)\}$ is

$$\mathcal{S} = \{\ \boldsymbol{j} = (j_1, j_2) \mid 0 \leq j_i \leq K_i,\ j_1 j_2 = 0\ \}.$$

For $i = 1, 2$, we also define

$$\mathcal{S}_i = \{\ \boldsymbol{j} \in \mathcal{S} \mid j_i = 0 \text{ and } j_k \neq 0 \text{ for } k \neq i\ \},$$

then we have $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{(0, 0)\}$. Define

$$Z(t) = Q_1(t) - Q_2(t), \tag{1}$$

the system state process can be represented by one-dimensional process $\{Z(t)\}$. That is, if $Z(t) > (<)\, 0$, there are no tokens at buffer 2 (1), and tokens at buffer 1 (2) have to wait for arrivals of paired tokens to buffer 2 (1). The rest case $Z(t) = 0$ means that both buffers are empty at time t.

Stream i is assumed to be a point process N_i, and a probability space for $\{Z(t)\}$, N_1 and N_2 is denoted by $(\Omega, \mathcal{F}, P)$. We also assume that the point process N_i has finite intensity $\lambda_i = E\{N_i(0,1]\}$, where the expectation is taken with respect to P, and has jumps at time $t_{i,n}$ where $N_i(\{t\}) = 0$ for $t \neq t_{i,n}$ and

$$\cdots < t_{i,-1} < t_{i,0} \leq 0 < t_{i,1} < \cdots < t_{i,n} < \cdots, \tag{2}$$

path wise for each $i = 1, 2$. In addition, these processes are assumed to be all simple and the superposed point process N_1+N_2 is also simple. We refer the above synchronization queue as

$$G_1/K_1 + G_2/K_2,$$

where G_i means the corresponding point process N_i is arbitrarily distributed.

3 System state processes

In this section we deal with the system state processes $\{Z(t)\}$ of synchronization queues (nodes) with finite buffers, that is, $G_1/K_1 + G_2/K_2$ where K_1 and K_2 are finite. Note that $-K_2 \leq Z(t) \leq K_1$. In this case, if $Z(t) = K_1$ $(-K_2)$, tokens arriving to buffer 1 (2) are blocked and rejected. We consider the stationary process $\{Z(t)\}$ and the stationary distribution $\{p(k)\}$ where $p(k) = P[Z(t) = k]$, $-K_2 \leq k \leq K_1$.

3.1 Poisson arrivals

Consider a synchronization queue with two Poisson streams, i.e., $M_1/K_1 + M_2/K_2$, where stream 1 and 2 are Poisson arrivals of rates λ_1 and λ_2, respectively. We then observe that $\{Z(t)\}$ becomes a birth-death process and get the balance equation for $\{p(k)\}$,

$$\begin{aligned}
&\lambda_1 p(-K_2) = \lambda_2 p(-K_2+1)\\
&(\lambda_1+\lambda_2)p(k) = \lambda_2 p(k+1) + \lambda_1 p(k-1), \quad -K_2 < k < K_1,\\
&\lambda_2 p(K_1) = \lambda_1 p(K_1 - 1).
\end{aligned}$$

These are rewritten by

$$\lambda_2 p(k) = \lambda_1 p(k-1), \qquad -K_2+1 \leq k \leq K_1.$$

Theorem 1 *For $M_1/K_1 + M_2/K_2$, the stationary distribution of the system state process is given by*

$$p(k) = \rho^k p(0), \qquad -K_2 \leq k \leq K_1, \tag{3}$$

where

$$\rho = \frac{\lambda_1}{\lambda_2} \quad \textit{and} \quad p(0) = \frac{1-\rho}{\rho^{-K_2} - \rho^{K_1+1}}\,.$$

Note that $\{Z(t)\}$ is time-reversible (see Ōsawa, 1985), that is, the relation $p(k)q(k,k') = p(k')q(k',k)$ for any states k and k' which means

$$P[\ Z(t)=k,\ Z(t')=k'\ ] = P[\ Z(t)=k',\ Z(t')=k\ ], \quad \text{for any } t,\ t',$$

where $q(\cdot,\cdot)$ are transition rates.

