ITRATIVE ORDINARY DIFFERENTIAL EQUATIONS ON SEMI-INFINITE DOMAIN

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ABSTRACT

There are two types of iterative differential equations, iterative ordinary differential equation and iterative partial differential equation. In this paper, we will concentrate on iterative ordinary differential equation on the semi-infinite domain $[0,\infty)$. Podisuk[4] proved the existence and uniqueness of the solution of the iterative ordinary differential equation in the closed interval $[0,a]$. Podisuk[6] also proved the existence and uniqueness of the solution of the simple iterative ordinary differential equation on the semi-infinite domain $[0,\infty)$. In this paper, we will give a proof of existence and uniqueness of the solution of the iterative ordinary differential equation on the semi-infinite domain $[0,\infty)$.

KEYWORDS: iterative, ordinary, differential, equation, semi-infinite domain

1. INTRODUCTION

The first order iteration ordinary differential equation of degree m on the semi-infinite domain $[0,\infty)$ is in the form

$$y'(x) = f(x, y(x), y^2(x), \dots, y^m(x)) \tag{1}$$

with the initial condition

$$y(0) = c \tag{2}$$

where c is a positive real number, m is a positive integer greater than 1 and

$$y^2(x) = y(y(x))$$
$$y^3(x) = y(y^2(x)) = y(y(y(x)))$$
$$\vdots$$
$$y^m(x) = y(y^{m-1}(x))$$

The first order simple iterative ordinary differential equation of degree m on the semi- infinite domain $[0,\infty)$ is in the form

$$y'(x) = y^m(x) \tag{3}$$

with the initial condition

$$y(0) = c \tag{4}$$

Podisuk[1-3] introduction the first order iterative ordinary differential equations of second order Podisuk[4] proved the existence and uniqueness of the solution of the first order iterative ordinary differential equation of degree m in the closed interval $[0,a]$. Podisuk[5] proved the existence and

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uniqueness of the solution of the first order iterative partial differential equation of degree m in the closed interval $[0,a]$ and Podisuk[6] proved the existence and uniqueness of the solution of the first order iterative ordinary differential equation of degree m on the semi-infinite domain $[0,\infty)$.

2. THEOREMS

Theorem 1 (Podisuk[4])

Let $f(x,z_1,z_2,...,z_m)$ be defined and continuous on the domain $[0,a]\times R^m$, say D, and let

$|f(x,z_1,z_2,....z_m)|\le K$

$|f(x,z_1,z_2,...,z_m)-f(x,\bar{z}_1,\bar{z}_2,...,\bar{z}_m)|\le M_1|z_1-\bar{z}_1|+M_2|z_2-\bar{z}_2|+...+M_m|z_m-\bar{z}_m|$

for all $(x,z_1,z_2,...,z_m),(x,\bar{z}_1,\bar{z}_2,...,\bar{z}_m)$ in D and $K,M_1,M_2,...,M_m$ in R^+

$P_m=M_1+KM_2+K^2M_3+...+K^{m-1}M_m$

$S_m=M_2+(K+1)M_3+\left(K^2+K+1\right)M_4+...+\left(K^{m-2}+K^{m-3}+...+K+1\right)M_m$

$T_m=P_m+S_m$.

If $aS_m<e^{aP_m}$ then there exists at most one solution to the problem

$$y'(x)=f(x,y(x),y^2(x),y^3(x),...,y^m(x))$$

with the initial condition $y(0)=c$.

Theorem 2 (Podisuk[4])

If $aT_m<1$ and f satisfies the conditions of theorem 1 then there exists at most one solution to the problem in theorem 1.

Theorem 3 (Podisuk[4])

Let $c+aK\le a$,

$$a\left[M_1+(K+1)M_2+\left(K^2+K+1\right)M_3+...+(K^{m-1}+K^{m-2}+...+K+1)M_m\right]<1$$

and

$$y_{1,n+1}(x)=c+\int_0^x f(t,y_{1,n}(t),y_{1,n}^2(t),....,y_{1,n+1}^m(t))dt$$

$$y_{2,n+1}(x)=c+\int_0^x f(t,y_{2,n}(t),y_{2,n}^2(t),....,y_{2,n+1}^{m-1}(t),y_{2,n}^{m-1}(y_{2,n+1}(t)))dt$$

$$\vdots$$

$$y_{m+1,n+1}(x)=c+\int_0^x f(t,y_{m+1,n+1}(t),y_{m+1,n}(y_{m+1,n+1}(t)),...,y_{m+1,n+1}^{m-1}(y_{m+1,n+1}(t)))dt$$

$n=0,1,2,3,...$ where $y_{1,0}(x),y_{2,0}(x),...,y_{m+1,0}(x)$ are fixed functions of class C^1 map $[0,a]$ to $[0,a]$ such that $|y'_{1,0}(x)|\le K,...,|y'_{m+1,0}(x)|\le K$.

If the above condition holds and f satisfies the conditions of theorem 1 then the above sequences converge uniformly to the unique solution of the problem in theorem 1.

Theorem 4 (Podisuk[4])

There exist a unique solution to the problem $y'(x)=y^m(x)$ with the initial condition $y(0)=\frac{1}{4}$.

