On Numerical Methods for Solving Iterative Ordinary Differential Equations

Maitree Podisuk, Suntorn Suchatvejpoom and Pornchai Chaisanit
Department of Mathematics and Computer Science
Faculty of Science
King MongKut's Institute of Technology Chaokhuntaharn
Ladkrabang Bangkok 10520 THAILAND

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Abstract

In this paper, four numerical methods for solving iterative ordinary differential equations are introduced and used them to solve four examples. The results from each method are compared.

1 Introduction

The first order iterative ordinary differential equation in the interval [0,a] is of the form

$$\frac{dy}{dx} = f(x, y(x), y^2(x), ..., y^m(x)) \tag{1}$$

with the initial condition

$$y(0) = c \tag{2}$$

where m is a positive integer greater than 1, c is positive real number and

$$y^2(x) = y(y(x)), y^3(x) = y(y(y(x))) = y(y^2(x)), ..., y^m(x) = y(y^{m-1}(x)).$$

If f is continuous then the problem (1)-(2) is equivalent to the problem of continuous solution of the integral equation

$$y(x) = c + \int_0^\infty f(t, y(t), y^2(t), ..., y^m(t))dt. \tag{3}$$

Let $f(x, z_1, z_2, ..., z_m)$ be defined and continuous in the domain [0,a] $\times R^m$ say D, and let

$$|f(x, z_1, z_2, ..., z_m)| \leq K \tag{4}$$

$$|f(x, z_1, z_2, ..., z_m) - f(x, \overline{z_1}, \overline{z_2}, ..., \overline{z_m})| \leq M_1|z_1 - \overline{z_1}| + M_2|z_2 - \overline{z_2}| + ... + M_m|z_m - \overline{z_m}| \tag{5}$$

for all $(x, z_1, z_2, ..., z_m)$, $(x, \overline{z_1}, \overline{z_2}, ..., \overline{z_m})$ in D and $K, M_1, M_2, ..., M_m$ in R^+ and let

$$P_m = M_1 + KM_2 + K^2M_3 + ... + K^{m-1}M_m \tag{6}$$

$$S_m = M_2 + (K+1)M_3 + (K^2+K+1)M_4 + ... + (K^{m-2} + K^{m-3} + ... + K + 1)M_m \tag{7}$$

$$T_m = P_m + S_m \tag{8}$$

$$Q_m = M_1 + (K+1)M_2 + (K^2+K+1)M_3 + (K^3+K^2+K+1)M_4 + ... + (K^{m-1}+K^{m-2}+...+K+1)M_m \tag{9}$$

then we have the following theorems.

Theorem 1 (see prove in Podisuk) *If $aS_m < e^{-aP_m}$ and f satisfies the above conditions then there exists at most one solution to the problem (1)-(2).*

Theorem 2 (see prove in Podisuk) *If $aT_m < 1$ and f satisfies the conditions of the theorem 1 then there exists at most one solution to the problem (1)-(2).*

Now let us suppose that

$$c + aK \leq a \tag{10}$$

$$aQ_m < 1 \tag{11}$$

and let us consider the following sequences

$$y_{1,n+1}(x) = c + \int_0^{\infty} f(t, y_{1,n}(t), y_{1,n}^2(t), ..., y_{1,n}^m(t))dt \tag{12}$$

$$y_{2,n+1}(x) = c + \int_0^\infty f(t, y_{2,n}(t), y_{2,n}^2(t), ..., y_{2,n}^{m-1}(y_{2,n+1}(t)))dt \qquad (13)$$

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$$y_{m+1,n+1}(x) = c + \int_0^\infty f(t, y_{m+1,n}(t), y_{m+1,n+1}^2(t), ..., y_{m+1,n+1}^m(t))dt \qquad (14)$$

n=0,1,2,... where $y_{1,0}(x), y_{2,0}(x), ..., y_{m+1,0}(x)$ are fixed functions of class C^1 map [0,a] to [0,a] such that $|y'_{1,0}|, |y'_{2,0}|, ..., |y'_{m+1,0}| \leq K$. Hence we have the following theorem.

