

## FREE CONTACT BETWEEN PLATE AND SAGGED SUPPORT UNDER UNIFORMLY DISTRIBUTED LOAD

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### ABSTRACT

This paper deals with the advancing contact between a rectangular plate simply supported on the two opposite edges and free on the others and an internal line sagged support located along the center under uniformly distributed load. The finite Hankel transform method is used and the correct singularity is introduced at the tips of contact. The problem is formulated as dual series equations and reduced to an inhomogeneous Fredholm integral equation of the second kind in term of an unknown auxiliary function which can be solved numerically by using Simpson's rule. The deflection, bending moment, and support reaction of the plate are expressed in the analytical form and presented graphically.

**KEYWORDS:** Advancing Contact / Fredholm Integral Equation / Hankel Transform / Rectangular Plate / Analytical Solution / Singularity

### 1. INTRODUCTION

Contact problems of plate under the lateral load have been studied in the past [1-5] which are the receding contact between plate and unilateral supports. The nature of this contact type is that the extent of contact depends on the geometry and elastic properties of the plate only. It is independent of the level of loading, and the support reactions are proportional to the applied load. For a plate resting on sagged supports, Dundurs et al. [6] examined the advancing contact of rectangular plate with the application of finite Hankel transform. The results show that the extent of contact depends on the level of loading and the support reactions are not proportional to the applied load. Although the both types of contact are opposite together, the singularity at the tip of contact is still to be in the order of an inverse square root in the shear.

Therefore, the objective of this paper is to investigate the analytical solution of advancing contact between a rectangular plate and a partially internal line sagged support. The different of this paper compared to the previous paper [6] is that the sagged support is located at the inner of the plate but those the sagged supports are at the plate edges. The problem formulation is based on the Levy-Nadai approach [7] in the form of dual series equations and finite Hankel transform is used to convert the dual series equations to an inhomogeneous Fredholm integral equation of the second kind. The physical result presents the deflection, bending moment, and support reaction of the plate with varying length of contact along the internal line sagged support.

### 2. FORMULATION

To simplify the analysis, the scaled plate configuration is shown in Fig. 1 whereas the actual length of rectangular plate is  $a$  and is scaled by the factor  $\pi/a$ . The equation governing the deflection  $w$  of the plate under the uniformly distributed load  $q$  in the transformed coordinates  $(x, y)$  is given by

$$D(w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}) = qa^4 / \pi^4, \quad (1)$$

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where  $D = Eh^3 / 12(1 - \nu^2)$ ,  $E$  = Young's modulus,  $\nu$  = Poisson's ratio, and  $h$  = plate thickness.

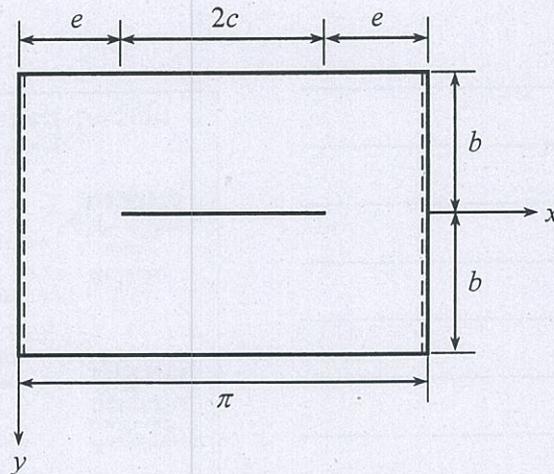


Fig. 1 Geometry of Scaled Rectangular Plate with an Internal Line Sagged Support

Due to the two-fold symmetry of the deflection, the boundary conditions need to be considered only on the lower left quadrant of the plate, and can be written as

$$w_{,yy} + \nu w_{,xx} = 0; \quad 0 \leq x \leq \pi/2, \quad y = b, \quad (2)$$

$$w_{,yyy} + (2 - \nu)w_{,xxy} = 0; \quad 0 \leq x \leq \pi/2, \quad y = b, \quad (3)$$

$$w_{,y} = 0; \quad 0 \leq x \leq \pi/2, \quad y = 0, \quad (4)$$

$$w = \delta : w_{,x} = w_{,xx} = 0; \quad e < x \leq \pi/2, \quad y = 0, \quad (5a,b,c)$$

$$w_{,yyy} + (2 - \nu)w_{,xxy} = 0; \quad 0 \leq x < e, \quad y = 0. \quad (6)$$

where  $\delta$  = the initial gap between the plate and internal line sagged support.

