On the Application of the Distribution $e^{\alpha t}\delta^{(k)}$ to the Electrical Networks

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Abstract

In this paper, the distribution $e^{\alpha t}\delta^{(k)}$ where α is a complex constant and $\delta^{(k)}$ is the Dirac -delta distribution with k-derivatives and $t \in [0, \infty)$ is to be applied in solving the solution of differential equation in the circuit theory.

1 Introduction

We know that the flow of electric current I in a simple series circuit with inductance L, resistance R and capacitance C is given by

$$L\frac{d}{dt}I(t) + RI(t) + \frac{1}{C}Q(t) = E(t)$$

where t is the time, Q is the total charge and E is the impressed voltage. Since $I(t) = \frac{d}{dt}Q(t)$, then we also obtain the second order equation

$$L\frac{d^2}{dt^2}Q(t) + R\frac{d}{dt}Q(t) + \frac{1}{C}Q(t) = E(t).$$

It is not difficult to find the solution $Q(t)$ of the equation and the charge $Q(t)$ is always an ordinary function which is continuous for $t \in [0, \infty)$.

But in this paper, we study the case $E(t)$ is replaced by the electromotive force $\sum_{k=0}^{m} c_k\delta^{(k)}(t)$ where $\delta(t)$ with its derivatives is the impulse function and c_k is a constant and $\delta^{(0)} = \delta$. Now we consider the equation

$$L\frac{d^2}{dt^2}Q(t) + R\frac{d}{dt}Q(t) + \frac{1}{C}Q(t) = \sum_{k=0}^{m} c_k\delta^{(k)}(t) \tag{1}$$

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It is found that the solution of the equation (1) need not be an ordinary function, it may be the distribution that is the solution in the space $\mathcal{D}'_{\mathbb{R}}$ of distribution whose supports are bounded on the left. All types of those solutions depending on the value of m are as the following cases :

1. If $m \geq 2$ then there exists the solutions of the equation (1) that belong to the space $\mathcal{D}'_{\mathbb{R}}$.

2. If $0 \leq m < 2$ then all solutions are ordinary functions that are continuous for $t \in [0, \infty)$ and it also follows that if $m \geq 1$ the current $I(t)$ is not an ordinary function but it is the distribution in the space $\mathcal{D}'_{\mathbb{R}}$, that mean the current $I(t)$ is not continuous and it occurs in very short time intervals.
   If $m = 0$ then the current $I(t)$ is continuous function for $t \in [0, \infty)$ that mean the current $I(t)$ flows continuously for the time $t \in [0, \infty)$.

In solving the solution of the equation (1) that is the charge $Q(t)$, we use the method of convolution of $e^{\alpha t}\delta^{(k)}$ with some distributions to apply and also the Laplace transform is needed. Before going to that point, the following properties of $e^{\alpha t}\delta^{(k)}$ are first introduced.

2 Some Properties of $e^{\alpha t}\delta^{(k)}$

Property 2.1 $e^{\alpha t}\delta^{(k)} = (D - \alpha)^k\delta$ *where* $D \equiv \frac{d}{dt}$ *and* $e^{\alpha t}\delta^{(k)}$ *is a tempered distribution of order* k *with support* $\{0\}$.

Proof By the definition of distribution and δ

$$
\begin{aligned}
< e^{\alpha t}\delta^{(k)}, \varphi > &= < \delta^{(k)}, e^{\alpha t}\varphi(t) > \\
&= < \delta, (-1)^k \sum_{\nu=0}^{k} {}^kC_\nu (e^{\alpha t})^{(\nu)}(\varphi(t))^{(k-\nu)} >
\end{aligned}
$$

for every $\varphi \in \mathcal{D}$ and also $e^{\alpha t}\varphi(t) \in \mathcal{D}$ where $\mathcal{D}$ is the space of continuous function of infinitely differentiable with bounded supports.
Hence

$$
\begin{aligned}
< \delta^{(k)}, e^{\alpha t}\varphi(t) > &= (-1)^k \sum_{\nu=0}^{k} {}^kC_\nu \alpha^\nu \varphi^{(k-\nu)}(0) \\
&= \sum_{\nu=0}^{k} (-1)^{\nu} {}^kC_\nu \alpha^\nu < \delta^{(k-\nu)}, \varphi > \\
&= < (D - \alpha)^k\delta, \varphi >
\end{aligned}
$$

where $D = \frac{d}{dt}$. It follow that $e^{\alpha t}\delta^{(k)} = (D-\alpha)^k\delta$.