3.2 Poisson and PH-renewal arrivals

Suppose that streams 1 and 2 form a Poisson process of rate λ_1 and a PH-renewal process with interarrival time distribution of phase type having representation $(\boldsymbol{\alpha}, \boldsymbol{T})$, respectively. This queue is represented by $M/K_1+PH/K_2$. We assume that the matrix $\boldsymbol{T}$ is of finite order m and $\boldsymbol{\alpha e} = 1$ where $\boldsymbol{e}$ is a column vector with all its components equal to unity.

For this model, the system state is represented by $\{(Z(t), J(t))\}$ where $Z(t)$ was defined by (1) and $J(t)$ is the phase state of the arrival process of stream 2 at time t. Then the process $\{(Z(t), J(t))\}$ is a Markov process with the state space

$$\{(k,j) \mid -K_2 \le k \le K_1,\ 1 \le j \le m\}.$$

and a generator matrix $\boldsymbol{Q}$ is given by

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{B}_0 & \lambda_1\boldsymbol{I} & \boldsymbol{O} & & & \cdots & & \\ \boldsymbol{T}^0\boldsymbol{\alpha} & \boldsymbol{A}_1 & \lambda_1\boldsymbol{I} & \boldsymbol{O} & & & \cdots & \\ \boldsymbol{O} & \boldsymbol{T}^0\boldsymbol{\alpha} & \boldsymbol{A}_1 & \lambda_1\boldsymbol{I} & \boldsymbol{O} & & \cdots & \\ \cdots & \boldsymbol{O} & \boldsymbol{T}^0\boldsymbol{\alpha} & \boldsymbol{A}_1 & \lambda_1\boldsymbol{I} & \boldsymbol{O} & & \cdots \\ & & & \cdots & & & & \\ & & & \cdots & & & & \\ & \cdots & & \boldsymbol{O} & \boldsymbol{T}^0\boldsymbol{\alpha} & \boldsymbol{A}_1 & \lambda_1\boldsymbol{I} & \boldsymbol{O} \\ & & \cdots & & \boldsymbol{O} & \boldsymbol{T}^0\boldsymbol{\alpha} & \boldsymbol{A}_1 & \lambda_1\boldsymbol{I} \\ & & & \cdots & & \boldsymbol{O} & \boldsymbol{T}^0\boldsymbol{\alpha} & \boldsymbol{T} \end{pmatrix},$$

where $\boldsymbol{I}$ and $\boldsymbol{O}$ are the identity and zero matrices of order m, respectively, $\boldsymbol{B}_0 = \boldsymbol{T}^0\boldsymbol{\alpha} + \boldsymbol{T} - \lambda_1\boldsymbol{I}$ and $\boldsymbol{A}_1 = \boldsymbol{T} - \lambda_1\boldsymbol{I}$.

Denote the stationary probabilities for $\{(Z(t), J(t))\}$ by

$$p(k,j) = P[\ (Z(t), J(t)) = (k,j)\ ], \qquad -K_2 \le k \le K_1,\ 1 \le j \le m,$$

and these are partitioned as

$$\boldsymbol{\pi} = (\ \boldsymbol{\pi}(-K_2), \dots, \boldsymbol{\pi}(-1), \boldsymbol{\pi}(0), \boldsymbol{\pi}(1), \dots,\ \boldsymbol{\pi}(K_1)\ ),$$

where $\boldsymbol{\pi}(k) = (p(k,1), p(k,2), \cdots, p(k,m))$ is the probability vector. Then $\boldsymbol{\pi}$ is solved by the equation $\boldsymbol{\pi Q} = 0$ which is given in block matrix form

$$\begin{aligned} &\boldsymbol{\pi}(-K_2)\boldsymbol{B}_0 + \boldsymbol{\pi}(-K_2+1)\boldsymbol{T}^0\boldsymbol{\alpha} = 0, \\ &\boldsymbol{\pi}(k-1)\lambda_1\boldsymbol{I} + \boldsymbol{\pi}(k)\boldsymbol{A}_1 + \boldsymbol{\pi}(k+1)\lambda_1\boldsymbol{I} = 0, \quad -K_2+1 \le k \le K_1-1, \\ &\boldsymbol{\pi}(K_1-1)\lambda_1\boldsymbol{I} + \boldsymbol{\pi}(K_1)\boldsymbol{T} = 0. \end{aligned}$$