Based on the Lévy-Nadai approach [7], the deflection satisfying the boundary conditions of simply supported edges at  $x = 0, \pi$  and the governing equation (1) is taken as

$$w = \frac{qa^4}{D} \sum_{m=1,3,5,\dots}^{\infty} \left[ \frac{4}{\pi^5 m^5} + A_m \cosh(my) + B_m my \sinh(my) + C_m \sinh(my) + D_m my \cosh(my) \right] \sin(mx). \quad (7)$$

where the constants  $A_m, B_m, C_m$  and  $D_m$  are determined from the boundary conditions (2), (3), and (4), that yield

$$A_m = -\frac{4\nu [\beta \cosh \beta - \eta'' \sinh \beta]}{\pi^5 m^5 [(3 + \nu) \sinh \beta \cosh \beta - (1 - \nu) \beta]} + D_m \frac{[(1 - \nu) \beta^2 + (1 + \nu) \eta'' + (3 + \nu) \cosh^2 \beta]}{[(3 + \nu) \sinh \beta \cosh \beta - (1 - \nu) \beta]}, \quad (8)$$

$$B_m = \frac{4\nu \sinh \beta}{\pi^5 m^5 [(3 + \nu) \sinh \beta \cosh \beta - (1 - \nu) \beta]} - \frac{D_m [2 + (3 + \nu) \sinh^2 \beta]}{[(3 + \nu) \sinh \beta \cosh \beta - (1 - \nu) \beta]}, \quad (9)$$

$$C_m = -D_m, \quad (10)$$

where  $\eta''$  and  $\beta$  are defined by  $(1 + \nu)/(1 - \nu)$  and  $mb$ , respectively.

The boundary conditions (5c) and (6) are mixed and can be written as in the form of dual series equations,

$$\sum_{m=1,3,5,\dots}^{\infty} m^2 P_m \sin(mx) = 0, \quad e < x \leq \frac{\pi}{2}, \quad (11)$$

$$\sum_{m=1,3,5,\dots}^{\infty} m^3 P_m (1 + F_m) \sin(mx) = \sum_{m=1,3,5,\dots}^{\infty} G_m \sin(mx), \quad 0 \leq x < e, \quad (12)$$

where

$$P_m = \frac{4[(3+\nu) \sinh \beta \cosh \beta - (1-\nu)\beta + \nu(\eta'' \sinh \beta - \beta \cosh \beta)]}{\pi^5 m^5 [(3+\nu) \sinh \beta \cosh \beta - (1-\nu)\beta]} + D_m \frac{[(1-\nu)\beta^2 + (1+\nu)\eta'' + (3+\nu) \cosh^2 \beta]}{[(3+\nu) \sinh \beta \cosh \beta - (1-\nu)\beta]}, \quad (13)$$

$$1 + F_m = \frac{[(3+\nu) \sinh \beta \cosh \beta - (1-\nu)\beta]}{[(1-\nu)\beta^2 + (1+\nu)\eta'' + (3+\nu) \cosh^2 \beta]}, \quad (14)$$

$$G_m = \frac{4\{(3+\nu) \sinh \beta \cosh \beta - (1-\nu)\beta + \nu[\eta'' \sinh \beta - \beta \cosh \beta]\}}{\pi^5 m^2 [(1-\nu)\beta^2 + (1+\nu)\eta'' + (3+\nu) \cosh^2 \beta]}, \quad (15)$$

Using the method of finite Hankel transform, the first dual series equations (11) can be automatically satisfied by introducing  $P_m$  in the form as

$$m^2 P_m = \int_0^e t \phi(t) J_1(mt) dt; \quad m = 1, 3, 5, \dots, \quad (16)$$

which occurs an inverse square root shear singularity at the tips of contact [6]. The functions  $\phi(t)$  and  $J_1(\ )$  are the unknown auxiliary function and Bessel function of the first kind and order 1, respectively.

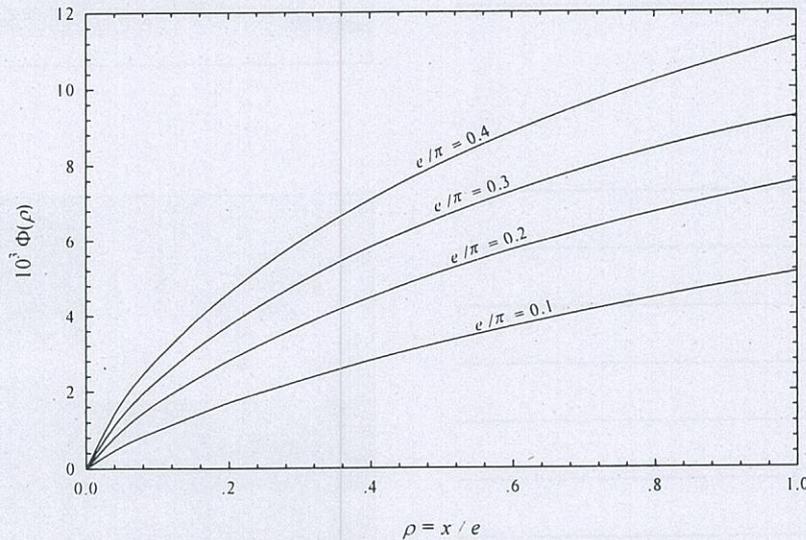


Fig. 2 Auxiliary-function  $\Phi(\rho)$  in Integral Equation

By integrating (12) once with respect to  $x$  and substituting  $P_m$  given in (16) with using the identity [8],