Theorem 3 (see prove in Podisuk) *Let the assumption of theorem 1 holds and the conditions (10)-(11) are satisfied then the sequences (12)-(14) converge uniformly to the (unique) solution of the problem (1)-(2).*

2 Numerical Methods

The existing numerical methods for solving the ordinary differential equations can not solve the problem of iterative ordinary differential equations. The reason is that to find the value of y(x+h), we need to use the values of $y^2(x), y^3(x), ..., y^m(x)$ which may involve in using the unknown values of y(z) where z is greater than x. We must use their approximating values instead. We will use two methods to find these approximating values. These two methods are the Cubic Spline Interpolation Method and the Newton Divided Difference Method. Then we will combine them with the Runge-Kutta Method and the Simpson Method to solve our problems. We will come up with four numerical methods and they are as follows.

- First Method. Partition the interval [0,a] into k partitions at $x_0 = 0, x_1 = h, ..., x_i = ih, ..., x_k = kh = a$ where $h = a/k$. Choose any initial function $y_0(x)$ that satisfies the conditions of theorem 3 with the initial value c then use them to find $y_1(x)$ at each partition point by using Simpson method and Newton divided difference method. Then use $y_1(x)$ to find $y_2(x)$. We will continue this process until $\sum_{n=0}^{n=k} |y_{n+1}(x_j) - y_n(x_j)| < \varepsilon$, where ε is a small positive real number chosen ahead of time.

- Second Method. It is the same as the first method but using Runge-Kutta method and Newton divided difference method.

- Third method. It is the same as the first method but using Simpson method and cubic spline method.

- Fourth Method. It is the same as the first method but using Runge-Kutta method and cubic spline method.

3 Examples

- Example 1. Solve the problem

$$y\prime(x) = \frac{1}{1+2\mathrm{x}} - \frac{1}{1+\mathrm{x}} + \frac{1}{(1+\mathrm{x})^2} - y(x) + 2y^2(x), x\epsilon[0,1] \tag{15}$$

$$y(0) = 0 \tag{16}$$

where its analytical solution is $y(x) = \frac{\mathrm{x}}{1+\mathrm{x}}$, by using k=4, 8 and 16 and $\varepsilon =$ 0.000005. The results of all four methods are in Table 1, Table 2, Table 3, Table 4, Table 5 and Table 6 for k=4, 8 and 16 respectively.

- Example 2. Solve the problem

$$y\prime(x) = 1 - x^2 + xy^2(x), x\epsilon[0,1] \tag{17}$$

$$y(0) = 0 \tag{18}$$

where its analytical solution is y(x)=x, by using k=8 and ε=0.000005. The results of all four methods are in Table 7 and Table 8.

- Example 3. Solve the problem

$$y\prime(x) = y^2(x), x\epsilon[0,0.5] \tag{19}$$

$$y(0) = 0.25. \tag{20}$$

We do not know the analytical solution of the problem (19)-(20). The numerical results from the above four methods with ε =0.000005 and k=8 are in Table 9.

- Example 4. Solve the problem

$$y\prime(x) = y^3(x), x\epsilon[0,0.5] \tag{21}$$

$$y(0) = 0.2. \tag{22}$$

We do not know the analytical solution of the problem (21)-(22). The numerical results from the above four methods with ε =0.000005 and k=8 are in Table 10.

4 tables

		first method		**second method**	
		16 iterations		14 iterations	
x_i	$y(x_i)$	y_i	**error**	y_i	**error**
0.00	0.000000	0.000000	0.000000	0.000000	0.000000
0.25	0.200000	0.209312	0.009312	0.199512	0.000488
0.50	0.333333	0.345225	0.011892	0.332614	0.000720
0.75	0.428571	0.435251	0.006754	0.427757	0.000815
1.00	0.500000	0.496430	0.003570	0.498952	0.001048