Theorem 2 (Neuts, 1981). *For $M/K_1+PH/K_2$, the stationary distribution of the system state process is given in matrix geometric form as follows:*

$$\begin{aligned}\boldsymbol{\pi}(k) &= \boldsymbol{\pi}(0)\boldsymbol{R}^k, \qquad -K_2 \le k \le K_1-1,\\ \boldsymbol{\pi}(K_1) &= \boldsymbol{\pi}(0)\boldsymbol{R}^{K_1-1}(-\lambda_1\boldsymbol{T}^{-1}).\end{aligned}$$

where $\boldsymbol{R}$ is the rate matrix for the $M/PH/1$ queue defined by

$$\boldsymbol{R} = \lambda_1(\lambda_1\boldsymbol{I}-\boldsymbol{T}-\lambda_1\boldsymbol{e}\boldsymbol{\alpha})^{-1}.$$

Remark 1 For following two cases of synchronization queues $M/K_1+M/K_2$ and $M/K_1+PH/K_2$ which one buffer has infinite capacities, it is well-known that the above results are available;

- $\rho<1,\ K_1=\infty$ and $K_2<\infty$,
- $\rho>1,\ K_1<\infty$ and $K_2=\infty$.

Remark 2 Neuts (1981) also studied the stationary distribution of the system state for a synchronization queue $PH_1/K_1+PH_2/K_2$. Further, for a synchronization queue $M/K_1+GI/K_2$, Srivastava et al. (1982) studied the stationary distribution $\{p(k)\}$ by using a supplementary variable method. However, their expressions of $p(k)$ are complicated.

3.3 General streams

In this section, we discuss the system state process of synchronization queues with two general streams and finite buffers, $G_1/K_1+G_2/K_2$. Suppose that $\{\boldsymbol{Q}(t)\}$, N_1 and N_2 are jointly stationary and ergodic, then we can define Palm probability measures P_1 and P_2 of P with respect to N_1 and N_2, respectively. Further, define

$$p(k)=P[Z=k],\ p_i^-(k)=P_i[Z^-=k],\ p_i^+(k)=P_i[Z^+=k],\qquad i=1,\ 2,$$

where $Z=Z(0)$, $Z^-=Z(0^-)$, $Z^+=Z(0^+)$ and 0^- is just prior time 0. It follows that $\{p(k)\}$, $\{p_i^-(k)\}$ and $\{p_i^+(k)\}$ are stationary distributions of the system state at an arbitrary point in time, just before and just after time points of N_i, respectively. The expectation with respect to P_i is denoted by E_i for $i=1$ and 2.

We now consider the relationship between distributions of the system state at arrival points in time of both streams. Define

$$X_k(t)=\begin{cases}1_{\{Z(t)\ge k\}}, & 1\le k\le K_1,\\ 1_{\{Z(t)\le k\}}, & -K_2\le k\le -1.\end{cases}$$

Applying the rate conservation law to $X_k(t)$, N_1 and N_2 (see Miyazawa, 1983, and Miyazawa and Yamazaki, 1992) yields

$$E[X_k'(0)]=\sum_{k=1}^{2}\lambda_i\left\{E_i[X_k(0^-)]-E_i[X_k(0^+)]\right\}, \tag{4}$$

where $X_k'(t)$ is a derivative of $X_k(t)$ at t.

Theorem 3 (Ōsawa, 2001). *The relation between distributions of the system state just before arrival points in time of both streams is given by*

$$\lambda_1 p_1^-(k) = \lambda_2 p_2^-(k+1), \qquad -K_2 \le k \le K_1 - 1. \tag{5}$$

Proof. Note that $X_k'(t) = 0$ $a.s.$ P for all k and t, then $E[X_k'(0)] = 0$.