$$\sum_{m=1,3,5,\dots}^{\infty} J_1(mt) \cos(mx) = \frac{1}{2}t^{-1} - \frac{1}{2}xt^{-1}(x^2 - t^2)^{-\frac{1}{2}} H(x-t) + \int_0^{\infty} [\exp(\pi s) + 1]^{-1} I_1(ts) \cosh(xs) ds, \quad x+t < \pi, \tag{17}$$

where  $H(\ )$  and  $I_1(\ )$  are the Heaviside function and modified Bessel function of the first kind and order 1, respectively. Therefore, the second dual series equations (12) can be reduced in the form of Abel's integral equation as

$$\int_0^x \frac{x\phi(t)}{\sqrt{x^2 - t^2}} dt = h(x), \quad 0 \leq x < e, \tag{18}$$

in which, with  $t = er$ ,

$$h(x) = e \int_0^1 \phi(er) \left\{ 1 - r^2 + 2er \int_0^{\infty} \frac{[\exp(\pi s) + 1]^{-1} [I_1(ser) - rI_1(se)]}{\cosh(xs)} ds \right\} dr + 2e^2 \int_0^1 r\phi(er) \sum_{m=1,3,5,\dots}^{\infty} F_m J_1(mer) \cos(mx) dr - 2 \sum_{m=1,3,5,\dots}^{\infty} m^{-1} G_m \cos(mx), \quad 0 \leq x < e. \tag{19}$$

The solution of (18) is found in the form

$$\phi(t) = \frac{2}{\pi} \frac{d}{dt} \int_0^t \frac{h(x)}{\sqrt{t^2 - x^2}} dx, \quad 0 < t < e. \tag{20}$$

Substituting (19) into (20), then the final result is reduced to an inhomogeneous Fredholm integral equation of the second kind

$$\Phi(\rho) + \int_0^1 K(\rho, r) \Phi(r) dr = f(\rho), \quad 0 \leq \rho \leq 1, \tag{21}$$

in which

$$\Phi(\rho) = \phi(e\rho), \quad \Phi(r) = \phi(er), \tag{22}$$

$$K(\rho, r) = 2e^2 r \left\{ \sum_{m=1,3,5,\dots}^{\infty} m F_m J_1(mer) J_1(me\rho) - \int_0^{\infty} s [\exp(\pi s) + 1]^{-1} I_1(ser) I_1(se\rho) ds \right\}, \tag{23}$$

$$f(\rho) = 2 \sum_{m=1,3,5,\dots}^{\infty} G_m J_1(me\rho), \tag{24}$$

where  $F_m$  and  $G_m$  are defined by (14) and (15), respectively. An integral equation presented in (21) can be solved numerically to obtain an auxiliary function  $\Phi(\rho)$  by using Simpson's rule as shown in Fig. 2.

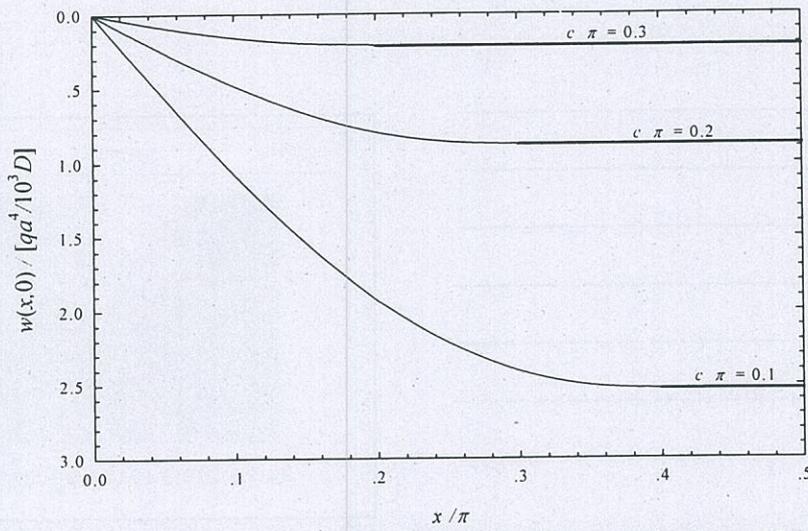


Fig. 3 Deflection along the Unsupported Portion of Internal Line Sagged Support

The deflection along the line outside an internal line sagged support is determined by substituting (16) into (7) and utilizing the identity [8]

