FREDHOLM INTEGRAL EQUATION SOLUTION FOR RECTANGULAR PLATE WITH MOMENT SINGULARITY

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ABSTRACT

This paper presents the finite Hankel transform method for a problem of rectangular plate clamped on the two opposite edges and simply supported on the remaining edges with an internal line support located along the center under uniformly distributed load. The proper singularity in the order of square root in moment is introduced at the tips of the internal support. The problem formulation is reduced to an inhomogeneous Fredholm integral equation of the second kind in term of an unknown auxiliary function which can be transformed to a set of algebraic simultaneous equations by using Simpson's rule. The solution results provide the unknown auxiliary function of the problem and the physical quantities, namely the deflection and bending moment of the plate with various lengths of internal line support.

KEYWORDS: Fredholm Integral Equation / Hankel Transform / Rectangular Plate / Singularity

1. INTRODUCTION

The solution of plate bending problem is governed by the fourth order partial differential equation. There are many approaches to solve the problem numerically such as the finite element and boundary element methods and recently the mesh free element method. These mentioned methods give the approximated solution in the acceptable accuracy where the problem with no singularity. The Navier and Levy-Nadai approaches [1] are the well known analytical solution methods to solve the static bending and eigenvalue problems of plate. However, the problem of plate with mixed boundary conditions is more difficult to solve analytically since Williams [2] had been pointed out that there is singularity at the point of discontinuity in the order of an inverse square root in moment. The problems of plate with various mixed boundary conditions were studied on the bending, vibration, and buckling [3-7]. The finite Hankel transform is one of the efficient methods to solve the mixed boundary value problem which leads to determine the solution of an integral equation.

The purpose of this paper is to analyze the mixed boundary conditions of a rectangular plate using the Levy-Nadai approach with assuming the proper finite Hankel transform which exhibits the inverse square root singularity in the moment at the tips of an internal line support of the plate. The obtained numerical result presents the unknown auxiliary function of Fredholm integral equation of the second kind, deflection and bending moment of the plate with varying the length of internal line support.

2. FORMULATION

Considering the geometry of scaled rectangular plate as shown in Fig. 1, the differential equation governing the deflection w of the plate under the uniformly distributed load q in the transformed coordinates (x, y) is given by

$$D\left(w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}\right) = qa^4/\pi^4, \quad (1)$$

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where a = actual length of plate which is scaled by the factor π/a, E = Young's modulus, $D = Eh^3/12\left(1-\nu^2\right)$, ν = Poisson's ratio, and h = plate thickness.

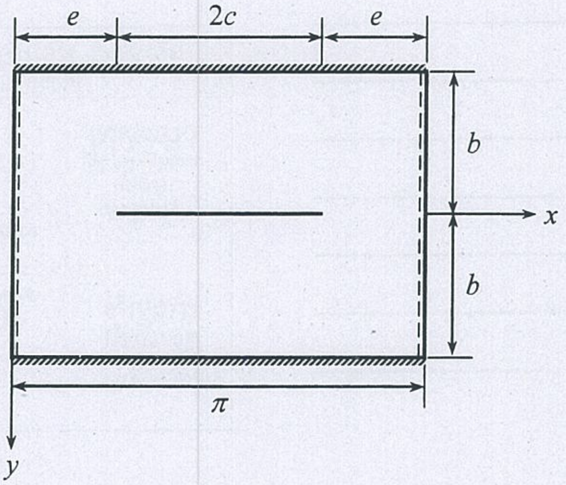

Fig. 1 Geometry of Scaled Rectangular Plate with an Internal line Support

Because of the symmetry of the deflection, the boundary conditions need only be written on the lower left quadrant of the plate, thus the boundary conditions are

$$w = w,_y = 0; \quad 0 \le x \le \pi/2, \quad y = b, \tag{2a.b}$$

$$w,_y = 0; \quad 0 \le x \le \pi/2, \quad y = 0, \tag{3}$$

$$w = w,_x = w,_{xx} = 0; \quad e < x \le \pi/2, \quad y = 0, \tag{4a.b.c}$$

$$w,_{yyy} + (2-\nu)\, w,_{xxy} = 0; \quad 0 \le x < e, \quad y = 0. \tag{5}$$

Utilizing the Levy-Nadai approach, the deflection satisfying the boundary conditions of simple support at $x = 0, \pi$ and the governing equation (1) is taken as

