Numerical Method for Solving Iterative Partial Differential Equations

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Abstract

In this paper, We introduced the Newton Divided Difference Method and Backward Difference Approximation to solve iterative partial differential equations in the case of the existing numerical methods can not be used.

1 Introduction

The content of this paper is to discuss the numerical method for solving iterative partial differential equations of the form

$$\frac{\partial^n u(x)}{\partial x_1, \partial x_2, \ldots, \partial x_n} = f\left(x, u(x), u^2(x), u^3(x), \ldots, u^m(x)\right) \tag{1}$$

with the initial conditions

$$u(x_1, x_2, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n) = g_{1,i}(x_1, x_2, \cdots x_{i-1}, x_{i+1}, \cdots, x_n) \qquad, i = 1,2,\cdots,n$$

$$\begin{aligned} &u(x_1, x_2, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_{j-1}, 0, x_{j+1}, \cdots x_n) \\ &\qquad = g_{2,i,j}(x_1, x_2, \cdots x_{i-1}, x_{i+1}, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) \\ &\qquad\quad , i \neq j, i, j = 1,2,\cdots,n \\ &\qquad\quad \vdots \end{aligned}$$

$$\begin{aligned} u(x_1, 0, 0, \cdots, 0) &= g_{n-1,2,3,\cdots,n}(x_1) \\ u(0, x_2, 0, \cdots, 0) &= g_{n-1,1,3,\cdots,n}(x_2) \\ &\vdots \\ u(0, 0, \cdots, 0, x_n) &= g_{n-1,1,2,\cdots,n-1}(x_n) \\ u(0,0,0,\cdots,0) &= c = \left[c_1, c_2, \cdots, c_n\right]^T \end{aligned} \tag{2}$$

or the compatibility initial condition

$$g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix}$$
$$= (-1)^2\left(g_{1,1}(x_2, x_3, \ldots, x_n) + \cdots + g_{1,n}(x_1, x_2, \ldots, x_{n-1})\right)$$
$$+ (-1)^3\left(g_{2,1,2}(x_3, x_4, \ldots, x_n) + \cdots + g_{2,n-1,n}(x_1, x_2, \ldots, x_{n-2})\right)$$
$$\vdots$$
$$+ (-1)^n\left(g_{n-1,2,3}(x_1) + \cdots + g_{n-1,1,2,\ldots,n-1}(x_n)\right) + (-1)^{n+1}c \qquad (3)$$

where m is a positive integer greater than 1 and

$$u^2(x) = u(u(x)), u^3(x) = u(u(u(x))) = u(u^2(x)), \cdots, u^m(x) = u(u^{m-1}(x)).$$

and $Z = [0 \times a_1] \times [0, a_2] \times \cdots \times [0, a_n]$

$$u_i : Z \to R, \quad f_i : Z \times R^{mn} \to R, \quad i = 1,2,\ldots,n$$
$$u : Z \to R^n, \quad f : Z \times R^{mn} \to R^n;$$

If f and g are continuous then the problem (1)-(2) is equivalent to the problem of solving continuous solution of the integral equation

$$u(x) = g(x) + \int_0^x f(t, u(t), u^2(t), u^3(t), \ldots, u^m(t))\, dt. \qquad (4)$$

Let $f(x, u(x), u^2(x), \ldots, u^m(x))$ be defined and continuous in the set $Z \times R^{mn}$,say D and g be defined and continuous in Z and let

$$\| f(x, z_1, z_2, \ldots, z_m) \| \le K \qquad (5)$$

$$\| f(x, z_1, z_2, \ldots, z_m) - f(x, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_m) \| \le M_1 \| z_1 - \bar{z}_1 \| + M_2 \| z_2 - \bar{z}_2 \| + \ldots$$
$$\ldots + M_m \| z_m - \bar{z}_m \| \qquad (6)$$

$$\| g(x) \| \le L \le K \qquad (7)$$

for all $(x, z_1, z_2, \ldots, z_m), (x, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_m)$ in D and for x in Z and $L, K, M_1, M_2, \ldots, M_m$ are in R^+ . The norm $\| \cdot \|$ is the Euclidean norm.