$$\sum_{m=1,3,5,\dots}^{\infty} m^{-2} J_1(mt) \sin(mx) = \left\{ \begin{array}{l} \frac{1}{4} \left[ \frac{x}{t} (t^2 - x^2)^{1/2} \right. \\ \left. + t \sin^{-1} \left( \frac{x}{t} \right) \right], \quad x < t \\ \frac{\pi}{8} t, \quad x \geq t \end{array} \right\}, \quad x+t < \pi, \quad (25)$$

which the final form becomes

$$w(x,0) = \frac{qa^4 e^3}{4D} \left\{ \begin{array}{l} \frac{\pi}{2} \int_0^{\xi} \rho^2 \Phi(\rho) d\rho \\ + \int_{\xi}^1 \left[ \xi \sqrt{\rho^2 - \xi^2} + \rho^2 \sin^{-1} \left( \frac{\xi}{\rho} \right) \right] \Phi(\rho) d\rho \end{array} \right\} \quad (26)$$

;  $\xi = x/e, \quad 0 \leq \xi, \rho \leq 1.$

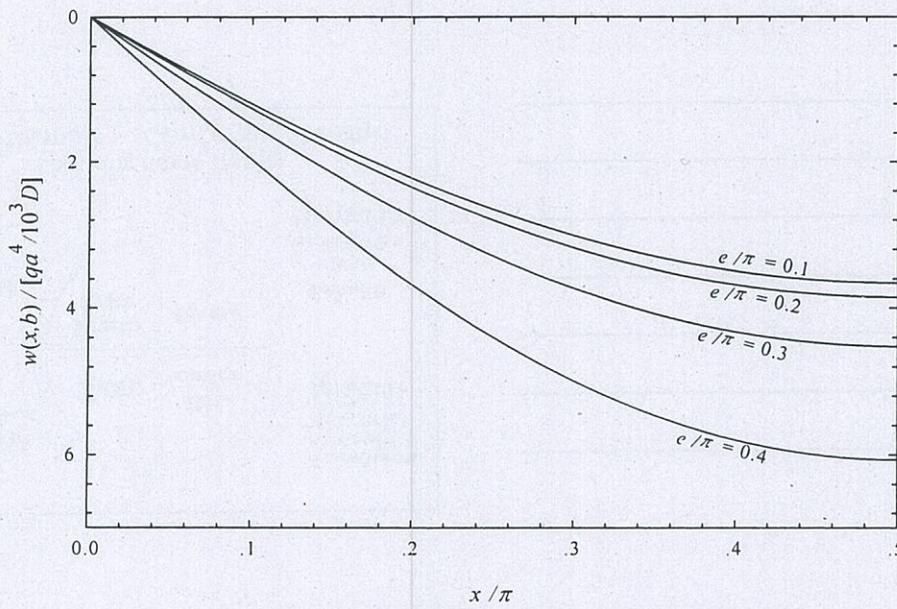
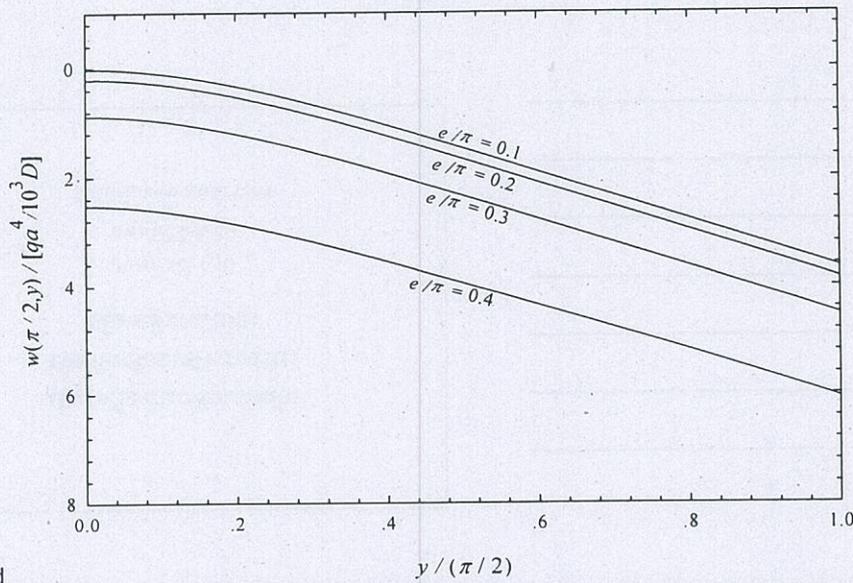