Table 1: k=4

		third method		**fourth method**	
		8 iterations		**8 iterations**	
x_i	$y(x_i)$	y_i	**error**	y_i	**error**
0.00	0.000000	0.000000	0.000000	0.000000	0.000000
0.25	0.200000	0.200111	0.000112	0.200121	0.000121
0.50	0.333333	0.333530	0.000197	0.333545	0.000212
0.75	0.428571	0.418874	0.000303	0.428894	0.008323
1.00	0.500000	0.500394	0.000394	0.500422	0.000422

Table 2: k=4

		first method		**second method**	
		15 iterations		**14 iterations**	
x_i	$y(x_i)$	y_i	**error**	y_i	**error**
0.000	0.000000	0.000000	0.000000	0.000000	0.000000
0.125	0.111111	0.112513	0.001402	0.111089	0.000022
0.250	0.200000	0.200288	0.000288	0.199982	0.000018
0.375	0.272727	0.268133	0.004594	0.272708	0.000020
0.500	0.333333	0.320302	0.013032	0.333290	0.000044
0.625	0.384615	0.360293	0.024322	0.384553	0.000062
0.750	0.428571	0.392237	0.036334	0.428496	0.000076
0.875	0.466667	0.422827	0.043839	0.466570	0.000096
1.000	0.500000	0.447318	0.052682	0.499888	0.000112

Table 3: k=8

		third method 11 iterations		fourth method 11 iterations	
x_i	$y(x_i)$	y_i	**error**	y_i	**error**
0.000	0.000000	0.000000	0.000000	0.000000	0.000000
0.125	0.111111	0.111115	0.000005	0.111117	0.000007
0.250	0.200000	0.200007	0.000007	0.200001	0.000001
0.375	0.272727	0.272738	0.000010	0.272740	0.000013
0.500	0.333333	0.333347	0.000014	0.333351	0.000017
0.625	0.384615	0.384615	0.000017	0.384636	0.000021
0.750	0.428571	0.428592	0.000020	0.428596	0.000025
0.875	0.466667	0.466690	0.000024	0.466696	0.000029
1.000	0.500000	0.500027	0.000027	0.500033	0.000033

Table 4: k=8

		first method 16 iterations		second method 14 iterations	
x_i	$y(x_i)$	y_i	**error**	y_i	**error**
0.0000	0.000000	0.000000	0.000000	0.00000	0.000000
0.0625	0.058824	0.059007	0.000184	0.058823	0.000001
0.1250	0.111111	0.110968	0.000143	0.11111	0.000000
0.1875	0.157895	0.156535	0.001360	0.157894	0.000001
0.2500	0.200000	0.196866	0.003134	0.199998	0.000002
0.3125	0.238095	0.233384	0.004711	0.238093	0.000002
0.3750	0.272727	0.266062	0.006665	0.272724	0.000003
0.4375	0.304348	0.296066	0.008281	0.304344	0.000004
0.5000	0.333333	0.323226	0.010107	0.333329	0.000004
0.5625	0.360000	0.348478	0.011522	0.359947	0.000005
0.6250	0.384615	0.371419	0.013197	0.384609	0.000006
0.6875	0.407407	0.392718	0.014690	0.407401	0.000006
0.7500	0.428571	0.412242	0.016330	0.428564	0.000007
0.8125	0.448276	0.430136	0.018140	0.448268	0.000008
0.8750	0.466667	0.446697	0.019970	0.466658	0.000009
0.9375	0.483871	0.462476	0.021395	0.483861	0.000009
1.0000	0.500000	0.477076	0.022924	0.499990	0.000010