$$w = \frac{qa^4}{D}\sum_{m=1,3,5,\ldots}^{\infty}\left[\begin{array}{l}\frac{4}{\pi^5 m^5} + A_m\cosh(my) + B_m my\sinh(my)\\ + C_m\sinh(my) + D_m my\cosh(my)\end{array}\right]\sin(mx). \tag{6}$$

where the constants A_m, B_m, C_m and D_m are determined from the boundary conditions (2) and (3). Therefore,

$$\begin{aligned}A_m = &-\frac{4\sinh(mb)\left[1 + mb\coth(mb)\right]}{\pi^5 m^5\left[mb + \sinh(mb)\cosh(mb)\right]}\\ &+ D_m\frac{\left\{\tanh(mb) + (mb)^2\left[\tanh(mb) - \coth(mb)\right]\right\}}{1-(mb)\left[\tanh(mb) - \coth(mb)\right]},\end{aligned} \tag{7}$$

$$B_m = \frac{4\,\mathrm{sech}(mb)}{\pi^5 m^5 \left\{1 - mb\left[\tanh(mb) - \coth(mb)\right]\right\}} - \frac{D_m \tanh(mb)}{\left\{1 - mb\left[\tanh(mb) - \coth(mb)\right]\right\}}, \tag{8}$$

$$C_m = -D_m. \tag{9}$$

The application of the remaining boundary conditions (4c) and (5) , then, they can be written as the following dual series equations,

$$\sum_{m=1,3,5,\ldots}^{\infty} m^2 P_m \sin(mx) = 0, \quad e < x \le \frac{\pi}{2}, \tag{10}$$

$$\sum_{m=1,3,5,\ldots}^{\infty} m^3 P_m (1 + F_m) \sin(mx) = \sum_{m=1,3,5,\ldots}^{\infty} G_m \sin(mx), \quad 0 \le x < e, \tag{11}$$

where

$$P_m = \frac{4\left[1 - \cosh(mb)\right]\left[mb - \sinh(mb)\right]}{\pi^5 m^5 \left[\sinh(mb)\cosh(mb) + mb\right]} + D_m \left\{ \frac{\tanh(mb) + (mb)^2\left[\tanh(mb) - \coth(mb)\right]}{1 - (mb)\left[\tanh(mb) - \coth(mb)\right]} \right\}, \tag{12}$$

$$1 + F_m = \frac{1 - (mb)\left[\tanh(mb) - \coth(mb)\right]}{\tanh(mb) + (mb)^2\left[\tanh(mb) - \coth(mb)\right]}, \tag{13}$$

$$G_m = \frac{4\left[1 - \cosh(mb)\right]\left[mb - \sinh(mb)\right]}{\pi^5 m^2 \left[\sinh(mb)\cosh(mb) + mb\right]} \times \left\{ \frac{1 - (mb)\left[\tanh(mb) - \coth(mb)\right]}{\tanh(mb) + (mb)^2\left[\tanh(mb) - \coth(mb)\right]} \right\}. \tag{14}$$

The solution to the dual series equations proceeds by choosing P_m to be of the appropriate finite Hankel transform as

$$m^2 P_m = \int_0^e t\phi(t)\left[J_1(mt) - \frac{t}{e} J_1(me) \right] dt, \quad m = 1,3,5,\ldots \tag{15}$$

which automatically satisfies (10) and produces an inverse square root moment singularity [2] at the tips of internal line support. The functions $\phi(t)$ and $J_1(\ )$ are the auxiliary function and Bessel function of the first kind of first order, respectively.