Fig. 4 Deflection along the Free Edge at  $y = b$

The deflection given in (26) is illustrated as in Fig. 3. The deflections along the free edge and along the center line normal to the internal line sagged support can be obtained by substituting  $y = b$  and  $x = \pi/2$  into (7), respectively, yield

$$w(x,b) = \frac{qa^4}{D} \sum_{m=1,3,5,\dots}^{\infty} \left\{ \begin{array}{l} \frac{4}{\pi^5 m^5} + A_m \cosh(mb) \\ + B_m m y \sinh(mb) \\ + C_m \begin{bmatrix} \sinh(mb) \\ -m y \cosh(mb) \end{bmatrix} \end{array} \right\} \sin(mx), \quad 0 \leq x \leq \pi/2, \quad (27)$$



and

Fig. 5 Deflection along the Center Line Normal to the Internal Line Sagged Support

$$w(\pi/2, y) = \frac{qa^4}{D} \sum_{m=1,3,5,\dots}^{\infty} \left\{ \begin{array}{l} \frac{4}{\pi^5 m^5} + A_m \cosh(my) \\ + B_m m y \sinh(my) \\ + C_m \begin{bmatrix} \sinh(my) \\ -m y \cosh(my) \end{bmatrix} \end{array} \right\} (-1)^{\frac{m-1}{2}}, \quad 0 \leq y \leq b, \quad (28)$$

in which  $A_m, B_m$  and  $C_m$  are defined in (8), (9), and (10), respectively. The results are shown in Figs. 4 and 5

The bending moments along the internal line sagged support and along the direction normal to the internal line sagged support are determined by substituting (7) into the formulae.

$$M_x = -\frac{\pi^2}{a^2} D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right), \quad (29)$$

$$M_y = -\frac{\pi^2}{a^2} D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right). \quad (30)$$

Therefore, the bending moments can be expressed as

$$\frac{M_x(x, 0)}{qa^2 \pi^2} = \left[ \begin{array}{l} \sum_{m=1,3,5,\dots}^{\infty} \nu S_m \sin(mx) \\ + \int_0^1 K(x, \rho) \Phi(\rho) d\rho \end{array} \right], \quad 0 \leq x \leq \pi/2, \quad (31)$$

where

$$K(x, \rho) = e^2 \rho \left\{ \begin{aligned} & \frac{(1+\nu)xH(e\rho-x)}{2e\rho(e^2\rho^2-x^2)^{1/2}} \\ & + 2\nu \sum_{m=1,3,5,\dots}^{\infty} H_m J_1(me\rho) \sin(mx) \end{aligned} \right\}, \quad (32)$$

$$S_m = \frac{4}{\pi^5 m^3} \left[ 1 - \frac{4\nu \sinh \beta + \pi^5 m^2 G_m [2 + (3+\nu) \sinh^2 \beta]}{2[(3+\nu) \sinh \beta \cosh \beta - (1-\nu)\beta]} \right], \quad (33)$$

$$1 + H_m = \frac{(1+F_m)[2 + (3+\nu) \sinh^2 \beta]}{(3+\nu) \sinh \beta \cosh \beta - (1-\nu)\beta}, \quad (34)$$

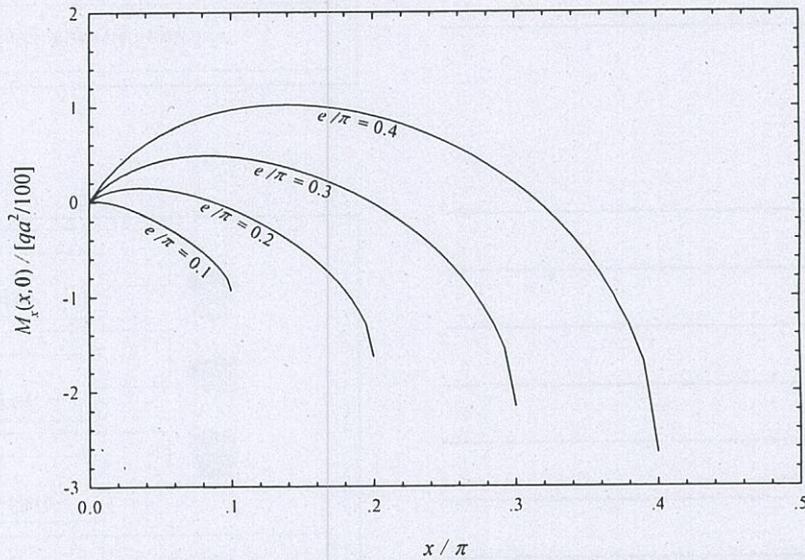


Fig. 6 Bending Moment along the Unsupported Portion of Internal Line Saggd Support and

$$\frac{M_y\left(\frac{\pi}{2}, y\right)}{qa^2 \pi^2} = \sum_{m=1,3,5,\dots}^{\infty} \left\{ \begin{aligned} & \frac{4\nu}{\pi^5 m^3} - m^2 A_m (1-\nu) \cosh(my) \\ & - m^2 B_m \left[ \begin{aligned} & (1-\nu) my \sinh(my) \\ & + 2 \cosh(my) \end{aligned} \right] \\ & + m^2 C_m \left[ \begin{aligned} & (1+\nu) \sinh(my) \\ & + (1-\nu) my \cosh(my) \end{aligned} \right] \end{aligned} \right\} \quad (35)$$

$$\times (-1)^{\frac{m-1}{2}}, \quad 0 \leq y \leq b.$$

It is seen that the bending moment given in (31) is nonsingular at  $x = e$ . The bending moments are presented graphically as in Figs. 6 and 7.