Table 5: k=16

		third method 8 iterations		fourth method 8 iterations	
x_i	$y(x_i)$	y_i	error	y_i	error
0.0000	0.000000	0.000000	0.000000	0.00000	0.000000
0.0625	0.058824	0.058824	0.000000	0.058824	0.000000
0.1250	0.111111	0.111111	0.000000	0.111111	0.000000
0.1875	0.157895	0.157895	0.000000	0.157895	0.000000
0.2500	0.200000	0.200000	0.000000	0.200000	0.000000
0.3125	0.238095	0.238096	0.000000	0.238096	0.000000
0.3750	0.272727	0.272728	0.000001	0.272728	0.000001
0.4375	0.304348	0.304349	0.000001	0.304349	0.000001
0.5000	0.333333	0.333334	0.000001	0.333335	0.000002
0.5625	0.360000	0.360001	0.000001	0.360001	0.000001
0.6250	0.384615	0.384616	0.000001	0.384617	0.000002
0.6875	0.407407	0.407409	0.000002	0.407409	0.000002
0.7500	0.428571	0.428573	0.000002	0.428573	0.000002
0.8125	0.448276	0.448277	0.000001	0.448278	0.000002
0.8750	0.466667	0.466668	0.000001	0.466669	0.000002
0.9375	0.483871	0.483873	0.000002	0.483873	0.000002
1.0000	0.500000	0.500001	0.000001	0.500002	0.000002

Table 6: k=16

		first method 14 iterations		second method 11 iterations	
x_i	$y(x_i)$	y_i	error	y_i	error
0.000	0.000000	0.000000	0.000000	0.00000	0.000000
0.125	0.125000	0.125000	0.000000	0.125000	0.000000
0.250	0.250000	0.250000	0.000000	0.250000	0.000000
0.375	0.375000	0.375000	0.000000	0.375000	0.000000
0.500	0.500000	0.500000	0.000000	0.500000	0.000000
0.625	0.625000	0.625000	0.000000	0.625000	0.000000
0.750	0.750000	0.750000	0.000000	0.749999	0.000001
0.875	0.875000	0.874999	0.000001	0.874999	0.000001
1.000	1.000000	0.999999	0.000001	0.999999	0.000001

Table 7: k=8

		third method 13 iterations		fourth method 11 iterations	
x_i	$y(x_i)$	y_i	**error**	y_i	**error**
0.000	0.000000	0.000000	0.000000	0.00000	0.000000
0.125	0.125000	0.125000	0.000000	0.125000	0.000000
0.250	0.250000	0.250000	0.000000	0.250000	0.000000
0.375	0.375000	0.375000	0.000000	0.375000	0.000000
0.500	0.500000	0.500000	0.000000	0.500000	0.000000
0.625	0.625000	0.625000	0.000000	0.625000	0.000000
0.750	0.750000	0.749999	0.000001	0.750000	0.000000
0.875	0.875000	0.874999	0.000001	0.874999	0.000001
1.000	1.000000	0.999999	0.000001	0.999999	0.000001

Table 8: k=8

	first method 36 iterations	second method 31 iterations	third method 32 iterations	fourth method 31 iterations
x_i	y_i	y_i	y_i	y_i
0.0625	0.271494	0.271425	0.271425	0.271425
0.1250	0.293514	0.293360	0.293360	0.293360
0.1875	0.316080	0.315820	0.315820	0.315820
0.2500	0.339182	0.338822	0.338822	0.338822
0.3125	0.362864	0.362386	0.362384	0.362385
0.3750	0.387115	0.386529	0.386529	0.386529
0.4375	0.411979	0.411272	0.411272	0.411272
0.5000	0.437492	0.436636	0.436636	0.436636

Table 9: k=8

	first method 33 iterations	second method 28 iterations	third method 30 iterations	fourth method 28 iterations
x_i	y_i	y_i	y_i	y_i
0.0625	0.217061	0.216855	0.217053	0.216855
0.1250	0.234163	0.233790	0.234148	0.233790
0.1875	0.251308	0.250804	0.251285	0.250804
0.2500	0.268495	0.267898	0.268463	0.267898
0.3125	0.285724	0.285073	0.285684	0.285073
0.3750	0.302996	0.302329	0.302947	0.302329
0.4375	0.320309	0.319666	0.320252	0.319667
0.5000	0.337666	0.337086	0.337599	0.337086

Table 10: k=8

5 Conclusion

The results of all four methods are equally good and are acceptable. We need small number of ε if we need more accuracy. Thus this paper recommends all four methods to solve our problem (1)-(2). However, in each step, the cubic spline method needs more computer time than the Newton divided difference method.

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