By integrating (11) once with respect to x and substituting P_m given in (15) with using the identity [4],

$$\sum_{m=1,3,5,...}^{\infty} J_1(mt)\cos(mx) = \frac{1}{2}t^{-1} - \frac{1}{2}xt^{-1}\left(x^2-t^2\right)^{-\frac{1}{2}} H(x-t) + \int_0^{\infty}\left[\exp(\pi s)+1\right]^{-1} I_1(ts)\cosh(xs)\,ds, \quad x+t<\pi, \tag{16}$$

where $H(\ )$ and $I_1(\ )$ are the Heaviside function and modified Bessel function of the first kind and order 1, respectively. Therefore, the second dual series equations (11) can be further reduced in the form of Abel's integral equation as

$$\int_0^x \frac{x\phi(t)}{\sqrt{x^2-t^2}}dt = h(x), \quad 0 \le x < e, \tag{17}$$

in which, with $t = er$,

$$h(x) = e\int_0^1 \phi(er)\left\{1-r^2+2er\int_0^{\infty}\left[\exp(\pi s)+1\right]^{-1}\left[I_1(ser)-rI_1(se)\right]\times\cosh(xs)\,ds\right\}dr + 2e^2\int_0^1 r\phi(er)\sum_{m=1,3,5,...}^{\infty} F_m\left[J_1(mer)-rJ_1(me)\right]\cos(mx)\,dr - 2\sum_{m=1,3,5,...}^{\infty} m^{-1}G_m\cos(mx), \quad 0\le x<e. \tag{18}$$

The solution of (17) takes the form

$$\phi(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{h(x)}{\sqrt{t^2-x^2}}dx, \quad 0<t<e. \tag{19}$$

Substituting (18) into (19), then the final result becomes an inhomogeneous Fredholm integral equation of the second kind

$$\Phi(\rho) + \int_0^1 K(\rho,r)\Phi(r)\,dr = f(\rho), \quad 0\le\rho\le 1, \tag{20}$$

in which

$$\Phi(\rho) = \phi(e\rho), \quad \Phi(r) = \phi(er), \tag{21}$$

$$K(\rho,r) = 2e^2 r\left\{\sum_{m=1,3,5,...}^{\infty} mF_m\left[J_1(mer)-rJ_1(me)\right]J_1(me\rho) - \int_0^{\infty} s\left[\exp(\pi s)+1\right]^{-1}\left[I_1(ser)-rI_1(se)\right]I_1(se\rho)\,ds\right\}, \tag{22}$$

$$f(\rho) = 2\sum_{m=1,3,5,...}^{\infty} G_m J_1(me\rho). \tag{23}$$

The auxiliary function $\Phi(\rho)$ in (20) can be solved numerically by using Simpson's rule as shown in Fig. 2.

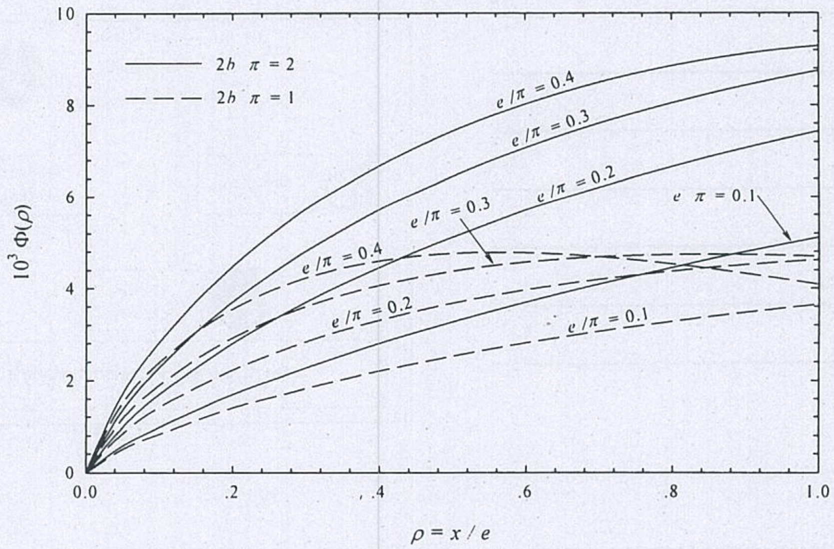

Fig. 2 Auxiliary function $\Phi(\rho)$ in Integral Equation

The deflection along the line outside an internal line support is determined by substituting (15) into (6) and utilizing the identity [4]

$$\sum_{m=1,3,5,\ldots}^{\infty} m^{-2}J_1(mt)\sin(mx)=\begin{cases}\dfrac{1}{4}\left[\dfrac{x}{t}\left(t^2-x^2\right)^{1/2}+t\sin^{-1}\left(x/t\right)\right], & x<t\\[2ex] \dfrac{\pi}{8}t, \quad x\geq t\end{cases},\quad x+t<\pi, \tag{24}$$