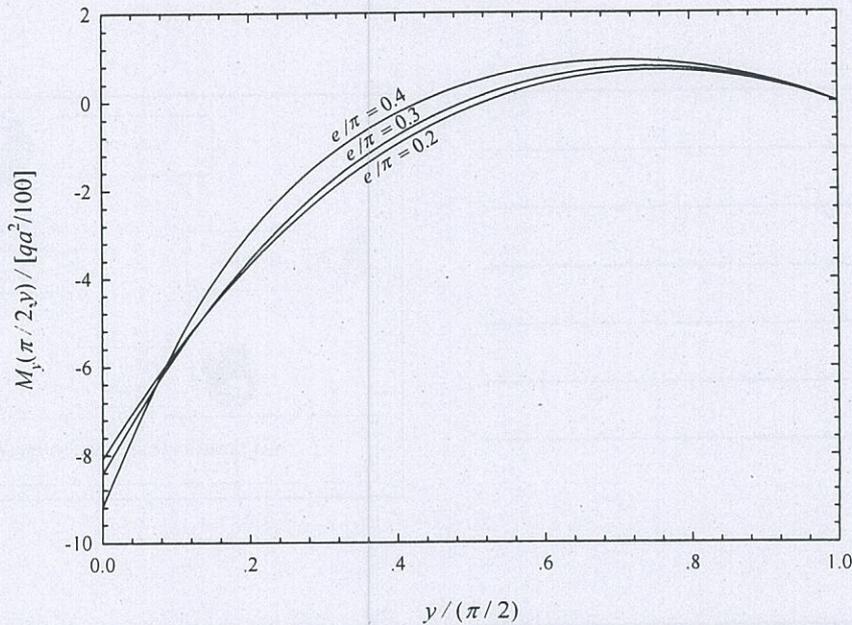


Fig. 7 Bending Moment along the Center Line Normal to the Internal Line Sagged Support

Although the problem has the singularity in order of an inverse square root in shear at the tips of contact, however, it is integrable. The support reactions can be determined from

$$V_x = -\frac{\pi^3 D}{\alpha^3} \left[ \frac{\partial^3 w}{\partial x^3} + (2-\nu) \frac{\partial^3 w}{\partial x \partial y^2} \right], \quad (36)$$

$$V_y = -\frac{\pi^3 D}{\alpha^3} \left[ \frac{\partial^3 w}{\partial y^3} + (2-\nu) \frac{\partial^3 w}{\partial x^2 \partial y} \right]. \quad (37)$$

By substituting (7) into (36) and after performing some manipulations, the support reaction along the  $y$ -axis can be expressed as, and the numerical result is shown in Fig. 8.

$$\frac{V_y(0, y)}{qa\pi^3} = \sum_{m=1,3,5,\dots}^{\infty} \left[ K_m^{(2)}(y) + K_m^{(1)}(y) m e^2 \int_0^1 \rho \Phi(\rho) J_1(m e \rho) d\rho \right], \quad 0 \leq y \leq b, \quad (38)$$

in which

$$K_m^{(1)}(y) = -(1-\nu) \cosh(my) + (1+H_m) \{ (1-\nu) m y \sinh(my) + 2(2-\nu) \cosh(my) \} - (1+F_m) \{ (3-\nu) \sinh(my) + (1-\nu) m y \cosh(my) \}, \quad (39)$$

$$K_m^{(2)}(y) = \frac{2}{\pi^5 m^2} \left\{ \begin{aligned} & 2 [1 + (1-\nu) \cosh(my)] \\ & - \frac{4\nu \sinh \beta + \pi^5 m^2 G_m [2 + (3+\nu) \sinh^2 \beta]}{2 [(3+\nu) \sinh \beta \cosh \beta - (1-\nu) \beta]} \\ & \times [(1-\nu) m y \sinh(my) + 2(2-\nu) \cosh(my)] \\ & + \frac{\pi^5 m^2 G_m}{2} [(3-\nu) \sinh(my) + (1-\nu) m y \cosh(my)] \end{aligned} \right\}. \quad (40)$$