We are looking for the solution $u(x)$ of the problem (1)-(2) or (4) where $u(x)$ belongs to Z for all x in Z (8)

$\| u(x) - u(y) \| \le N \| x - y \|$ for all x and y in Z and N in R^+. (9)

Let

$$S_1 = M_1 + M_2 N + M_3 N^2 + \cdots M_m N^{m-1}$$
$$S_2 = M_2 + M_3 N + M_4 N^2 + \cdots M_m N^{m-2}$$
$$\vdots$$
$$S_{m-1} = M_{m-1} + M_m N$$
$$S_m = M_m$$
$$d = (S_1)^{1/m}, d > 0, \; a = a_1 a_2 a_3 \cdots a_n, \; b = S_2 + S_3 + \cdots + S_m, \; h = 2L + 2K,$$
$$A = h, \; B = ab, \; c = a_1 + a_2 + \cdots + a_n$$

thus we have the following theorem.

Theorem 1 (see prove in Podisuk) *If* $Be^{dC} < 1$, f *and* g *satisfy the above conditions then there exist at most one solution to the problem (1)-(2).*

Now let us suppose that

$$\| g(x) \| + aK \le a \quad \text{for all } x \text{ in } Z \text{ and} \tag{10}$$

$$a\lfloor M_1 + (N+1)M_2 + (N^2 + N + N)M_3 + \cdots + (N^{m-1} + N^{m-2} + \cdots + N + 1)M_m \rfloor < 1 \tag{11}$$

and let consider the following sequences

$$u_{1,k+1}(x) = g(x) + \int_0^x f(t, u_{1,k}(t), u_{1,k}^2(t), \ldots, u_{1,k}^m(t))\, dt \tag{12.1}$$

$$u_{2,k+1}(x) = g(x) + \int_0^x f(t, u_{2,k}(t), u_{2,k}^2(t), \ldots, u_{2,k}^{m-1}(t), u_{2,k}^{m-1}(u_{2,k+1}(t)))\, dt \tag{12.2}$$

$$\vdots$$

$$u_{m+1,k+1}(x) = g(x) + \int_0^x f(t, u_{m+1,k+1}(t), u_{m+1,k}(u_{m+1,k+1}(t)), u_{m+1,k}^2(u_{m+1,k+1}(t)), u_{m+1,k}^{m-1}(u_{m+1,k+1}(t))\, dt \tag{12.m+1}$$

where $u_{1,0}(x), u_{2,0}(x), \ldots, u_{m+1,0}(x)$ are fixed functions of the class C^1 map Z to Z such that

$$\left\| \frac{\partial^n u_{i,0}(x)}{\partial x_1, \partial x_2, \ldots \partial x_n} \right\| \le K \quad i = 1,2,\ldots,m+1$$

Hence we have the following theorem.

Theorem 2 (see prove Podisuk) *Let the condition of theorem 1 holds and the conditions (10)-(11) be satisfied then the sequences (12)-(14) converge uniformly to the (unique) solution* $u = u(x)$ *of the problem (1)-(2).*

2 Numerical Methods

The exist numerical methods for solving the partial differential equations can not solve the problem of iterative partial differential equations. The reason is that to find the value of $u(x+h)$, we need to use the values of $u^2(x), u^3(x), \ldots, u^m(x)$ which may involve in using the unknown values of $u(r)$ where r is not equal x_i. We must use their approximating values instead. We will use Newton Divided Difference method to find the approximating values. Then we will combine them with the Backward Difference Approximation to solve our problems. We will come up with the numerical method as follows. We divided the interval $[0, a_i]$ into k_i partitions at $x_{i0}, = 0, x_{i1} = h_i, \ldots, x_{ij} = jh, \ldots, x_{ik} = k_i h_i = a_i$ where $h_i = 1/k_i$ for $i = 1,2,\ldots,n$. Choose any initial function $u^{(0)}(x)$ that satisfies the conditions of theorem 2 with the initial function $g(x)$ then use them to find $u^{(1)}(x)$ at each grid point by Newton Divided Difference method and Backward Difference Approximation. Then use $u^{(1)}(x)$ to find $u^{(2)}(x)$. We will continue this process until