For $1 \le k \le K_1$, since

$$E_i[X_k(0^-)] = P_i[Z^- \ge k], \qquad E_i[X_k(0^+)] = P_i[Z^+ \ge k], \qquad i = 1, 2,$$

(4) is equivalent to

$$0 \;=\; \lambda_1\left\{P_1[\,Z^- \ge k\,] - P_1[\,Z^+ \ge k\,]\right\} + \lambda_2\left\{P_2[\,Z^- \ge k\,] - P_2[\,Z^+ \ge k\,]\right\}. \tag{6}$$

Here, using the relation

$$\{Z^+ \ge k\} = \{Z^- \ge k-1\}\ a.s.\ P_1 \text{ and } \{Z^+ \ge k\} = \{Z^- \ge k+1\}\ a.s.\ P_2,$$

(6) is rewritten as

$$\begin{aligned} 0 &= \lambda_1\left\{P_1[\,Z^- \ge k\,] - P_1[\,Z^- \ge k-1\,]\right\} + \lambda_2\left\{P_2[\,Z^- \ge k\,] - P_2[\,Z^- \ge k+1\,]\right\} \\ &= -\lambda_1 p_1^-(k-1) + \lambda_2 p_2^-(k). \end{aligned}$$

Similarly, for $-K_2 \le k \le -1$, we get

$$\begin{aligned} 0 &= \lambda_1\left\{P_1[\,Z^- \le k\,] - P_1[\,Z^+ \le k\,]\right\} + \lambda_2\left\{P_2[\,Z^- \le k\,] - P_2[\,Z^+ \le k\,]\right\} \\ &= \lambda_1\left\{P_1[\,Z^- \le k\,] - P_1[\,Z^- \le k-1\,]\right\} + \lambda_2\left\{P_2[\,Z^- \le k\,] - P_2[\,Z^- \le k+1\,]\right\} \\ &= \lambda_1 p_1^-(k) - \lambda_2 p_2^-(k+1). \end{aligned}$$

Therefore, the theorem holds. □

Corollary 4 *The relation between distributions of the system state just after arrival points in time of both streams is given by*

$$\lambda_1 p_1^+(k) = \lambda_2 p_2^+(k-1), \qquad -K_2 + 1 \le k \le K_1.$$

Remark 3 Suppose that stream 1 forms a Poisson arrival of rate λ_1, then $PASTA$ (Poisson Arrivals See Time Average) property says $p_1^-(k) = p(k)$ for $-K_2 \le k \le K_1$ and thus (5) is rewritten as

$$\begin{aligned} p_2^-(k) &= \rho p(k-1), \qquad -K_2 + 1 \le k \le K_1, \\ p_2^-(-K_2) &= 1 - \rho + \rho p(K_1). \end{aligned}$$

Further, in a synchronization queue $M/K_1 + M/K_2$, we have

$$p_1^-(k) = p_2^-(k) = p(k), \qquad -K_2 \le k \le K_1.$$

4 Outputs from queues with infinite buffer

Let N_d be the output process of the synchronization queue. Then we have

$$N_d(\{t\}) = N_1(\{t\})1_{\{Z(t^-)<0\}} + N_2(\{t\})1_{\{Z(t^-)>0\}},$$

where t^- is just prior time t and 1_A is an indicator function of a set A.

In this section we are interested in the output process of a synchronization queue with a finite and an infinite capacity buffers. We consider a synchronization queue $G_1/K_1 + G_2/\infty$, that is, buffer 2 has an infinite capacity. Suppose that the synchronization operation starts at time 0 and the system is empty at that time, i.e., all buffers are empty.