Note that the initial condition $y(0)$ in theorem 4 must be in the interval $(0,0.25]$ and if $m\to\infty$ then the solution $y(x)\to\frac{1}{4}+\frac{x}{2}$.

By the solution of the problem (1) and (2), we mean a function $y \in C^1[0,\infty)$ satisfying (1) and (2) in $[0,\infty)$. Thus the problem (1)-(2) is equivalent to the problem of finding a continuous solution of the integral equation

$$y(x) = c + \int_0^x f(t, y(t), y^2(t), y^3(t), \ldots, y^m(t))dt \, . \qquad (5)$$

The following theorems will be the main result of this paper.

Theorem 5

Suppose

(a) f is defined and continuous in

$$D = \{(x, y_1, y_2, \ldots, y_m) : x \geq 0, 0 \leq (y_1, y_2, \ldots, y_m) < \infty\}$$

(b) there exist positive constants K, L, M, s and a real-valued non-negative function g continuous in $[0,s]$ such that $2L < M$ and

1. $|f(x, y_1, y_2, \ldots, y_m)| \leq K$ for

$$0 \leq y_i \leq \max\{M, M - L + \max\{g(x) : 0 \leq x \leq s\}\}, j = 1, 2, 3, \ldots, m$$

2. for arbitrary real-valued function $y_j(x), j = 1, 2, 3, \ldots, m$ defined and continuous in $[0,\infty)$ fulfilling the equalities

$$0 \leq y_j(x) \leq h(x), x \geq 0, j = 1, 2, 3, \ldots, m$$

where $h(x) = g(x) + M - L$ for $x \in [0, s]$ and $h(x) = M$ for $x > s$, we have

$$-L \leq \int_0^x f(t, y_1(t), y_2(t), y_3(t), \ldots, y_m(t))dt \leq g(x) \quad \text{for} \quad x \in [0,s]$$

$$\left| \int_0^x f(t, y_1(t), y_2(t), y_3(t), \ldots, y_m(t))dt \right| \leq L \quad \text{for} \quad x \geq s \, .$$

Under these assumptions, for every real number c such that $L \leq c \leq M - L$ there exists a solution (5) of class C^1 on the interval $[0,\infty)$.

Proof.

Let B be a linear space of real-valued function u define and continuous for $x \geq 0$ such that

$$\sup\{|u(x)| \cdot \exp(-x) : x \geq 0\} < \infty.$$

Define norm on B by

$$|u| = \sup\{|u(x)| \cdot \exp(-x) : x \geq 0\}. \qquad (6)$$

The space B with norm (6) is complete. Define the space A by

$$A = \{u \in B \mid 0 \leq u \leq h(x), \ for \ x \geq 0\}.$$

The space A with norm (6) is nonempty, convex, closed and bounded. Let $T : A \to B$ be define as follows; for $u \in A$,

$$(Tu)(x) = c + \int_0^x f(t, u(t), u^2(t), u^3(t), \ldots, u^m(t))dt \qquad for \quad x \geq 0. \qquad (7)$$

For each $u \in A$, Tu is continuous and

$$0 \leq (Tu)(x) \leq c + g(x) \qquad for \ x \in [0,s]$$

$$0 \leq (Tu)(x) \leq c + L \qquad for \ x \in [0,s]$$

Since $c + g(x) \leq M - L + g(x)$ and $c + L \leq M$, we have $T(A) \subset A$, thus T maps A into A. We put $V = T(A)$. We need to prove that T is continuous with respect to norm (6).

Let $\{u_n\}$ be a sequence of functions such that $u_n \in A, u_n \to u$ as $n \to \infty$. We put $v_n = Tu_n$ and we shall prove that $v_n \to v = Tu$ as $n \to \infty$.

Now we have

$$|v_n(x)-v(x)| \le \int_0^x \left| f(t,u_n(t),u_n^2(t),u_n^3(t),...,u_n^m(t)) - f(t,u(t),u^2(t),u^3(t),...,u^m(t)) \right| dt \tag{8}$$

For $x \ge 0$, $n = 0,1,2,3,...$ and

$$|v_n(x)-v(x)| \le K|x-t| \tag{9}$$

For $x \ge 0, t \ge 0$ and $n = 0,1,2,3,...$.

Let λ be a positive real number. In the interval $[0,\lambda]$, the sequence $\{u_n\}$ converges uniformly to u so that for each $\eta > 0$ there exists a number $N(\eta,\lambda)$ such that $|u_n(x)-u(x)| < \eta$ for $\eta > N(\eta,\lambda)$ and for $x \in [0,s]$. In particular, for each $\eta > 0$ there exists a number $N(\eta,\overline{M})$ such that $|u_n(t)-u(t)| < \eta$ for $\eta > N(\eta,\overline{M})$ and $t \in [0,\overline{M}]$, where $\overline{M} = \max\{M, c + \max\{|g(x)| : 0 \le x \le s\}\}$. Now since f being uniformly continuous in the set of $[0,\lambda]\times[0,\overline{M}]\times[0,\overline{M}]\times...\times[0,\overline{M}]$, for each $\varepsilon > 0$ there exists $\sigma > 0$ such that if $|y_j - \overline{y_j}| < \sigma, |y_j| < \overline{M}, |\overline{y_j}| < \sigma$ and $n = 0,1,2,3,...,m$ then

$$\left| f(x,y_1,y_2,y_3,...,y_m) - f(x,\overline{y_1},\overline{y_2},\overline{y_3},...,\overline{y_m}) \right| < \varepsilon \ for\ x \in [0,\lambda].$$