which leads to

$$w(x,0)=\frac{qa^4e^3}{4D}\left\{\begin{aligned}&\frac{\pi}{2}\int_0^{\xi}\rho^2\Phi(\rho)\,d\rho\\&+\int_{\xi}^{1}\left[\xi\sqrt{\rho^2-\xi^2}+\rho^2\sin^{-1}\left(\frac{\xi}{\rho}\right)\right]\Phi(\rho)\,d\rho\\&-\int_0^1\rho^2\left[\xi\sqrt{1-\xi^2}+\sin^{-1}\xi\right]\Phi(\rho)\,d\rho\end{aligned}\right\}, \tag{25}$$

where $\xi = x/e,\ 0\leq\xi,\rho\leq 1$. The deflection computed from (25) is plotted numerically as in Fig.3.

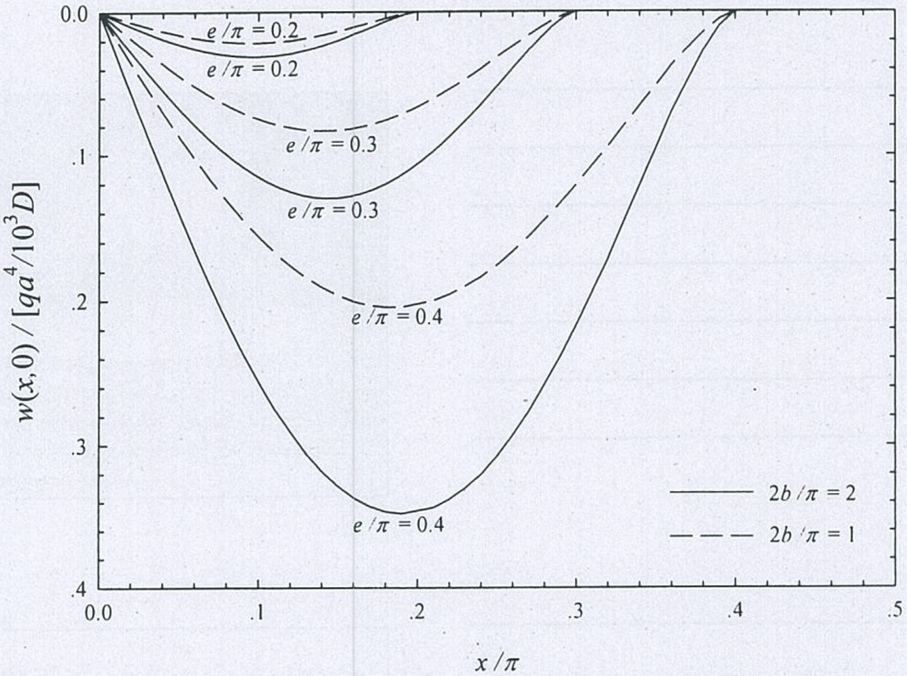

Fig. 3 Deflection along the Unsupported Portion of Internal Line Support

The deflection along the center line normal to the internal line support can be obtained by considering (6) and setting $x = \pi / 2$, yields the expression as

$$w(\pi/2, y) = \frac{qa^4}{D} \sum_{m=1,3,5,...}^{\infty} \left\{ \begin{array}{l} \frac{4}{\pi^5 m^5} + A_m \cosh(my) \\ + B_m my \sinh(my) \\ + C_m \left[\begin{array}{l} \sinh(my) \\ -my\cosh(my) \end{array} \right] \end{array} \right\} (-1)^{\frac{m-1}{2}}, \quad 0 \le y \le b, \quad (26)$$

in which A_m, B_m and C_m are defined in (7), (8), and (9), respectively. The numerical result is illustrated in Fig. 4.