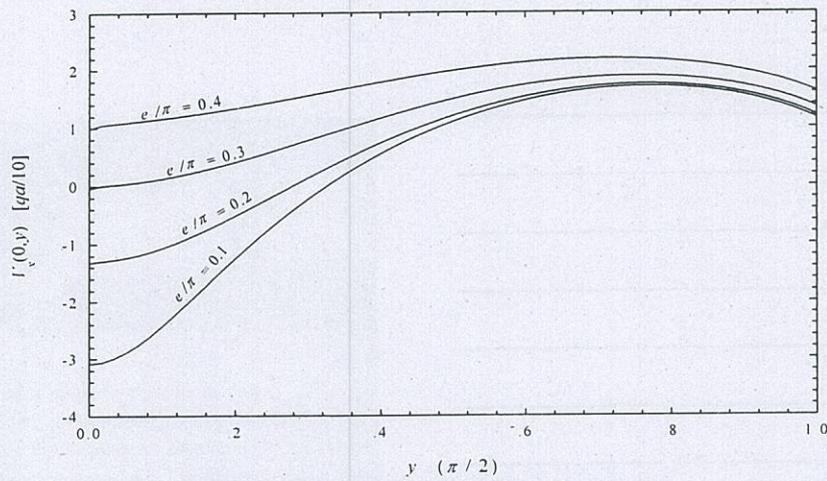


Fig. 8 Support Reaction along the  $y$ -axis

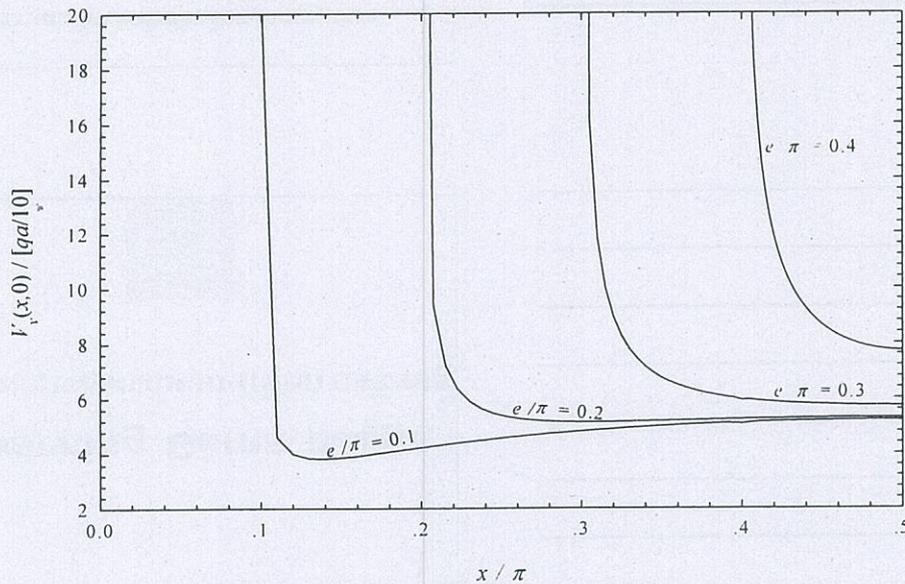
Similarly, the support reaction along the internal line sagged support can be determined from (37) which is

$$\frac{V_y(x,0)}{qa\pi^3} = 2 \left\{ \sum_{m=1,3,5,\dots}^{\infty} G_m \sin(mx) + \int_0^1 T(x,\rho) \Phi(\rho) d\rho \right\}, \quad e < x \leq \frac{\pi}{2}, \quad (41)$$

where

$$T(x,\rho) = e^2 \rho \left\{ \begin{aligned} & \frac{e\rho}{2(x^2 - e^2\rho^2)^{3/2}} + \int_0^{\infty} \frac{[\exp(\pi s) + 1]^{-1} I_1(se\rho) \sinh(xs) ds}{\dots} \\ & - \sum_{m=1,3,5,\dots}^{\infty} m F_m J_1(m e \rho) \sin(mx) \end{aligned} \right\}, \quad (42)$$

and the result is illustrated in Fig. 9.



Support Reaction along the Internal Line Sagged Support

Fig. 9

### 3. RESULTS AND DISCUSSION

The inhomogeneous Fredholm integral equation of the second kind governing the problem solution presented in (21) can be treated numerically using Simpson's rule of numerical integration. The improper infinite integral in the kernel given in (23) was determined by a 16-point Gauss-Legendre quadrature formula which the integrand of infinite integral increases monotonically up to some maximum value and then decays exponentially. The infinite series in the kernel and the function  $f(\rho)$  given in (24) were calculated to a relative error of 0.00001, i.e., the series were terminated when the ratio of the absolute value of the last term to the absolute value of the summation of all previous terms became less than 0.00001. The numerical results was restricted to only for the case of a square plate of actual length  $a$  or scaled length  $\pi$  and the Poisson's ratio was taken as 0.3.