$\sum_{l=0}^{l=k} \| u^{(l+1)}(x) - u^{(l)}(x) \| < \varepsilon$, where ε is a small positive real number chosen a head of time.

3 Examples

- **Examples 1.** Find the solution, in $Z = [0,1] \times [0,1]$, of the equation

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \begin{bmatrix} 0 \\ \frac{1}{2} - \frac{xy}{8} \end{bmatrix} - \frac{xy}{4} u(x,y) + u^2(x,y) \qquad (15)$$

with the initial conditions

$$u(x,0) = u(0,y) = u(0,0) = [\tfrac{1}{2} \quad 0]^T \qquad (16)$$

where its analytical solution is $u(x,y) = \begin{bmatrix} \frac{1}{2} + \frac{xy}{2} \\ \frac{xy}{2} \end{bmatrix}$, by using k = h = 4, 8 and 16 and $\varepsilon = 0.0000005$. The results are in Table 1.

	k=4	k=8	k=16
$u_1(x,y)$			
The number of Iterations	6	5	5
Total Error	0.0000003	0.0000004	0.0000003
$u_2(x,y)$			
The number of Iterations	6	5	5
Total Error	0.0000003	0.0000004	0.0000003

Table 1.

- **Examples 2.** Find the solution, in $Z = [0,1] \times [0,1]$, of the equation

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \begin{bmatrix} -\frac{7}{60} - \frac{x^2y^2}{45} - \frac{1}{10} u_1(x,y) + u_1(u_1,u_2) \\ \frac{1}{5} - \frac{x^2y^2}{75} - \frac{1}{10} u_2(x,y) + u_2(u_1,u_2) \end{bmatrix} \qquad (19)$$

with the initial conditions

$$u(x,0) = u(0,y) = u(0,0) = [\tfrac{1}{2} \quad 0]^T \qquad (20)$$

where its analytical solution is $u(x,y) = \begin{bmatrix} \frac{1}{2} + \frac{xy}{3} \\ \frac{xy}{5} \end{bmatrix}$, by using k = h = 4, 8 and 16 and $\varepsilon = 0.0000005$. The results are in Table 2.

	k=4	k=8	k=16
$u_1(x,y)$			
The number of Iterations	4	4	5
Total Error	0.0000002	0.0000000	0.0000000
$u_2(x,y)$			
The number of Iterations	4	4	5
Total Error	0.0000001	0.0000000	0.0000000

Table 2.

- **Examples 3.** Find the solution, in $Z = [0,1] \times [0,1] \times [0,1]$, of the equation

$$\frac{\partial^3 u(x,y,z)}{\partial x \partial y \partial z} = \begin{bmatrix} \frac{1}{4} + \frac{23}{96} xyz - \frac{1}{96} x^2 y^2 z^2 + u_1(u_1, u_2, u_3) - u_1(x,y,z) \\ 0 \\ \frac{1}{2} - \frac{23}{48} xyz - \frac{1}{48} x^2 y^2 z^2 + u_2(u_1, u_2, u_3) - u_2(x,y,z) \end{bmatrix} \quad (21)$$

with the initial conditions

$$u(0,y,z) = u(x,0,z) = u(x,y,0) = u(0,0,z) = u(0,y,0) = u(x,0,0) = u(0,0,0) = [\tfrac{1}{4} \ \tfrac{1}{3} \ 0]^T \quad (22)$$

where its analytical solution is $u(x,y,z) = \begin{bmatrix} \frac{1}{4} + \frac{xyz}{4} \\ \frac{1}{3} \\ \frac{xyz}{2} \end{bmatrix}$, by using k = h = 16 and $\varepsilon = 0.0000005$. The results are in Table 3.