Since $\delta^{(k)}$ is a tempered distribution by L. Schwartz [1], hence so is $(D-\alpha)^k$ and it follows that $e^{\alpha t}\delta^{(k)}$ is a tempered as required.

Now

$$e^{\alpha t}\delta^{(k)} = (D-\alpha)^k\delta = \sum_{\nu=0}^{k}(-1)^{\nu k}C_\nu \alpha^\nu \delta^{(k-\nu)},$$

this mean that $e^{\alpha t}\delta^{(k)}$ is a finite linear combination of Direc-delta distribution and its derivative up to order k. Hence, by A.H. Zemanian [2](Theorem 3.5-2, p98)$e^{\alpha t}\delta^{(k)}$ is of order k with a point support $\{0\}$.

Property 2.2 *(The convolution of $e^{\alpha t}\delta^{(k)}$ with some distributions)*

1. $(e^{\alpha t}\delta^{(k)}) * f = (D-\alpha)^k f$ *where* $D \equiv \frac{d}{dt}$ *and* f *is some distributions in the space* $\mathcal{D}$ *of distributions.*
2. $\left[(e^{\alpha_1 t}\delta^{(k_1)}) * \cdots * (e^{\alpha_n t}\delta^{(k_n)})\right] * f = (D-\alpha_1)^{k_1}\cdots(D-\alpha_n)^{k_n} f$

where $\alpha_1, \ldots, \alpha_n$ *are complex constants,* $D \equiv \frac{d}{dt}$ *and* $k_1, \ldots, k_n$ *are positive integers and* $f \in \mathcal{D}'$.

Proof 1. Since $(e^{\alpha t}\delta^{(k)}) * f = (D-\alpha)^k * f$, by Property 2.1.

Hence

$$\begin{aligned}(e^{\alpha t}\delta^{(k)}) * f &= \left[\sum_{\nu=0}^{k}(-1)^{\nu\, k}C_\nu \alpha^\nu \delta^{(k-\nu)}\right] * f \\ &= \sum_{\nu=0}^{k}(-1)^{\nu\, k}C_\nu \alpha^\nu (\delta^{(k-\nu)} * f) \\ &= \sum_{\nu=0}^{k}(-1)^{\nu\, k}C_\nu \alpha^\nu f^{(k-\nu)} \\ &= (D-\alpha)^k f \quad \text{since } \delta^{(k)} * f, = f^{(k)}.\end{aligned}$$

2. Since $(e^{\alpha_i t}\delta^{(k_i)})$ is a tempered, then we can take the Laplace transform to the convolution $(e^{\alpha_1 t}\delta^{(k_1)}) * \cdots * (e^{\alpha_n t}\delta^{(k_n)})$, that is

$$\begin{aligned}L\left[(e^{\alpha_1 t}\delta^{(k_1)}) * \cdots * (e^{\alpha_n t}\delta^{(k_n)})\right] &= < (e^{\alpha_1 t}\delta^{(k_1)}), e^{-st} > \cdots < (e^{\alpha_n t}\delta^{(k_n)}), e^{-st} > \\ &= < \delta, (s-\alpha_1)^{k_1}e^{-st} > \cdots < \delta, (s-\alpha_n)^{k_n}e^{-st} > \\ &= (s-\alpha_1)^{k_1}\cdots(s-\alpha_n)^{k_n}.\end{aligned}$$

Take the inverse Laplace transform, we obtain

$$(e^{\alpha_1 t}\delta^{(k_1)}) * \cdots * (e^{\alpha_n t}\delta^{(k_n)}) = (D-\alpha_1)^{k_1}\cdots(D-\alpha_n)^{k_n}\delta,$$

and similarly, as 1.,

$$[(e^{\alpha_1 t}\delta^{(k_1)}) * \cdots * (e^{\alpha_n t}\delta^{(k_n)})] * f = (D-\alpha_1)^{k_1}\cdots(D-\alpha_n)^{k_n} f.$$

That completes the proof.