The deflections along the unsupported portion of internal line sagged support, along the free edge, and the deflections along the center line normal to the internal line sagged support are illustrated in Figs. 3, 4 and 5, respectively. It is noted that for the case of  $e/\pi = 0.1$  in Fig. 3, the deflection is very small which cannot be plotted on graph. The bending moments given in (31) and (35) are plotted on Figs. 6 and 7, respectively. It is seen that the bending moment along the internal line sagged support is not singular at the tip of contact which is different from the case of an internal line no sag support. For the case of  $e/\pi = 0.1$ , the bending moment presented in Fig. 7 is closed to the case of  $e/\pi = 0.2$ .

Figs. 8 and 9 show the support reactions along the  $y$ -axis and internal line sagged support, respectively. At the tip of contact between plate and sagged support, it is seen that the support reaction is singular. Fig. 10 presents the deformed shape of a square plate with the contact length  $2c/\pi = 0.4$  under the uniformly distributed load which is bounded by the region  $0 \leq x \leq \pi/2$  and  $0 \leq y \leq \pi/2$ .

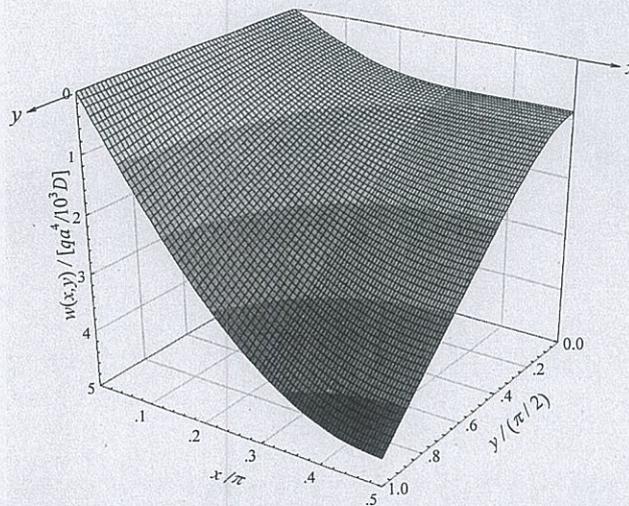


Fig. 10 Deflection Surface of a Square Plate with Contact Length  $2c/\pi = 0.4$

### 4. CONCLUSIONS

The advancing contact problem of a rectangular plate simply supported on the two opposite edges and free on the other edges with an internal line sagged support located at the center is examined. The problem is formulated as dual series equations and can be reduced to an inhomogeneous Fredholm integral equation of the second kind by choosing the proper finite Hankel transform which satisfies the inverse square root singularity in the shear at the tip of contact. The integral equation can be solved numerically to obtain the discretized unknown auxiliary function using Simpson's rule. The advantage of present method is that the singularity is isolated and can be treated analytically. The physical

quantities namely, the deflection, bending moment, and support reaction are expressed in the closed form and the numerical results are given graphically.

### REFERENCES

- [1] Keer, L.M. and Mak, A.F. **1981** Loss of Contact in the Vicinity of a Right-Angle Corner for a Simply Supported, Laterally Loaded Plate, *Journal of Applied Mechanics*, 48, 597-600.
- [2] Dempsey, J.P., Keer, L.M., Patel, N.B. and Glasser, M.L. **1984** Contact Between Plates and Unilateral Supports. *Journal of Applied Mechanics*, 51, 324-328.
- [3] Dempsey, J.P. and Li, H. **1986** Rectangular Plates on Unilateral Edge Supports: Part 1-Theory and Numerical Analysis. *Journal of Applied Mechanics*, 53, 146-150.
- [4] Salamon, N.J., Pawlak, T.P. and Mahmoud, F.F. **1986** Plates in Unilateral Contact With Simple Supports: Pressure Loading. *Journal of Applied Mechanics*, 53, 141-145.
- [5] Hu, C. and Hartley, G.A. **1993** Boundary Element Analysis of Thin Plates Unilaterally Edge Supported. *Engineering Analysis with Boundary Elements*, 12, 47-55.
- [6] Dundurs, J., Kiattikomol, K. and Keer, L.M. **1974** Contact Between Plates and Sagged Supports. *Journal of the Engineering Mechanics Division*, 100(EM2), 445-456.
- [7] Timoshenko, S.P. and Woinowsky-Krieger, S. **1959** *Theory of Plates and Shells*. 2<sup>nd</sup> Edition. Singapore, McGraw-Hill.
- [8] Stahl, B. and Keer, L.M. **1972** Vibration and Buckling of a Rectangular Plate with an Internal Support, *Quarterly Journal of Mechanics and Applied Mathematics*. 25, 467-478.