Let $\varepsilon > 0$ be arbitrary then there exists $\sigma > 0$ such that if

$$|u_n(x)-u(x)| < \sigma, n > N_1(\sigma,\lambda),$$

$$|u_n(u(x))-u(u(x))| < \sigma, n > N_{21}(\sigma,\bar{M}),$$

$$|u_n(u_n(x))-u_n(u(x))| < \sigma, n > N_{22}(\frac{\sigma}{K},\lambda),$$

$$|u_n(u^2(x))-u^3(x)| < \sigma, n > N_{31}(\sigma,\bar{M}),$$

$$|u_n^3(x)-u_n(u^2(x))| < \sigma, n > N_{32}(\frac{\sigma}{K},\lambda),$$

$$\vdots$$

$$|u_n^m(x)-u_n(u^{m-1}(x))| < \sigma, n > N_{mm}(\frac{\sigma}{K},\lambda),$$

then for $x \in [0,\lambda], n > N_\varepsilon = \max\{N_1, N_{21}, N_{22}, ..., N_{mm}\}$ we have

$$\left| f(x,u_n(x),u_n^2(x),...,u_n^m(x)) - f(x,u_n(x),u_n^2(x),...,u_n^{m-1}(x),u_n^{m-1}(u(x))) \right| < \frac{\varepsilon}{m}.$$

$$\left| f(x,u_n(x),u_n^2(x),...,u_n^{m-1}(x),u_n^{m-1}(u(x))) - f(x,u_n(x),u_n^2(x),...,u_n^{m-2},u_n^{m-2}(u^2(x))) \right| < \frac{\varepsilon}{m},$$

$$\vdots$$

$$\left| f(x,u_n(x),u_n^2(x),...,u_n(u^{m-1}(x))) - f(x,u_n(x),u_n^2(x),...,u^m(x)) \right| < \frac{\varepsilon}{m}.$$

Thus we have

$$\left| f(x,u_n(x),u_n^2(x),u_n^3(x),...,u_n^m(x)) - f(x,u(x),u^2(x),u^3(x),...,u^m(x)) \right| < \frac{\varepsilon}{m} + \frac{\varepsilon}{m} + \frac{\varepsilon}{m} + ... + \frac{\varepsilon}{m} = \varepsilon. \tag{10}$$

Hence
$$|v_n(x)-v(x)|.\exp(-x) \le \varepsilon.\lambda.\exp(-x) \tag{11}$$

for $x \in [0,\lambda]$ and $n \ge N_\varepsilon$. And for $x \ge s$, we have $|v_n(x)-v(x)| \le 2M$ and therefore, if $\lambda \ge s$ we get

$$|v_n(x)-v(x)| \cdot \exp(-x) \le 2M \cdot \exp(-x) \tag{12}$$

for $x \ge \lambda$ and for each n.

Let μ be positive real number. There exists $\lambda \ge 0$ such that $2M \cdot \exp(-x) < \mu$ for $x \ge \lambda$ and We can assume that $\lambda \ge s$. For this λ and for each $\varepsilon > 0$, we can find N such that for $n \ge N$ and $x \in [0,\lambda]$.

$$|v_n(x)-v(x)| \cdot \exp(-x) < \varepsilon \cdot \lambda \cdot \exp(-x) < \varepsilon \cdot \lambda. \tag{13}$$

Putting $\varepsilon = \frac{\mu}{\lambda}$, we have $|v_n(x)-v(x)| \cdot \exp(-x) < \mu$ for $n \ge N$ and $x \ge 0$. Thus T is continuous.

Now, let $\{v_n\}$ be a sequence of functions in V . For each n ,we have $|v_n(x)| \le \overline{M}$ and $|v_n(x) - v_n(t)| < K|x-t|$. Hence by Arzela-Ascoli theorem , we can find a subsequence $\{v_n^1\}$ converging uniformly in $[0,1]$ to a continuous v^1. Similary we can find a subsequence $\{v_n^2\}$ of $\{v_n^1\}$ which converges uniformly in $[0,2]$ to a continuous function v^2. Finally we can find a subsequence $\{v_n^m\}$ of $\{v_n^{m-1}\}$ which converges uniformly in $[0,m]$ to a continuous function v^m. Hence , the sequence $\{v_n^n\}$ converges uniformly in each closed subinterval $[0,\infty')$ to a function $v \in V$ such that $v \in v^m$ in $[0,m]$. Let $\varepsilon > 0$ be arbitrary positive real number. We put $k = \ln\left(\frac{4\overline{M}}{\varepsilon}\right)$, we have

$$\sup_{x\ge0}\left|v(x) - v_n^n(x)\right|\cdot\exp(-x) \le \sup_{0\le x\le k}\left|v(x) - v_n^n(x)\right| + 2\overline{M}\cdot\exp(-k). \tag{14}$$

Thus for n large enough, we get

$$\sup_{x\ge0}\left|v(x) - v_n^n(x)\right|\cdot\exp(-x) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{15}$$

Hence V is compact. Thus by Schauder fixed point theorem, we obtain a solution (5).