Theorem 5 (Prabhakar, Bambos and Mountford, 2000). *For the synchronization queue $G_1/K_1+G_2/\infty$, if buffer 1 has a stream with intensity less than that of stream 2, i.e., $\lambda_1 < \lambda_2$, the output process converges strongly to the process N_1.*

Proof. Using theory of point processes, since $\lambda_1 < \lambda_2$, then there is a finite random time τ such that

$$N_1(0, \tau + t] < N_2(0, \tau + t] \qquad t > 0.$$

Thus we have

$$Q_2(\tau + t) \geq N_2(0, \tau + t] - N_1(0, \tau + t] > 0, \qquad t > 0,$$

which means buffer 2 can never be empty after τ. Therefore we get

$$N_d(\{t\}) = N_1(\{t\}) \ a.s. \ P, \qquad t > \tau,$$

This implies that buffer 2 has nothing to do with the output instants after τ. From this observation and synchronization operations, the theorem is obtained. □

Remark 4 The more general results for the output process of a synchronization queue with some finite or infinite capacity buffers have been shown by Prabhakar et al. (2000) and Ōsawa (2001).

Remark 5 In a queue $G_1/K_1 + G_2/\infty$ with $\lambda_1 > \lambda_2$, there exists the stationary distribution and the output process seems to be complicated.

5 Outputs from queues with finite buffers

In this section, we consider outputs from a synchronization queue with two finite buffers, $G_1/K_1 + G_2/K_2$.

5.1 System state at output point in time

We now define a Palm probability measure P_d of P with respect to N_d and distributions $\{p_d^-(k)\}$ and $\{p_d^+(k)\}$ where

$$p_d^-(k) = P_d[Z^- = k],\ p_d^+(k) = P_d[Z^+ = k], \qquad -K_2 \leq k \leq K_1.$$

Note that $\{p_d^-(k)\}$ and $\{p_d^+(k)\}$ mean stationary distributions of the system state just before and just after time points of N_d, respectively. We also denote the expectation with respect to P_d by E_d.

We study the system state at an output point in time and derive its relationship with ones at arrival time points. For our purpose, define the residual arrival time $U_i(t)$ of stream i and the residual output time $D(t)$ at time t:

$$D(t) = \begin{cases} U_2(t)\,, & Z(t) > 0, \\ U_1(t)\,, & Z(t) < 0, \\ \max(U_1(t), U_2(t)), & Z(t) = 0. \end{cases}$$

Denote the intensity of point process N_d by $\lambda_d = E\{N_d(0,1]\}$, then λ_d is the output rate. Let $D^+ = D(0^+)$ be the output interval, where 0^+ and D^+ mean time just after an output, we then have

$$\lambda_d E_d[e^{-\theta D^+}, Z^- = k] = \begin{cases} \lambda_2 E_2[e^{-\theta D^+}, Z^- = k], & 1 \leq k \leq K_1, \\ \lambda_1 E_1[e^{-\theta D^+}, Z^- = k], & -K_2 \leq k \leq -1, \end{cases} \tag{7}$$

from the ergodicity of the process. By setting $\theta = 0$ in (7) and using (5), we have the following theorem.

Theorem 6 (Ōsawa, 2001). *Relations between stationary distributions $\{p_d^-(k)\}$ and $\{p_i^-(k)\}$ are given by*

$$\lambda_d p_d^-(k) = \begin{cases} \lambda_1 p_1^-(k-1) = \lambda_2 p_2^-(k), & 1 \leq k \leq K_1, \\ \lambda_1 p_1^-(k) = \lambda_2 p_2^-(k+1), & -K_2 \leq k \leq -1. \end{cases}$$

Further, the output rate λ_d satisfies

$$\lambda_d = \lambda_1\{1 - p_1^-(K_1)\} = \lambda_2\{1 - p_2^-(-K_2)\}.$$

Remark 6 Suppose that stream 1 forms a Poisson arrival of rate λ_1, then using *PASTA* property yields

$$\begin{aligned} \lambda_d p_d^-(k) &= \begin{cases} \lambda_1 p(k-1), & 1 \leq k \leq K_1, \\ \lambda_1 p(k), & -K_2 \leq k \leq -1, \end{cases} \\ \lambda_d &= \lambda_1\{1 - p(K_1)\}. \end{aligned}$$