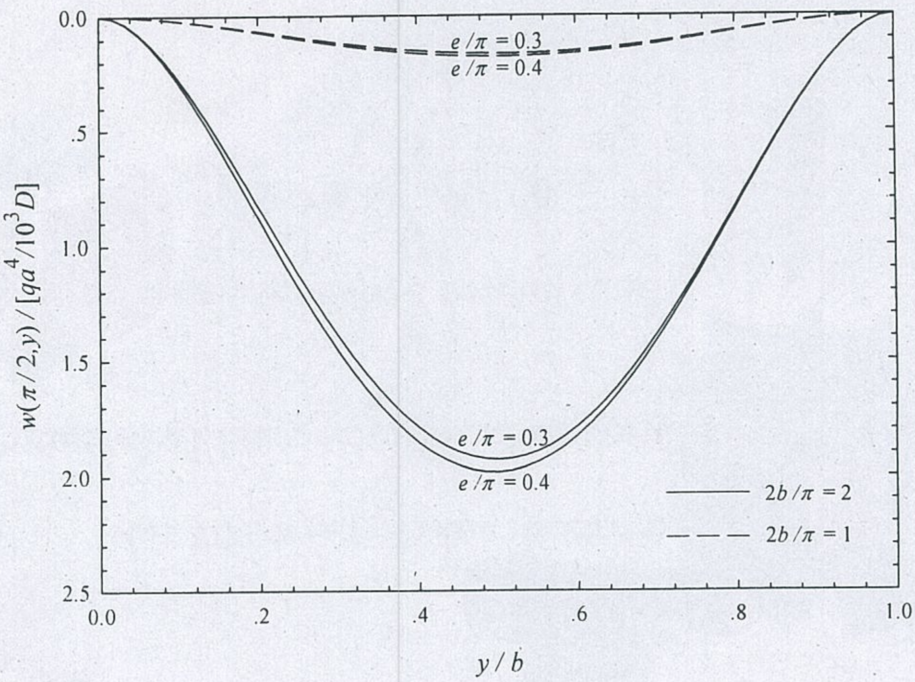

Fig. 4 Deflection along the Center Line Normal to the Internal Line Support

Similarly, the bending moments along the internal line support and along the direction normal to the internal line support are determined by substituting (6) into the formulae,

$$M_x = -\frac{\pi^2}{a^2} D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right), \quad (27)$$

$$M_y = -\frac{\pi^2}{a^2} D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right). \quad (28)$$

After performing some manipulations, the final results become

$$\frac{M_x(x,0)}{qa^2\pi^2} = \begin{bmatrix} \sum_{m=1,3,5,\ldots}^{\infty} \nu S_m \sin(mx) \\ + \int_0^1 K(x,\rho)\Phi(\rho)\,d\rho \end{bmatrix}, \quad \begin{matrix} 0 \le x < e, \\ e < x \le \pi/2, \end{matrix} \quad (29)$$

where

$$K(x,\rho) = e^2\rho \left\{ \begin{matrix} -\dfrac{(1+\nu)x\rho H(e-x)}{2e(e^2-x^2)^{1/2}} + \dfrac{(1+\nu)xH(e\rho-x)}{2e\rho(e^2\rho^2-x^2)^{1/2}} \\ +2\nu \sum_{m=1,3,5,\ldots}^{\infty} H_m\left[J_1(me\rho) - \rho J_1(me)\right]\sin(mx) \end{matrix} \right\}, \quad (30)$$

$$S_m = \frac{4}{\pi^5 m^3}\left[1 - \frac{4\,\mathrm{sech}(mb) + \pi^5 m^2 G_m \tanh(mb)}{2\left\{1-(mb)\left[\tanh(mb) - \coth(mb)\right]\right\}}\right], \quad (31)$$

$$1+H_m=\frac{(1+F_m)\tanh(mb)}{1-(mb)\left[\tanh(mb)-\coth(mb)\right]},\qquad (32)$$

and

$$\frac{M_y\left(\frac{\pi}{2},y\right)}{qa^2\pi^2}=\sum_{m=1,3,5,\ldots}^{\infty}\left\{\begin{array}{l}\frac{4\nu}{\pi^5 m^3}-m^2A_m(1-\nu)\cosh(my)\\ -m^2B_m\left[\begin{array}{l}(1-\nu)my\sinh(my)\\ +2\cosh(my)\end{array}\right]\\ +m^2C_m\left[\begin{array}{l}(1+\nu)\sinh(my)\\ +(1-\nu)my\cosh(my)\end{array}\right]\end{array}\right\}\qquad (33)$$

$$\times(-1)^{\frac{m-1}{2}},\quad 0\le y\le b.$$

It is noted that the bending moment given in (29) is singular at $x=e$ by considering the first term of $K(x,\rho)$ in (30). For the bending moment presented in (33), there is no singularity. The bending moments are presented graphically as in Figs. (5) and (6).