Next lemma will help us prove the uniqueness of the solution of the problem (1)-(2). Let K^*, L^*, M^* be positive constants and let a^* be the solution of the equation $M^*a^* = \exp(-L^*a^*)$. Let c be a real number such that $|c| < a^*$ and w a real-valued function continuous in $[0,\infty)$ such that

$$|w(x)| \le |c| + x \qquad for \quad x \in [0,\infty). \tag{16}$$

Lemma

If v is a function continuous in $[0,\infty)$ such that

$$v(x) \le L^*\int_0^x v(t)dt + M^*\int_0^x v(w(t))dt \qquad for \quad x \ge 0 \tag{17}$$

then $v(x) = 0$ for $x \ge 0$.

Proof.

Let a be arbitrary real number in the open interval $(0,a^*)$. Hence $M^*a < \exp(-L^*a)$. For $K^* = 1$, we obtain $v(x) = 0$ for $x \in [0,a)$ and by continuity of v,

$$v(x) = 0 \qquad for \quad x \in [0,a^*]. \tag{18}$$

Let $b_1 = a^* - |c|$. We have $0 < b_1 < a^*$ and we have $v(w(x)) = 0$ for $x \in [0,b_1]$. Hence for $x \in [b_1, b_1 + a^*]$,

$$v(x) \le L^*\int_0^x v(t)dt + M^*\int_0^x v(w(t))dt = L^*\int_0^x v(t)dt + M^*\int_{b_1}^x v(w(t))dt \tag{19}$$

thus $v(x) = 0$ for $x \in [b_1, b_1 + a^*]$ then $v(x) = 0$ for $x \in [0, b_1 + a^*]$.

Putting $b_n = b_{n-1} - |c|$, we can show, by the same method, that $v(x) = 0$ for $x \in [0, b_1 + a^*]$ and thus $v(x) = 0$ for $x \ge 0$.

Theorem 6

Suppose that $M_1, M_2, M_3, ..., M_m, N, c$ be constants such that $|c| < a^*$ where a^* is a solution of the equation $M_1a^* = \exp(-Ma^*), M = M_1 + M_2 + M_3 + ... + M_m$. If f is defined in D theorem 5 .

$$\left|f(x,y_1,y_2,...,y_n)-f(x,\overline{y}_1,\overline{y}_2,...,\overline{y}_n)\right|\le M_1\left|y_1-\overline{y}_1\right|+M_2\left|y_2-\overline{y}_2\right|+...+M_m\left|y_m-\overline{y}_m\right|$$

and $\left|f(x,y_1,y_2,y_3,...,y_m)\right|<1$ for every $(x,y_1,y_2,y_3,...,y_m),(x,\overline{y_1},\overline{y_2},\overline{y_3},...,\overline{y_m})\in D$ then there exists at most one function defined and continuous for $x\ge 0$ fulfilling (5).

Proof.

Let $y(x)$ and $z(x)$ be two continuous functions fulfilling (5) for $x\ge 0$. Now $|z(x)|\le|c|+x$ for $x\ge 0$, we have

$$|y(x)-z(x)|\le\int_0^x\left(M_1|y(t)-z(t)|+M_2\left|y^2(t)-z^2(t)\right|+...+M_m\left|y^m(t)-z^m(t)\right|\right)dt$$

$$\le M_1\int_0^x|y(t)-z(t)|\,dt+M_2\int_0^x\left|y^2(t)-z^2(t)\right|dt+...+M_m\int_0^x\left|y^m(t)-z^m(t)\right|dt.$$

But

$$\left|y^2(t)-z^2(t)\right|\le|y(y(t))-y(z(t))|+|y(z(t))-z(z(t))|$$

$$\left|y^3(t)-z^3(t)\right|\le|y(y(y(t)))-y(y(z(t)))|+|y(y(z(t)))-y(z(z(t)))|+|y(z(z(t)))-z(z(z(t)))|$$

$$\vdots$$

$$\left|y^m(t)-z^m(t)\right|\le\left|y^m(t)-y^{m-1}(z(t))\right|+\left|y^{m-1}(z(t))-y^{m-2}(z^2(t))\right|+...+\left|y\left(z^{m-1}(t)\right)-z^m(t)\right|$$

Thus

$$|y(x)-z(x)|\le L\int_0^x|y(t)-z(t)|\,dt+M_2\int_0^x|y(z(t))-z(z(t))|\,dt+...+M_m\int_0^x\left|y(z^{m-1}(t))-z^m(t)\right|dt$$

where $L=M_1+M_2+2M_3+3M_4+...+(m-1)M_m$. Now if we put $v(x)=|y(x)-z(x)|$ and $w(x)=z(x)$, then all assumptions of previous lemma are satisfied thus $v(x)=0$ for $x\ge 0$. That is $y(x)=z(x)$.

The details in the proof of the theorems and lemmas in Podisuk[6] suggest us to come up with the other proof of the existence and uniqueness of the solution of the iterative ordinary differential equation in the semi-infinite domain $[0,\infty)$.

First look at the function $R(x,y(x),y^2(x),...,y^m(x))$ of the form

$$2^k\,p(x)(y(x))^{k_1}\,(y^2(x))^{k_2}\,...(y^m(x))^{k_m} \tag{20}$$

where $k_1, k_2, ..., k_m$ are positive integers and $k=k_1+k_2+...+k_m$.