5.2 Output interval

Denote the stationary distribution of the output interval by $F_d(x) = P_d[D^+ \le x]$. Since

$$D^+ = \begin{cases} t_{2,1}, & Z^- > 1, \\ t_{1,1}, & Z^- < -1, \\ \max(t_{1,1}, t_{2,1}), & Z^- = 1 \text{ or } -1, \end{cases} \qquad a.s.\ P_d,$$

where $t_{1,1}$ and $t_{2,1}$ have been defined by (2), we have

$$\begin{aligned} F_d(x) &= \sum_{k=2}^{K_1} P_d\left[N_2(0,x] > 0 | Z^- = k\right] p_d^-(k) + \sum_{k=-2}^{-K_2} P_d\left[N_1(0,x] > 0 | Z^- = k\right] p_d^-(k) \\ &\quad + P_d\left[N_1(0,x] > 0, N_2(0,x] > 0, Z^- = 1\right] \\ &\quad + P_d\left[N_1(0,x] > 0, N_2(0,x] > 0, Z^- = -1\right]. \end{aligned} \tag{8}$$

Suppose that both streams are renewal point processes with distribution function $G_i(x)$ of the arrival time, $i = 1, 2$. Then (8) becomes

$$\begin{aligned} F_d(x) &= G_2(x) \sum_{k=2}^{K_1} p_d^-(k) + G_1(x) \sum_{k=-2}^{-K_2} p_d^-(k) \\ &\quad + G_2(x) P_d[U_1 \le x] p_d^-(1) + G_1(x) P_d[U_2 \le x] p_d^-(-1). \end{aligned} \tag{9}$$

Remark 7 In a queue $M_1/K_1 + M_2/K_2$, since

$$p_d^-(k) = \frac{1-\rho}{\rho^{-K_2} - \rho^{K_1}} \rho^k, \qquad -K_2 \le k \le -1, 1 \le k \le K_1,$$

$F_d(x)$ can be calculated by (9). Som et al. (1994) derived $F_d(x)$ by a direct calculation.

For $M/K_1 + PH/K_2$, the stationary distribution $\{p_d^-(k)\}$ is given by

$$p_d^-(k) = \begin{cases} \nu \boldsymbol{\pi}(0) \boldsymbol{R}^{k-1} \boldsymbol{e}, & 1 \le k \le K_1, \\ \nu \boldsymbol{\pi}(0) \boldsymbol{R}^{k} \boldsymbol{e}, & -K_2 \le k \le -1, \end{cases}$$

where $\nu = \lambda_1/\lambda_d$. Applying this result, it is shown that $F_d(x)$ has a PH distribution with some representation. Takahashi et al. (2000) derived it by a direct calculation.

5.3 Remaining output time

Let $D = D(0)$, $D^- = D(0^-)$ and $F(x) = P[D \le x]$, then we should note that $F(x)$ is the stationary distributions of the remaining output time at an arbitrary point in time. We consider the relationship between $F(x)$ and $F_d(x)$. For our purpose, define

$$X_k^d(t) = \begin{cases} 1_{\{Z(t) \ge k\}} e^{-\theta D(t)}, & 1 \le k \le K_1, \\ 1_{\{Z(t) \le k\}} e^{-\theta D(t)}, & -K_2 \le k \le -1. \end{cases}$$

Applying again the rate conservation law to $X_k^d(t)$, N_1 and N_2, we have the following lemma.

Lemma 7 *For* $1 \le k \le K_1$,

$$\begin{aligned}\theta E[e^{-\theta D}, Z \ge k] &= -\lambda_1 E_1[e^{-\theta D}, Z^- = k-1] + \lambda_d\{P_d[Z^- \ge k] \\ &\quad - E_d[e^{-\theta D^+}, Z^- \ge k+1]\}. \qquad (10)\end{aligned}$$

For $-K_2 \le k \le -1$,

$$\begin{aligned}\theta E[e^{-\theta D}, Z \le k] &= -\lambda_2 E_2[e^{-\theta D}, Z^- = k+1] + \lambda_d\{P_d[Z^- \le k] \\ &\quad - E_d[e^{-\theta D^+}, Z^- \le k-1]\}. \qquad (11)\end{aligned}$$