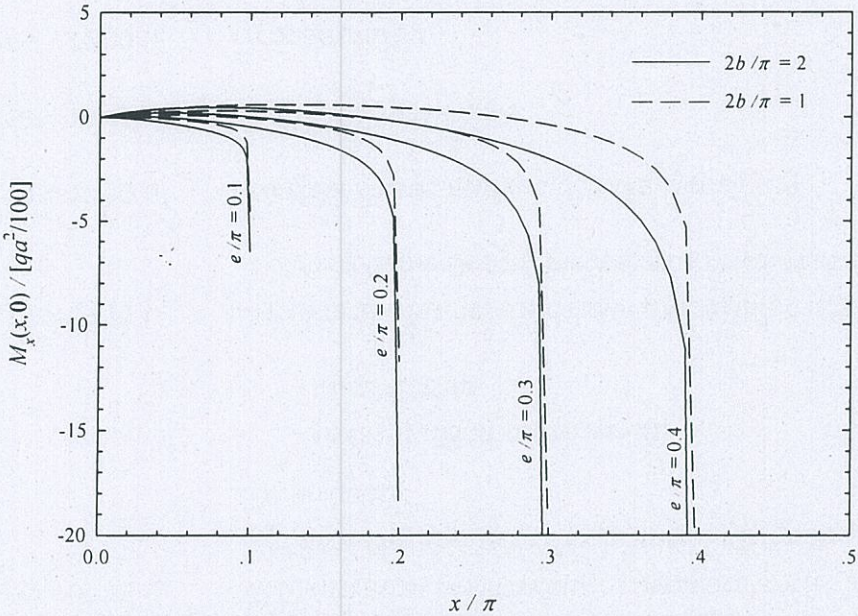

Fig. 5 Bending Moment along the Unsupported Portion of Internal Line Support

3. RESULTS AND DISCUSSION

The integral equation presented in (20) can be solved numerically to obtain the discretized value of unknown auxiliary function $\Phi(\rho)$ as shown in Fig. 2 by using Simpson's rule to transform to a system of linear algebraic equations. The improper infinite integral in the kernel given in (22) was determined by a 16-point Gauss-Legendre quadrature formula. The infinite series in the kernel and the function $f(\rho)$ given in (23) were calculated to a relative error of 0.00001. The numerical results were carried out for the cases of plate with the aspect ratios $2b/\pi$ of 1 and 2. The unsupported lengths of internal line support e/π were taken as 0.1, 0.2, 0.3, and 0.4 in each aspect ratio of plate, respectively. For all of analysis, the Poisson's ratio was taken as 0.3.

Fig. 3 illustrates the deflections along the unsupported portion of internal support and Fig. 4 presents the deflections along the center line normal to the internal line support. It is noted that when the ratios of e/π are less than 0.2, the deflections shown in Fig. 3 are very small and for the cases of $e/\pi < 0.3$, deflections shown in Fig. 4 are very closed to the case $e/\pi = 0.3$. The bending moments given in (29) and (33) are plotted on Figs. 5 and 6, respectively. It is seen that the last point of all curves shown in Fig. 5 approaches infinity because the moment is singular at $x = e$. In each aspect ratio of the plate, the values of bending moment presented in Fig. 6 are closed to the same value with various e/π-ratios.