Assume that

1. $0\le p(x)\le\frac{1}{2}$
2. $0\le R(x,y(x),y^2(x),...,y^m(x))\le\frac{1}{2}$ for $0\le x\le\frac{1}{2}$.

If we let $y_0(x)=\frac{1}{4}$ and $y_n(x)=\frac{1}{4}+\int_0^x y_n^j(t)dt$ for $j=1,2,...,m$ which $y_n^1(x)=y_n(x)$. Then we have $y_n(x)$ converges uniformly to a unique function $y(x)$ and $y(x)\le\frac{1}{2}$ for $0\le x\le\frac{1}{2}$ and $y(x)\le\frac{e^x}{2}-\frac{1}{4}$. Thus if we let $y_0(x)=\frac{1}{4}$ then

$$y_1(x)=\frac{1}{4}+\int_0^x R(t,y_0(t),y_0^2(t),...,y_0^m(t))dt\le\frac{1}{4}+\int_0^x\frac{1}{2}dt=\frac{1}{4}+\frac{x}{2}.$$

But $R(\frac{1}{4}+\frac{x}{2},y(\frac{1}{4}+\frac{x}{2}),y^2\left(\frac{1}{4}+\frac{x}{2}\right),...,y^m(\frac{1}{4}+\frac{x}{2}))\le\frac{1}{4}+\frac{x}{2}$ for $0\le x\le\frac{1}{2}$ thus

$$y_2(x)=\frac{1}{4}+\int_0^x R(t,y_0(t),y_0^2(t),...,y_0^{m}(t))dt \le \frac{1}{4}+\int_0^x\left(\frac{1}{4}+\frac{x}{2}\right)dt=\frac{1}{4}+\frac{x}{2}+\frac{x^2}{4}.$$

By the same argument, we can see that $y_n(x)$ converges uniformly to unique function $y(x)$ in $C^1\left[0,\frac{1}{2}\right]$ where $y(x)\le\frac{e^x}{2}-\frac{1}{4}$ and by the same argument as in the proof in Podisuk[6], we have $y(x)\in C^1[0,\infty)$. Thus we have the following three theorems which we will state without proof.

Theorem 7

If the above conditions are satisfied then there exists at most one solution of the problem

$$y'(x)=R(x,y(x),y^2(x),...,y^{m}(x)) \tag{21}$$

with the initial condition

$$y(0)=c \tag{22}$$

where $0\le c\le\frac{1}{4}$.

Theorem 8

Let $Q(x)=\sum_{i=1}^{j}\frac{1}{j}R_i(x,y(x),y^2(x),...,y^{m}(x))whereR_i(x,y(x),y^2(x),...,y^{m}(x))$ satisfies the above conditions then there exists at most one solution to the problem

$$y'(x)=Q(x) \tag{23}$$

with the initial condition

$$y(0)=c \tag{24}$$

where $0\le c\le\frac{1}{4}$.

In case $c=0$ we have theorem 9.

Theorem 9

Let $F\left(x,y(x),y^2(x),...,y^{m}(x)\right)=f(x)+Q(x,y(x),y^2(x),...,y^{m}(x))$. If f is a position function on $[0,\infty)$ with

$$0<f(0)<\frac{1}{2} \tag{25}$$

$$0\le f(0)\le\frac{1}{2}\ \text{ for }\ x\in\left[0,\frac{1}{2}\right] \tag{26}$$

then there exist at most one solution to the problem

$$y'(x)=F\left(x,y(x),y^2(x),...,y^{m}(x)\right) \tag{27}$$

with the initial condition

$$y(0)=0. \tag{28}$$

Note. R and f in theorem 7 and theorem 9, respectively, may be negative. In this case we will have the unique solution $y(x)$ of the problem in $(-\infty,-\infty)$ as you can see in example 6.

We may use the same type of the above proof to prove the existence and uniqueness of the system of n first order iterative ordinary differential equation of degree m on the semi-infinite domain:

$$y'(x)=f\left(x,y(x),y^2(x),...,y^{m}(x)\right) \tag{29}$$

with the initial condition

$$y(0)=c \tag{30}$$

where

$$y(x)=\begin{pmatrix} y_1(x)\\ y_2(x)\\ \vdots \\ y_n(x)\end{pmatrix}, y'(x)=\begin{pmatrix} y_1'(x)\\ y_2'(x)\\ \vdots \\ y_n'(x)\end{pmatrix}, c=\begin{pmatrix} c_1\\ c_2\\ \vdots \\ c_n\end{pmatrix}$$

$$f\left(x,y(x),y^2(x),...,y^m(x)\right)=\begin{pmatrix} f_1\left(x,y(x),y^2(x),...,y^m(x)\right)\\ f_2\left(x,y(x),y^2(x),...,y^m(x)\right)\\ \vdots \\ f_n\left(x,y(x),y^2(x),...,y^m(x)\right)\end{pmatrix}$$

Then we obtain the following two theorem.