Proof. Since $D'(t) = -1$ on $\{Z(t) \ge k\}$, by the rate conservation law to $X_k^d(t)$, N_1 and N_2 for $1 \le k \le K_1$, we have

$$\begin{aligned}\theta E[e^{-\theta D}, Z \ge k] &= \lambda_1\{E_1[e^{-\theta D^-}, Z^- \ge k] - E_1[e^{-\theta D^+}, Z^+ \ge k]\} \\ &\quad + \lambda_2\{E_2[e^{-\theta D^-}, Z^- \ge k] - E_d[e^{-\theta D^+}, Z^+ \ge k]\}, \\ &= \lambda_1\{E_1[e^{-\theta D}, Z^- \ge k] - E_1[e^{-\theta D}, Z^- \ge k-1]\} \\ &\quad + \lambda_2\{P_2[Z^- \ge k] - E_2[e^{-\theta D^+}, Z^- \ge k+1]\}, \\ &= -\lambda_1 E_1[e^{-\theta D}, Z^- = k-1]\} \\ &\quad + \lambda_2\{P_2[Z^- \ge k] - E_2[e^{-\theta D^+}, Z^- \ge k+1]\}, \qquad (12)\end{aligned}$$

where we should note that $D^- = D^+$ on $\{Z \ge 1\}$ *a.s.* P_1 and $D^- = 0$ on $\{Z \ge 1\}$ *a.s.* P_2. Using (7) and Lemma 6, (12) becomes (10).

In the same way, for $-K_2 \le k \le -1$, we have (11). □

Remark 8 For $k = K_1$ and $-K_2$, (10) and (11) are rewritten as follows:

$$\begin{aligned}\theta E[e^{-\theta D}, Z = K_1] &= -\lambda_1 E_1[e^{-\theta D}, Z^- = K_1 - 1] + \lambda_d P_d[Z^- = K_1], \\ \theta E[e^{-\theta D}, Z = -K_2] &= -\lambda_2 E_2[e^{-\theta D}, Z^- = -K_2 + 1] + \lambda_d P_d[Z^- = -K_2].\end{aligned}$$

Lemma 8

$$\begin{aligned}\theta E[e^{-\theta D}, Z = 0] &= \lambda_1 E_1[e^{-\theta D^-}, Z^- = 0] + \lambda_2 E_2[e^{-\theta D^-}, Z^- = 0] \\ &\quad - \lambda_d\{E_d[e^{-\theta D^+}, Z^- = -1] + E_d[e^{-\theta D^+}, Z^- = 1]\}. \qquad (13)\end{aligned}$$

Proof. Define

$$X_0^d(t) = 1_{\{Z(t)=0\}} e^{-\theta D(t)},$$

and apply the rate conservation law to $X_0^d(t)$, N_1 and N_2. Then we have

$$\begin{aligned}\theta E[e^{-\theta D}, Z = 0] &= \lambda_1\{E_1[e^{-\theta D^-}, Z^- = 0] - E_1[e^{-\theta D^+}, Z^+ = 0]\} \\ &\quad + \lambda_2\{E_2[e^{-\theta D^-}, Z^- = 0] - E_2[e^{-\theta D^+}, Z^+ = 0]\}, \\ &= \lambda_1\{E_1[e^{-\theta D^-}, Z^- = 0] - E_1[e^{-\theta D^+}, Z^- = -1]\} \\ &\quad + \lambda_2\{E_2[e^{-\theta D^-}, Z^- = 0] - E_2[e^{-\theta D^+}, Z^- = 1]\}, \\ &= \lambda_1 E_1[e^{-\theta D^-}, Z^- = 0] + \lambda_2 E_2[e^{-\theta D^-}, Z^- = 0] \\ &\quad - \lambda_d\{E_d[e^{-\theta D^+}, Z^- = -1] + E_d[e^{-\theta D^+}, Z^- = 1]\}.\end{aligned}$$

□

Let $F_d^{(e)}(x)$ be the stationary excess time distribution of $F_d(x)$. ¿From the above arguments, we have the following result.