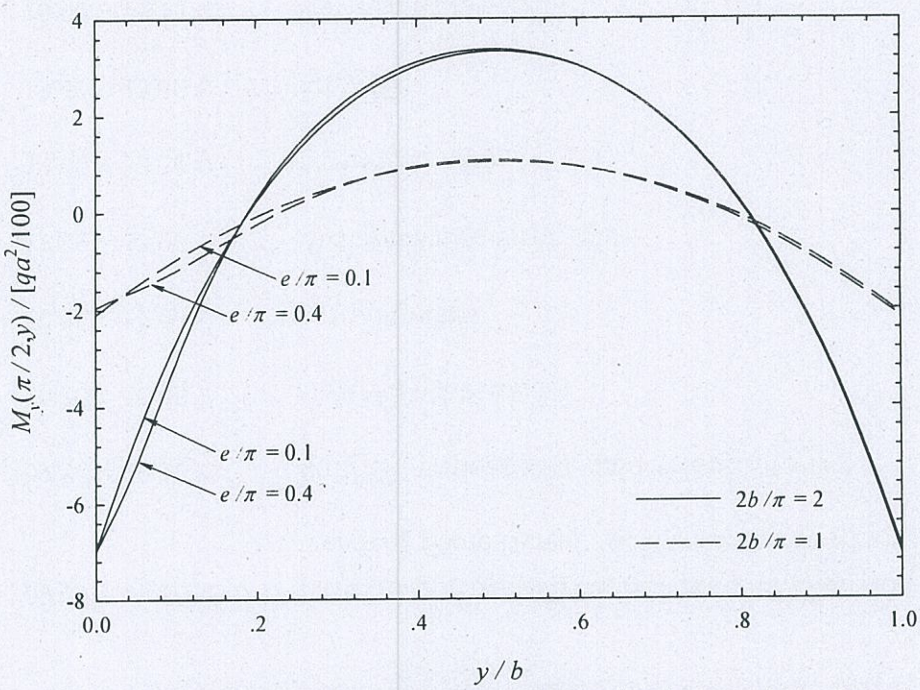

Fig. 6 Bending Moment along the Center Line Normal to the Internal Line Support

Fig. 7 shows an example of deformed shape of rectangular plate with aspect ratio 2 and $e/\pi = 0.3$ under the uniformly distributed load which is bounded by the region $0 \le x \le \pi/2$ and $0 \le y \le b$. All results of deflection and bending moment were computed on the summation of infinite series to be accurate to four and three significant digits, respectively.

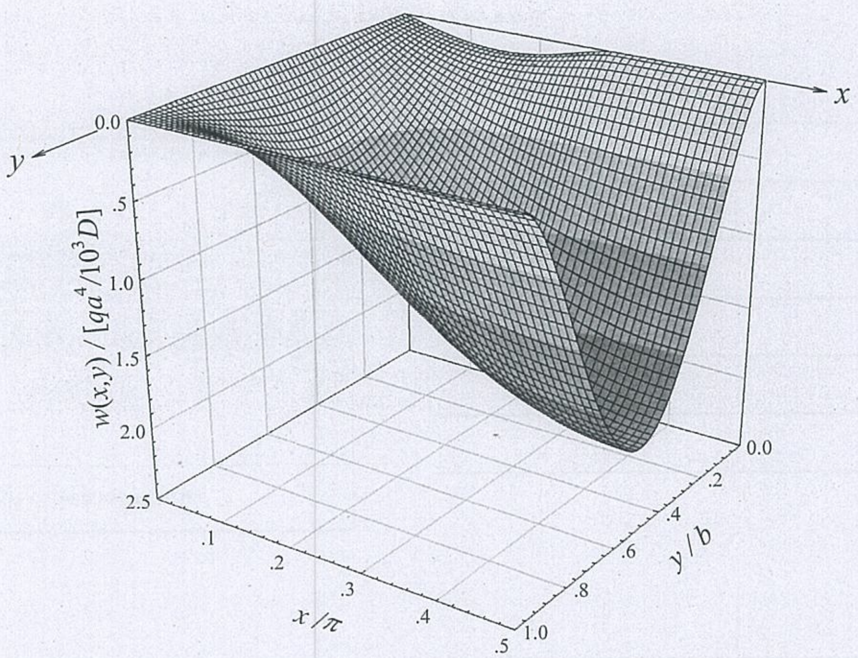

Fig. 7 Deflection Surface of a Rectangular Plate with Aspect Ratio 2 and $e/\pi = 0.3$

4. CONCLUSIONS

The method of finite Hankel transform is an efficient technique to analyze the mixed boundary conditions problem of rectangular plate. The plate is clamped on the two opposite edges and simply supported on the remaining edges with an internal line support located at the center. The problem considered has the moment singularity in order of an inverse square root at the tips of the internal line support. The dual series equations which results from the mixed boundary conditions along the line of internal support are derived and can be reduced to an inhomogeneous Fredholm integral equation of the second kind. The advantage of present method is that the singularity is isolated and treated analytically. The physical quantities namely, the deflection and bending moment can be expressed in the closed form and the numerical results are conveniently evaluated and illustrated graphically.

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