3 The Application of $e^{\alpha t}\delta^{(k)}$

Recall that the equation

$$L\frac{d^2}{dt^2}Q(t) + R\frac{d}{dt}Q(t) + \frac{1}{C}Q(t) = \sum_{k=0}^{m} c_k\delta^{(k)}(t) \tag{2}$$

Now (2) can be written as the form

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right)Q(t) = \frac{1}{L}\sum_{k=0}^{m} c_k\delta^{(k)}(t)$$

where $D = \frac{d}{dt}$; or

$$\left(D - (-\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}})\right)\left(D - (-\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}})\right)Q(t) = \frac{1}{L}\sum_{k=0}^{m} c_k\delta^{(k)}(t)$$

For simplicity, let

$$\omega_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}},$$

and

$$\omega_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}.$$

By applying the property 2.2(2) to (2) with $k_1 = k_2 = 1$ and $\alpha_1 = \omega_1$, $\alpha_2 = \omega_2$ then (2) can be written as the form

$$[(e^{\omega_1 t}\delta^{(1)}) * (e^{\omega_2 t}\delta^{(1)})] * Q(t) = \frac{1}{L}\sum_{k=0}^{m} c_k\delta^{(k)}(t) \tag{3}$$

Actually, $e^{\omega_1 t}$ and $e^{\omega_2 t}$ are the solutions of homogeneous equation of (2) with the right-hand side vanishes.

Now we can find the charge $Q(t)$ in (3) by convolving both sides of (3) with the inverse of $(e^{\omega_1 t}\delta^{(1)}) * (e^{\omega_2 t}\delta^{(1)})$. By making use of the convolution algebra, Kananthai A. [3] has proved that the inverse of $(e^{\omega_1 t}\delta^{(1)}) * (e^{\omega_2 t}\delta^{(1)})$ is

$$\frac{1}{\omega_1 - \omega_2}(e^{\omega_1 t} - e^{\omega_2 t})H(t)$$

where $H(t)$ is a Heaviside function, that is

$$H(t) = \begin{cases} 1 & \text{for } t \in [0, \infty), \\ 0 & \text{for } t \in (-\infty, 0). \end{cases}$$

Now convolving both sides of (3) by $\frac{1}{\omega_1 - \omega_2}(e^{\omega_1 t} - e^{\omega_2 t})H(t)$, we then obtain the charge

$$Q(t) = \left(\frac{1}{\omega_1 - \omega_2}(e^{\omega_1 t} - e^{\omega_2 t})H(t)\right) * \left(\frac{1}{L}\sum_{k=0}^{m} c_k \delta^{(k)}(t)\right) \tag{4}$$

which is the solution of (2). By computing directly

$$\begin{aligned} Q(t) = &\frac{1}{L(\omega_1 - \omega_2)}\sum_{k=0}^{m} c_k(\omega_1)e^{\omega_1 t}H(t) \\ &+ \frac{1}{L(\omega_1 - \omega_2)}\sum_{k=1}^{m}\sum_{r=0}^{k-1} c_k(\omega_1)^r \delta^{(k-1-r)}(t) \\ &- \frac{1}{L(\omega_1 - \omega_2)}\sum_{k=0}^{m} c_k(\omega_2)^k e^{\omega_2 t}H(t) \\ &- \frac{1}{L(\omega_1 - \omega_2)}\sum_{k=1}^{m}\sum_{r=0}^{k-1} c_k(\omega_2)^r \delta^{(k-1-r)}(t). \end{aligned} \tag{5}$$

Now consider the following cases :

(1) If $m \geq 2$ the the right-hand side of (5) contains the Dirac-delta distribution and its derivatives. That means that the charge $Q(t)$ is not an ordinary function but it is the distribution in the space $\mathcal{D}'_{\mathbb{R}}$.

(2) If $0 \leq m < 2\,(m = 0, 1)$ then for $m = 1$, from (5) we obtain

$$\begin{aligned} Q(t) &= \frac{1}{L(\omega_1 - \omega_2)}\left[c_0 e^{\omega_1 t}H(t) + c_1\omega_1 e^{\omega_1 t}H(t) + c_1\delta(t)\right] \\ &- \frac{1}{L(\omega_1 - \omega_2)}\left[c_0 e^{\omega_2 t}H(t) + c_1\omega_2 e^{\omega_2 t}H(t) + c_1\delta(t)\right] \\ &= \frac{1}{L(\omega_1 - \omega_2)}c_0 H(t)\left[e^{\omega_1 t} - e^{\omega_2 t}\right] + \frac{1}{L(\omega_1 - \omega_2)}c_1 H(t)\left[\omega_1 e^{\omega_1 t} - \omega_2 e^{\omega_2 t}\right] \end{aligned}$$