Theorem 9 (Ōsawa, 2001). *In the synchronization queue $G_1/K_1 + G_2/K_2$ with finite capacity buffers, the stationary distribution of the residual output time at an arbitrary point in time is equivalent to $F_d^{(e)}(x)$.*

Proof. For $j = 1$ in (10) and $j = -1$ in (11), we have

$$\begin{aligned}\theta E[e^{-\theta D}, Z \geq 1] &= \lambda_d E_d[1 - e^{-\theta D^+}, Z^- \geq 2] + \lambda_1 E_1[1 - e^{-\theta D}, Z^- = 0],\\ \theta E[e^{-\theta D}, Z \leq -1] &= \lambda_d E_d[1 - e^{-\theta D^+}, Z^- \leq -2] + \lambda_2 E_2[1 - e^{-\theta D}, Z^- = 0].\end{aligned}$$

Using these equations and (13), we get

$$\begin{aligned}\theta E[e^{-\theta D}] &= \lambda_d E_d[1 - e^{-\theta D^+}, Z^- \geq 2] + \lambda_d\{p_d^-(1) - E_d[e^{-\theta D+}, Z^- = 1]\}\\ &+ \lambda_d E_d[1 - e^{-\theta D^+}, Z^- \leq -2] + \lambda_d\{p_d^-(-1) - E_d[e^{-\theta D+}, Z^- = -1]\}\\ &= \lambda_d E_d[1 - e^{-\theta D^+}].\end{aligned}$$

Since we now have

$$E[e^{-\theta D}] = \lambda_d E_d\left[\frac{1 - e^{-\theta D^+}}{\theta}\right],$$

the theorem holds. □

References

[1] Bhat, U.N. (1986). Finite capacity assembly-like queues, *Queueing Systems* 1, 85-101.

[2] Baccelli, F., Massey, W.A. and Towsley, D. (1989). Acyclic fork-join queueing networks, *J. of A.C.M.* 36, 615-642.

[3] Hopp, W.J. and Simon, J.T. (1989). Bounds and heuristics for assembly-like queues, *Queueing Systems* 4, 137-156.

[4] Latouche, G. (1981). Queues with paired customers, *Journal of Applied Probability* 18, 684-696.

[5] Lipper, E.H. and Sengupta, B. (1986). Assembly-like queues with finite capacity: bounds, asymptotic and approximations, *Queueing Systems* 1, 67-83.

[6] Miyazawa, M. (1983). The derivation of invariance relations and their applications, *Advances in Applied Probability* 15, 874-885.

[7] Miyazawa, M. and Yamazaki, G. (1992). Relationships in stationary jump processes with countable state space and their applications to queues, *Stochastic Processes and their Applications* 43, 619-630.

[8] Neuts, M.F. (1981). *Matrix-Geometric Solutions in Stochastic Models*, The John Hopkins University Press, Baltimore, Maryland.

[9] Ōsawa, H. (2001). On output processes of synchronization queues, *International Transactions in Operational Research* 8, 193-201.

[10] Prabhakar, B., Bambos, N. and Mountford, T.S. (2000). The synchronization of Poisson processes and queueing networks with service and synchronization nodes, *Advances in Applied Probability*.

[11] Som, P., Wilhelm, W.E. and Disney, R.L. (1994). Kitting process in a stochastic assembly system, *Queueing Systems* 17, 471-490.

[12] Srivastava, H.M. and Kashyap, B.R.K. (1982). *Special Functions in Queueing Theory and Related Stochastic Process*, Academic Press.

[13] Takahashi, M., Ōsawa, H. and Fujisawa, T. (1998). A stochastic assembly system with resume levels, *Asia-Pacific Journal of Operational Research* 15, 127-146.

[14] Takahashi, M., Ōsawa, H. and Fujisawa, T. (2000). On the synchronization queue with two finite buffers, *Queueing Systems* 36, 107-123.

Hideo Ōsawa, E-mail: *ohsawa@eco.nihon-u.ac.jp*