	$u_1(x,y)$	$u_2(x,y)$
The number of Iterations	25	25
Total Error	0.0000000	0.0000000

Table 3: k=16

- **Examples 4.** Find the solution, in $Z = [0, \frac{1}{2}] \times [0, \frac{1}{2}]$, of the equation

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = u^3(x,y) \qquad (17)$$

with the initial conditions

$$u(x,0) = u(0,y) = u(0,0) = \begin{bmatrix} \frac{1}{8} & \frac{1}{16} \end{bmatrix}^T \qquad (18)$$

We do not know the analytical solution of the problem (17) - (18). The numerical results from the above method with $\varepsilon = 0.0000005$. The results show in Figure1 and 2

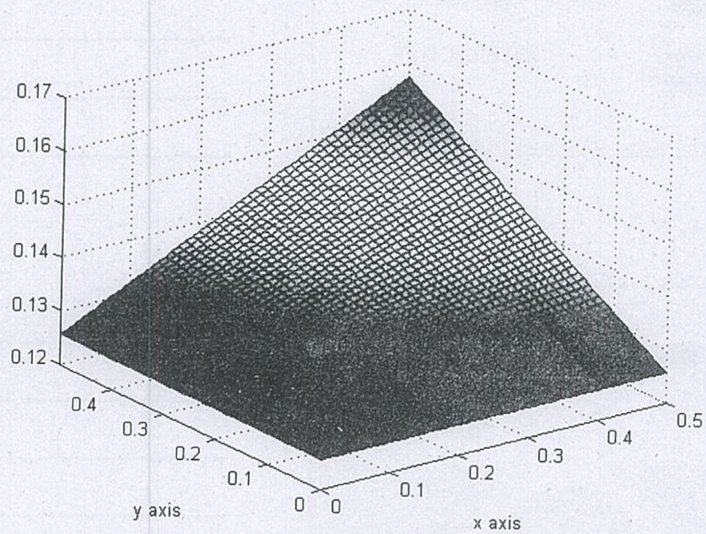

Figure1. Graph of the solution, $u_1(x,y)$

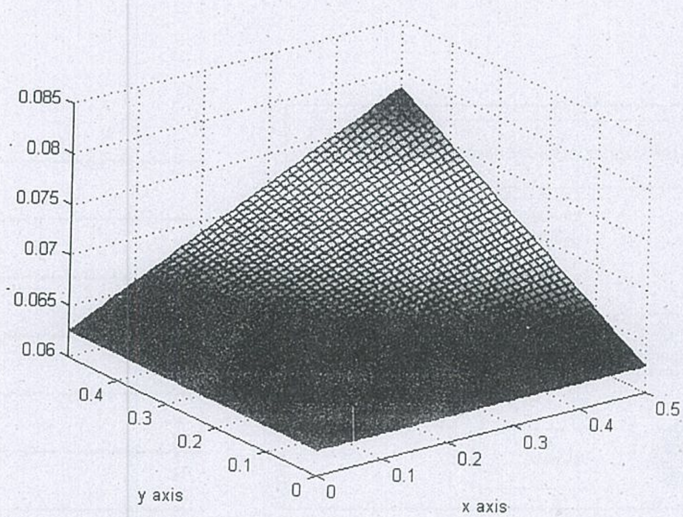

Figure2. Graph of the solution, $u_2(x,y)$

4. Conclution

The paper has shown that Newton Divided Difference Method for solving iterative partial differential equation is good and acceptable. The more accurate we want the result to be, the smaller the value of ε we need. For further studies may use other methods such as Linear interpolation and Lagrange interpolation to solve this kind of problem. Then compare their results with result of this Newton Divided Difference Method present in this paper in term of the number of iteration or the time used to find solution.

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