Theorem 10 Suppose

(a) f_i is defined and continuous in

$$D=\left\{(x,y_1,y_2,...,y_m): y_1=\begin{pmatrix} y_{11}\\ y_{12}\\ \vdots \\ y_{1n}\end{pmatrix}, y_2=\begin{pmatrix} y_{21}\\ y_{22}\\ \vdots \\ y_{2n}\end{pmatrix},..., y_m=\begin{pmatrix} y_{m1}\\ y_{m2}\\ \vdots \\ y_{mn}\end{pmatrix}, and 0\le x, 0\le y_{ij}<\infty, 1\le j\le m, 1\le k\le n\right\}$$

for $i=1,2,...,n$

(b) there exist positive K_i, s and vectors $M=\begin{pmatrix} M_1\\ M_2\\ \vdots \\ M_n\end{pmatrix}, L=\begin{pmatrix} L_1\\ L_2\\ \vdots \\ L_n\end{pmatrix}$ where M_i and L_i are positive

constants, $i=1,2,...,n$ and a real-valued non-negative function $g_i(x)$ continuous on

$[0,s], g(x)=\begin{pmatrix} g_1(x)\\ g_2(x)\\ \vdots \\ g_n(x)\end{pmatrix}$ such that $2L_i<M_i$, $i=1,2,...,n$ and

1. $\left|f_i(x,y_{11},y_{12},...,y_{1n},y_{21},y_{22},...,y_{2n},...,y_{m1},y_{m2},...,y_{mn})\right|\le K_i$ and

$$\left\|f_i(x,y_{11},y_{12},...,y_{1n},y_{21},y_{22},...,y_{2n},...,y_{m1},y_{m2},...,y_{mn})\right\|\le K \text{ where } K=\max_{1\le i\le n}\{K_i\}$$

for $0\le y_{ji}\le \max\left\{M_i, M_i-L_i+\max\left\{g_i(x): 0\le x\le s\right\}\right\} i=1,2,...,n, j=1,2,...,m$

2. for arbitrary real-valued functions $y_{ji}(x)(i=1,2,...,n, j=1,2,...,m)$ defined and continuous on $[0,\infty)$ fulfilling the inequalities $0\le y_{ji}\le h_i(x)(x\ge 0, i=1,2,...,n, j=1,2,...,m)$ where $h_i(x)=g_i(x)+M_i-L_i$, for $x\in[0,s]$ and $h_i(x)=M_i$ for $x>s$, we have

$$-L_i\le \int_0^x f_i(t,y(t),y^2(t),...,y^m(t))dt\le g_i(x)$$

for $x\in[0,s]$ and

$$\left|\int_0^x f_i(t,y(t),y^2(t),...,y^m(t))dt\right|\le L_i$$

for $x \geq s$ for $i = 1,2,...,n$. Under the above assumptions, for every real number c_i such that $L_i \leq c_i \leq M_i - L_i, i = 1,2,...,n$ there exist a solution of (29)-(30) of the class C^1 on the whole half-axis x , $x \geq 0$.

Theorem 11 Suppose that $M_1, M_2, ..., M_m$ be constants and c is the n dimensional vector such that $\|c\| < a$ where a is a solution of the equation

$$(M_2 + 2M_3 + 3M_4 + ... + (m-1)M_n)\bar{a} = \exp(-M\bar{a})$$

$$M = M_1 + M_2 + ... + M_n. \text{ If } f(x, y_1, y_2, ..., y_m) = \begin{pmatrix} f_1(x, y_1, y_2, ..., y_m) \\ f_2(x, y_1, y_2, ..., y_m) \\ . \\ . \\ . \\ f_m(x, y_1, y_2, ..., y_m) \end{pmatrix} \text{ and}$$

$f_i(x, y_1, y_2, ..., y_m), i = 1,2,...,n$ is defined in D of theorem 10 and $\|f(x, y_1, y_2, ..., y_m) - f(x, \overline{y_1}, \overline{y_2}, ..., \overline{y_m})\| \leq M_1\|y_1 - \overline{y_1}\| + M_2\|y_2 - \overline{y_2}\| + ... + M_m\|y_m - \overline{y_m}\|$ and $\|f(x, y_1, y_2, ..., y_m)\| \leq 1$ that is $K = 1$ for every $(x, y_1, y_2, ..., y_m), (x, \overline{y_1}, \overline{y_2}, ..., \overline{y_m}) \in D$ then there exist at most one function defined and continuous for $x \geq 0$ fulfilling (19)-(30).

3. EXAMPLE

Example 1. find the solution in the interval $[0, \infty)$ of the equation

$$y'(x) = \frac{1}{4} + \frac{x^2}{4} + xy^2(x), y(0) = 0.$$

Let $y_0(x) = 0$ then from the first sequence of theorem 3 we get

$y_1(x) = 0.25x + 0.0833333x^3$

$y_2(x) = 0.25x + 0.104167x^3 + 0.00442708x^5 + 0.000186012x^7$ + higher power terms

$y_3(x) = 0.25x + 0.104167x^3 + 0.00553385x^5 + 0.000449371x^7$ + higher power terms

$y_4(x) = 0.25x + 0.104167x^3 + 0.00553385x^5 + 0.000489053x^7$ + higher power terms.

We can see that $y_n(x)$ will converges.

