The cut locus of a surface of revolution *

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For investigating the behavior of geodesics emanating from a point p on a complete Riemannian manifold, we need study the structure of the cut locus of p. The very important property of the cut locus is that any two distinct geodesic segments emanating from p do not meet again if they do not intersect the cut locus of p. Let (M, g) be a complete Riemannian manifold and $\gamma : [0, t_0] \to M$ a minimal geodesic segment emanating from a point $p := \gamma(0)$. The endpoint $\gamma(t_0)$ of the geodesic segment is called a *cut point* of p along γ if any extended geodesic segment $\tilde{\gamma} : [0, t_1] \to M$ of γ, where $t_1 > t_0$, is not a minimizing arc joining p to $\gamma(t_1)$ anymore. The *cut locus* C_p of the point p is defined by the set of the cut points along all geodesic segments emanating from p. It is known that the cut locus of a point p on a complete 2-dimensional Riemannian manifold is a local tree (see [2] or [5]), i.e., for any $q \in C_p$ and any neighborhood U around q in M, there exists an open neighborhood $V \subset U$ around q such that any two points in V can be joined by a unique rectifiable Jordan arc in V. Here a *Jordan arc* is an arc homeomorphic to the interval $[0, 1]$. The cut locus of a point p on a complete Riemannian manifold is also defined as the closure of the set of all points q on M admitting two minimal geodesic segments joining q to p. It is difficult in general to determine the structure of a cut locus on a complete Riemannian manifold.

We introduce some structure theorems of the cut loci of a surface of revolution.

Theorem 1 (Elerath [1]) *Let M be a 2-sheeted hyperboloid $z = \frac{1}{a}\sqrt{x^2 + y^2 + b}$, $(a, b > 0)$ or a paraboloid $z = a(x^2 + y^2)$, $(a > 0)$. Then the cut locus of a point $p = (x_0, 0, z_0)$ with $x_0 > 0$ is a subset of the opposite meridian $\{(-x, 0, z) \in M\ ;\ x > 0\}$.*

Theorem 2 (Sinclair-Tanaka) *Let p be a point on an ellipsoid of revolution. Then the cut locus of the point p is an arc (or a point)*

Theorem 3 (Gravesen-Markvosen-Sinclair-Tanaka [3]) *The cut locus of a point p =*

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$(x_0, 0, z_0)$ with $x_0 > 0$ on the standard torus in Euclidean space defined by

$$(\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2 \quad (R > r > 0)$$

is the union of

(i) *the opposite meridian $y = 0, x < 0$,*

(ii) *a (piecewise C^1) Jordan curve which intersects the opposite meridian at a single point and is freely homotopic to each meridian,*

(see Figure 1) and, if p is sufficiently far from the inner equator, i.e., if $x_0 > c_2$ for some positive constant $c_2 (> R - r)$,

(iii) *a pair of subarcs of the parallel $z = -z_0$, each with a conjugate point of p as one endpoint and joining*

- *only the Jordan curve of (ii) if $c_2 < x_0 < c_1$ for some c_1, (see Figure 2)*
- *only the meridian of (i) if $c_1 < x_0$, (see Figure 3) or*
- *both of the above if $x_0 = c_1$ (see Figure 4)*

at their other endpoint.

Remark The four pictures of the cut locus was drawn by the computer program *Loki*, which was made by R.Sinclair ([4]). Therefore, this implies that Loki and the result proved theoretically above has reached the same conclusion.

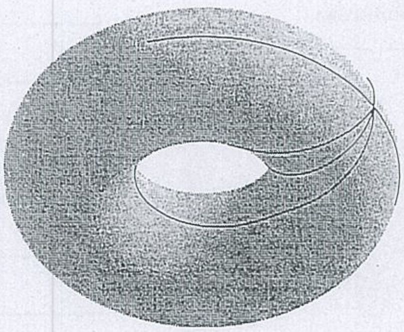

Figure 4: Cut loci on the surface $(\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1$. This is $C_{p(u_1)}$.

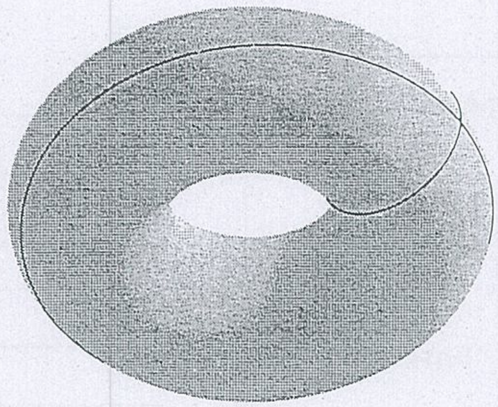

Figure 1: Cut loci on the surface $(\sqrt{x^2+y^2}-2)^2+z^2=1$, for which $u_* \approx 0.63514$, $u_2 \approx 2.43309$ and $u_1 \approx 2.98009$. This illustration is of $C_{p(u_*)}$.

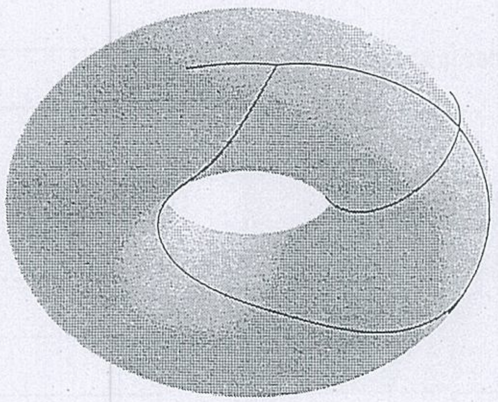

Figure 2: Cut loci on the surface $(\sqrt{x^2+y^2}-2)^2+z^2=1$. This is $C_{p(2.7)}$.

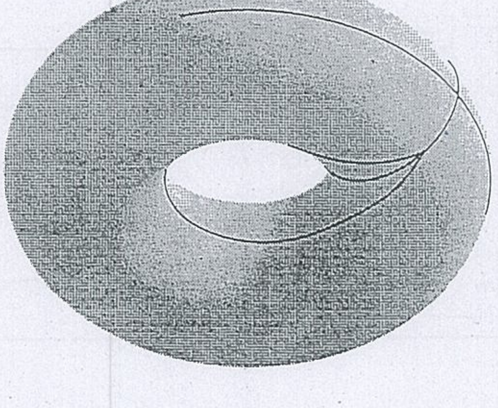

Figure 3: Cut loci on the surface $(\sqrt{x^2+y^2}-2)^2+z^2=1$. This is $C_{p(3.0)}$.

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