That $Q(t)$ is the continuous function for $t \in [0, \infty)$.
For $m = 0$, then

$$Q(t) = \frac{1}{L(\omega_1 - \omega_2)} \left[c_0 e^{\omega_1 t} H(t) - c_0 e^{\omega_2 t} H(t)\right]$$
$$= \frac{c_0 H(t)}{L(\omega_1 - \omega_2)} \left[e^{\omega_1 t} - e^{\omega_2 t}\right].$$

and also is the continuous function for $t \in [0, \infty)$.
Now consider the current $I(t)$, we know that $I(t) = \frac{d}{dt}Q(t)$, hence by (4)

$$I(t) = \frac{1}{\omega_1 - \omega_2} \frac{d}{dt} \left[\left((e^{\omega_1 t} - e^{\omega_2 t}) H(t) \right) * \left(\frac{1}{L} \sum_{k=0}^{m} c_k \delta^{(k)}(t) \right) \right]$$
$$= \frac{1}{\omega_1 - \omega_2} \left[(\omega_1 e^{\omega_1 t} - \omega_2 e^{\omega_2 t}) H(t) + (e^{\omega_1 t} - e^{\omega_2 t}) \delta \right] * \left(\frac{1}{L} \sum_{k=0}^{m} c_k \delta^{(k)}(t) \right)$$
$$= \frac{1}{\omega_1 - \omega_2} \left[(\omega_1 e^{\omega_1 t} - \omega_2 e^{\omega_2 t}) H(t) \right] * \left(\frac{1}{L} \sum_{k=0}^{m} c_k \delta^{(k)}(t) \right).$$

Since $H'(t) = \delta$ and $(e^{\omega_1 t} - e^{\omega_2 t})\delta = 0$.
By computing directly,

$$I(t) = \frac{\omega_1}{L(\omega_1 - \omega_2)} \sum_{k=0}^{m} c_k (\omega_1)^k e^{\omega_1 t} H(t)$$
$$+ \frac{\omega_1}{L(\omega_1 - \omega_2)} \sum_{k=1}^{m} \sum_{r=0}^{k-1} c_k (\omega_1)^r \delta^{(k-1-r)}(t)$$
$$- \frac{\omega_2}{L(\omega_1 - \omega_2)} \sum_{k=0}^{m} c_k (\omega_2)^k e^{\omega_2 t} H(t)$$
$$- \frac{\omega_2}{L(\omega_1 - \omega_2)} \sum_{k=1}^{m} \sum_{r=0}^{k-1} c_k (\omega_2)^r \delta^{(k-1-r)}(t). \quad (6)$$

Now consider the following cases :

(1) If $m \geq 2$ then we see that the current $I(t)$ contains the Dirac-delta distribution and its derivatives, that means $I(t)$ is not an ordinary function but it is the distribution in the space $\mathcal{D}'_{\mathbb{R}}$. It follows that the current $I(t)$ is not continuous and it occurs in very short time intervals.

(2)If $0 \leq m < 2$ $(m = 0, 1)$ then for $m = 1$, (6) becomes

$$I(t) = \frac{c_0 H(t)}{L(\omega_1 - \omega_2)} \left[\omega_1 e^{\omega_1 t} - \omega_2 e^{\omega_2 t}\right] + \frac{c_1 H(t)}{L(\omega_1 - \omega_2)} \left[\omega_1^2 e^{\omega_1 t} - \omega_2^2 e^{\omega_2 t}\right] + \frac{c_1}{L}\delta.$$

It follows that the current $I(t)$ is the same the case (1).
For $m = 0$, (6) becomes

$$I(t) = \frac{c_0 H(t)}{L(\omega_1 - \omega_2)} [\omega_1 e^{\omega_1 t} - \omega_2 e^{\omega_2 t}]$$

$$= \frac{c_0 H(t)}{\sqrt{R^2 - 4\frac{L}{C}}} e^{-\frac{R}{2L}} \left[(-\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}) e^{(\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}})t} + (\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}) e^{(\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}})t} \right]$$

by substitution for ω_1, ω_2.
That $I(t)$ is continuous function for $t \in [0, \infty)$. It follow that the current $I(t)$ flows continuously for the time $t \in [0, \infty)$.

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