VIEWING AN ANGLE FROM VARIOUS VIEW-POINTS

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1. INTRODUCTION. We always get lots of information from outside. An angle configured in 3D Euclidean space changes its visual angle by the viewpoint of observers. This visual angle also gives us some information. For example, it is well known that four angles of a rectangle $ABCD$ teach us its field of vision that is $A + B + C + D - \pi$.

In this note, let us focus on only one angle. Generally, for a given angle φ, its visual angle θ_φ takes a value from 0 to π. Fix the viewpoint and the vertex of angle with vertical angle φ, and configure randomly the open direction of the angle. How much is the expectation of its visual angle θ_φ? It is simply equal to the original angle φ. Using spherical geometry, we can derive this equation in two different ways: direct calculation and application Santaló's chord theorem to polar triangle.

2. STATEMENT OF THE RESULT.

Theorem 1. *For any angle φ, the expectation of its visual angle θ_φ is equal to φ, i.e.,*

$$E(\theta_\varphi) = \varphi. \tag{1}$$

Let us establish this theorem with spherical geometry. Set a viewpoint P on the unit sphere centered at the origin O, and make an angle φ as $\angle AOB$ where $A = \left(\cos\frac{\varphi}{2}, -\sin\frac{\varphi}{2}, 0\right)$ and $B = \left(\cos\frac{\varphi}{2}, \sin\frac{\varphi}{2}, 0\right)$. Then, it is easy to know that the visual angle θ_φ of the angle φ from P corresponds to $\angle APB$ of the spherical triangle $\triangle ABP$. Therefore, the expectation is calculated on the unit sphere with uniform density.

3. ONE PROOF. In this section, we prove Theorem 1 by the integral of θ_φ along a latitude of the unit sphere.

Proof of Theorem 1. First, equation (1) holds trivially when $\varphi = 0$ or

π, because θ_φ are almost surely equal to φ in each case. For the case of $\varphi = \pi/2$, we prepare the following proposition.

Proposition 2. *For any* φ,

$$E(\theta_\varphi) + E(\theta_{\pi-\varphi}) = \pi. \tag{2}$$

proof of Proposition 2. Let $-A$ be the antipodal point of A. $\angle(-A)PB$ corresponds to the visual angle of $\angle(-A)OB = \pi - \varphi$. The equation $\angle APB + \angle(-A)PB = \pi$ and the linearity of expectation imply equation (2). ■

Therefore, in particular, substitution of $\pi/2$ in φ yields the equation $E(\theta_{\pi/2}) = \pi/2$. If $\varphi \neq 0, \pi/2, \pi$, however, equation (1) is not trivial. From equation (2), let us assume that $\varphi \in [0, \pi/2]$ in the following argument. We can also put restriction on $P(x, y, z)$ such that both x, y, and z are positive, because $P_1(-x, y, z), P_2(x, -y, z)$, and $P_3(x, y, -z)$ have the same visual angles as that of $P(x, y, z)$. Take polar coordinates such as $P(x, y, z) = P(\sin s \cos t, \sin s \sin t, \cos s)$ where $s, t \in [0, \pi/2]$. By the spherical cosine law for sides,

$$\cos \widehat{AB} = \cos \widehat{AP} \cos \widehat{BP} + \sin \widehat{AP} \sin \widehat{BP} \cos \angle APB,$$

hence we obtain

$$\theta_\varphi = \cos^{-1}\left(\frac{\cos\varphi - c\cos 2t}{\sqrt{(1 - c\cos(2t-\varphi))(1 - c\cos(2t+\varphi))}}\right)$$

where

$$c = \frac{1 - \cos^2 s}{1 + \cos^2 s} \in [0, 1].$$

Figure 1 shows the contour lines of θ_φ and note that θ_φ is indefinite at A, B, $-A$, and $-B$.

Let us consider the expectation on a latitude. For the completeness of the proof of Theorem 1, it suffices to show the following identical equation with respect to c and φ ($c \in [0, 1], \varphi \in [0, \pi/2]$):

$$\int_0^{\pi/2} \cos^{-1}\left(\frac{\cos\varphi - c\cos 2t}{\sqrt{(1 - c\cos(2t-\varphi))(1 - c\cos(2t+\varphi))}}\right) dt = \frac{\pi}{2}\varphi, \tag{3}$$

because equation (3) means that the expectation of θ_φ on any fixed latitude is equal to φ.