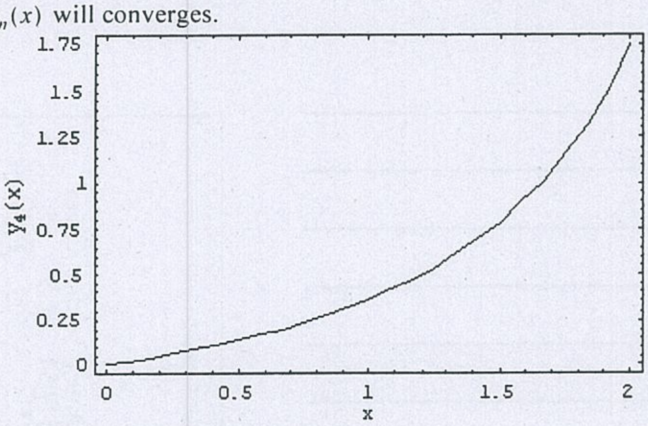

Figure 1 contains the graph of $y_4(x)$ for Example 1.

Example 2. Find the solution in the interval $[0, \infty)$ of the equation

$$y'(x) = 4(0.25 + x^2)y(x)y^2(x)$$

with initial condition $y(0) = 0.2$.

By letting $y_0(x) = 0.2$, we have

$$y_1(x) = 0.2 + 0.04x + 0.0533333x^3$$

$$y_2(x) = 0.2 + 0.0416853x + 0.00435413x^2 + 0.0556086x^3 + \text{ higher power terms}$$

$$y_2(x) = 0.2 + 0.04179504x + 0.00456606x^2 + 0.00056079x^3 + \text{ higher power terms}$$

We can see that $y_n(x)$ will converges.

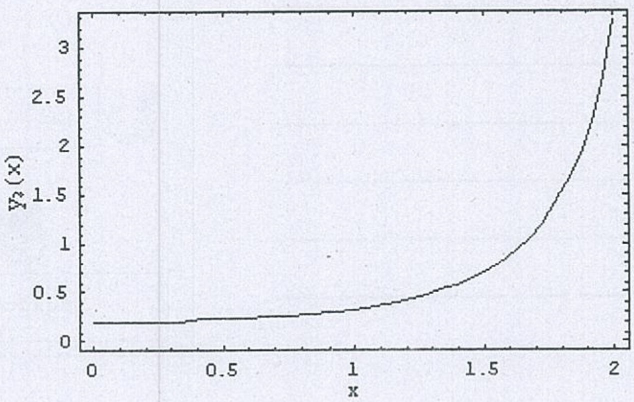

Figure 2 contains the graph of $y_3(x)$ for Example 2.

Example 3. Find the solution in the interval $[0,\infty)$ of the equation

$$y'(x) = 32(\frac{1}{4} + \frac{x}{2})(y^2(x))^3(y^3(x))^2 + 4y(x)y^2(x)y^3(x)$$

with initial condition $y(0) = 0.25$.

By letting $y_0(x) = 0.25$, we have

$$y_1(x) = 0.25 + 0.0703125x + 0.0078125x^2$$

$$y_2(x) = 0.25 + 0.083449 + 0.0224389x^2 + 0.00144319x^3 + \text{ higher power terms.}$$

We can see that $y_n(x)$ will converges.

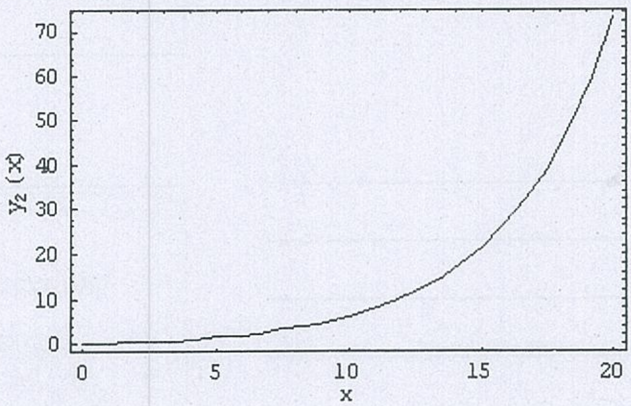

Figure 3 contains the graph of $y_2(x)$ for Example 3.

Example 4. Find the solution in the interval $[0,\infty)$ of the equation

$$y'(x) = y^3(x)$$

With initial condition $y(0) = 0.09$.

By letting $y_x(0) = 0.09$, we have

$$y_1(x) = 0.09 + 0.09x$$

$y_2(x) = 0.09 + 0.098829x + 0.0003645x^2$

$y_3(x) = 0.09 + 0.09977755x + 0.0004833313x^2 + 1.31734 \times 10^{-6}x^3 +$ higher power terms

$y_4(x) = 0.09 + 0.09988110x + 0.0004758x^2 + 1.78313126 \times 10^{-6}x^3 +$ higher power terms.

We can see that $y_n(x)$ will converges.

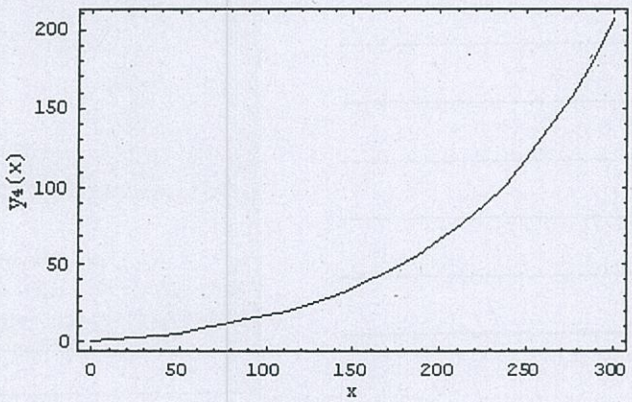

Figure 4 contains the graph of $y_4(x)$ for Example 4.

Example 5. Find the solution in the interval $[0, \infty)$ of the equation

$$y'(x) = y^2(x)(1 + x + xy^2(x))$$

with initial condition $y(0) = 0.01$.

By letting $y_0(x) = 0.2$, we have

$y_1(x) = 0.01 + 0.01x + 0.00505$

$y_2(x) = 0.01 + 0.0101005x + 0.00515177 + 0.0000515219x^3 +$ higher power terms

$y_3(x) = 0.01 + 0.0101015x + 0.00515331x^2 + 0.0000527451 +$ higher power terms.

We can see that $y_n(x)$ will converges.

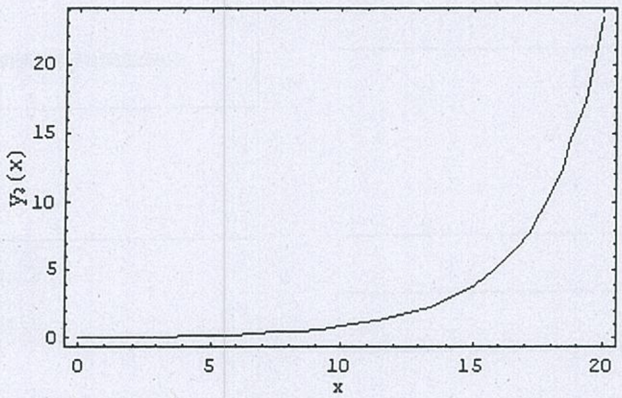

Figure 5 contains the graph of $y_3(x)$ for Example 5.

Example 6. Find the solution in the interval $[0, \infty)$ of the equation

$$y'(x) = xy(x)y^2(x) - \frac{1}{4} - x^2$$

With initial condition $y(0) = 0.25$.

By letting $y_0(x) = 0.25$, we have

$y_1(x) = 0.25 - 0.25x + 0.03125x^2 - 0.333333x^3$

$y_2(x) = 0.25 - 0.25x + 0.023306x^2 + 0.342502x^3 +$ higher power terms

$y_3(x) = 0.25 - 0.25x + 0.0229453x^2 + 0.342314x^3 +$ higher power terms.

We can see that $y_n(x)$ will converges.

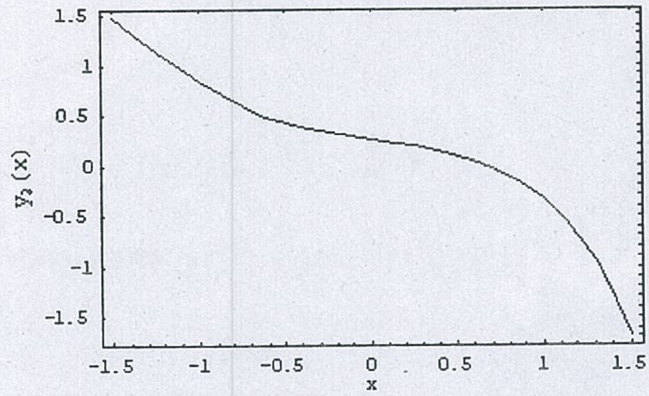

Figure 6 contains the graph of $y_3(x)$ for Example 6.

3. CONCLUSION

We may prove the existence and uniqueness of the above problem for the **infinite domain** $(-\infty, \infty)$. We may use the numerical method introduced by *Podisuk-Suchatvejpoom-Chaisanit*[7] **to find the** numerical solution of the first order iterative ordinary differential equation of m degree

$$y'(x) = f(x, y(x), y^2(x), \dots, y^m(x)) \tag{31}$$

with the initial condition

$$y(0) = c. \tag{32}$$

However, we have the new way to find this numerical solution, we call " The polynomial Approximation Method" , by letting the approximated polynomial of degree k be

$$y_c(x) = a_0 + a_1x + a_2x^2 + \dots + a_kx^k. \tag{33}$$

We get the value a_0 by putting $x = 0$ in $y_c(x)$ in (31) and substitute the value of x by the value of x at mesh points $x_1, x_2, \dots, x_k \in [0, a]$ then we obtain k equations which we may solve for $a_1, a_2, \dots, a_k$. Then use $y_c(x)$ to find the approximated value of the solution of (29)-(30). For example, for the problem $y'(x) = y^2(x) - 0.25y(x), x \in [0, 1]$ with the initial condition $y(0) = 0.25$.

If we let $y_c(x) = a_0 + a_1x + a_2x^2$ with the mesh points $x_0 = 0, x_1 = 0.05$ and $x_2 = 0.1$ the we obtain $a_0 = 0.25, a_1 = 0.25$ and $a_2 = 0$ that is $y_c(x) = 0.25 + 0.25x$. It still remain, finding the application in the real world problems using the iterative ordinary different equations. The author is thinking about the problem of **Growth and Decay** problems which will be helpful in the area of environmental science.

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