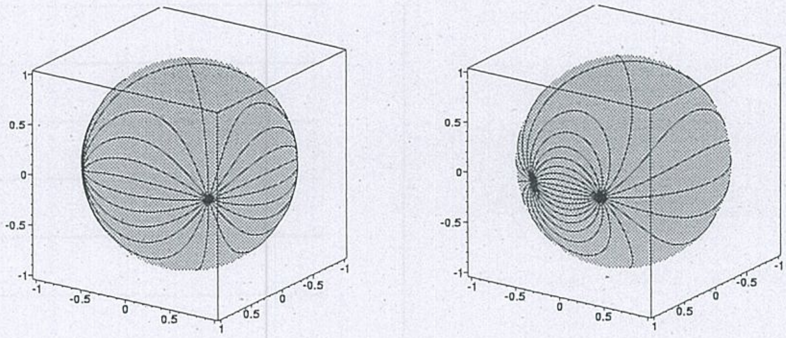

Figure 1: Contour lines of θ_φ for $\varphi = \pi/2$ (left) and $\varphi = \pi/3$ (right).

Equation (3) is trivial when $c = 0$ or 1. Let $c \in (0,1)$ and fix it. Differentiating the left side of equation (3),

$$\begin{aligned}
\frac{d}{d\varphi}\int_0^{\pi/2} \theta_\varphi \, dt &= \int_0^{\pi/2} \frac{d}{d\varphi}\theta_\varphi \, dt \\
&= \int_0^{\pi/2} \frac{\sqrt{1-c^2}(1 - c\cos\varphi\cos 2t)}{(1 - c\cos(2t-\varphi))(1 - c\cos(2t+\varphi))} \, dt \\
&= \int_0^{\pi/2} \frac{\sqrt{1-c^2}}{2}\left(\frac{1}{1 - c\cos(2t-\varphi)} + \frac{1}{1 - c\cos(2t+\varphi)}\right) dt \\
&= \int_{-\varphi/2}^{\pi/2-\varphi/2} + \int_{\varphi/2}^{\pi/2+\varphi/2} \frac{\sqrt{1-c^2}}{2(1 - c\cos 2t)} \, dt \\
&= \int_0^{\pi/2} \frac{\sqrt{1-c^2}}{1 - c\cos 2t} \, dt \;=\; \frac{\pi}{2}.
\end{aligned}$$

Therefore the left side of equation (3) is written as $\pi/2 \cdot \varphi + C$ where C is a constant. On the other hand, if $\varphi = 0$,

$$\int_0^{\pi/2} \theta_0 \, dt = 0$$

for any c, it implies $C = 0$. Thus the proof of Theorem 1 is complete. ■

4. ANOTHER PROOF. We can easily prove Theorem 1 by Santaló's chord theorem [1].

Proof of Theorem 1. P is a random point on the unit sphere. Let us consider the polar(dual) triangle $\triangle A^*B^*P^*$ of the spherical triangle $\triangle ABP$:

$$\triangle A^*B^*P^* = H(A) \cap H(B) \cap H(P)$$

where $H(A)$ is the hemisphere including A as the pole, and so on. Then,

$$\begin{aligned} \widehat{AB} + \angle A^*P^*B^* &= \pi, \\ \widehat{A^*B^*} + \angle APB &= \pi, \end{aligned}$$

hence $\angle A^*P^*B^* = \pi - \varphi$ and $\widehat{A^*B^*} = \pi - \theta_\varphi$. Let M denote the spherical lune $H(A) \cap H(B)$ with angle $\pi - \varphi$. Note that the boundary of $H(P)$ is a random great circle including $\widehat{A^*B^*}$ which is the arc cut out from M. From Santaló's chord theorem,

$$E(\widehat{A^*B^*}) = \frac{\text{area}(M)}{2},$$

therefore $E(\pi-\theta_\varphi) = \pi-\varphi$, that is, $E(\theta_\varphi) = \varphi$. ∎

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REFERENCES

1. L. A. Santaló, Integral formulas in Crofton's style on the sphere and some inequalities referring to spherical curves, *Duke Math. J.* 9 (1942) 707-722.