OPEN FORMULA OF INTEGRATION METHOD

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ABSTRACT

Integration method is the method that uses the numerical integration formula for finding the definite integral to find the numerical solution of the initial value problem of ordinary differential equation by applying those formula to the integral equation which equivalent to the given differential equation. In papers [1] and [2], we used the closed formulas of the numerical integration formula for this purpose. However they lead to the problem of solving system of equations. Thus we come up with the open formula to prevent such problem.

KEYWORD: Open-Formula Integral-Equation Integration-Method

1. INTRODUCTION

The initial value problem of the ordinary differential equation is of the form

(1) $y'(x) = f(x, y), x \in [a, b]$

with the initial condition

(2) $y(a) = c$

which equivalent to the integral equation

(3) $$y(x) = c + \int_a^x f(x, y)dx .$$

Instead of finding the numerical solution of the problem (1)-(2) by the Runge-Kutta method, we will find the numerical solution of the equation (3). We may use the Newton-Cotes formulas and Gaussian Quadrature formulas that already existed or we may build the new formulas for this purpose.

2. PROBLEM FORMULATION

In this paper, We will introduce five formulas of the type of open formula. These five formulas are one point formula, two points formula, three points formula, four points formula and five points formula.

We divide the interval $[a, b]$ in to n equally subintervals which length h that is $h = \frac{b-a}{n}$.

One Point Formula:

From mid point formula, we obtain the following one point formula

(4) $y_{m+2} = y_m + 2hf(x_{m+1}, y_{m+1})$.

If we let $s = 2h$ then the error of the above formula is of the form $O(s^3)$.

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Two Points Formula: We use the points x_{m+1} and x_{m+2} to be the points of the numerical integral formula for the interval $[x_m, x_{m+3}]$. We obtain the following two points formula

(5) $$y_{m+3} = y_m + 1.5h(f(x_{m+1}, y_{m+1}) + f(x_{m+2}, y_{m+2})).$$

If we let $s = 3h$ then the error of the above formula is of the form $O(s^3)$.

Three Points Formula: We use the points x_{m+1}, x_{m+2} and x_{m+3} to be the points of the numerical integral formula for the interval $[x_m, x_{m+4}]$. We obtain the following three points formula

(6) $$y_{m+4} = y_m + \frac{4}{3}h(2f(x_{m+1}, y_{m+1}) - f(x_{m+2}, y_{m+2}) + 2f(x_{m+3}, y_{mk+3})).$$

If we let $s = 4h$ then the error of the above formula is of the form $O(s^5)$.

Four Points Formula: We use the points $x_{m+1}, x_{m+2}, x_{m+3}$ and x_{m+4} to be the points of the numerical integral formula for the interval $[x_m, x_{m+5}]$. We obtain the following four points formula

(7) $$\begin{aligned} y_{m+5} = y_m + \frac{5}{24}h(11f(x_{m+1}, y_{m+1}) + f(x_{m+2}, y_{m+2}) + f(x_{m+3}, y_{m+3}) \\ + 11f(x_{m+4}, y_{m+4})) \end{aligned}$$

If we let $s = 5h$ then the error of the above formula is of the form $O(s^5)$.

Five Points Formula: We use the points $x_{m+1}, x_{m+2}, x_{m+3}, x_{m+4}$ and x_{m+5} to be the points of the numerical integral formula for the interval $[x_m, x_{m+6}]$. We obtain the following five formula

(8) $$\begin{aligned} y_{m+6} = y_m + \frac{3}{10}h(11f(x_{m+1}, y_{m+1}) - 14f(x_{m+2}, y_{m+2}) + 26f(x_{m+3}, y_{m+3}) \\ - 14f(x_{m+4}, y_{m+4}) + 11f(x_{m+5}, y_{m+5})) \end{aligned}$$

If we let $s = 6h$ then the error of the above formula is of the form $O(s^7)$.

We shall denote the above formulas as:
OP1 for One point formula
OP2 for Two points formula
OP3 for Three points formula
OP4 for Four points formula
OP5 for Five points formula.

3. EXAMPLES

There will be six examples in this section. We will use the above five formulas to find the numerical solutions of these six examples.

Example 1

Find the numerical solution of the equation

(9) $$y' = \frac{4}{x(x-4)}, \; x \in [5,6]$$ with the initial condition

(10) $$y(5) = 5.$$

The analytical solution of the above equation is

$$y(x) = 5 + \ln(5) - \ln(x) + \ln(x-4).$$

The ten exact values are:

$$y(5.1) = 5.0755075525$$

$$y(5.2) = 5.1431008437$$

$$y(5.3) = 5.2040953564$$
$$y(5.4) = 5.2595111955$$
$$y(5.5) = 5.3101549284$$
$$y(5.6) = 5.3566749440$$
$$y(5.7) = 5.3995999887$$
$$y(5.8) = 5.4393666599$$
$$y(5.9) = 5.4763275867$$
$$y(6.0) = 5.5108133286.$$

The numerical results are in following table 1 to table 5.

OP1	h = 0.01	
x	Calculated y	Error
5.1	5.0755046862	2.8663271×10^{-6}
5.2	5.1430958019	5.041729×10^{-6}
5.3	5.2040886257	6.730697×10^{-6}
5.4	5.2595031282	8.067364×10^{-6}
5.5	5.3101457857	9.142699×10^{-6}
5.6	5.3566649238	1.002015×10^{-5}
5.7	5.3995892437	1.074504×10^{-5}
5.8	5.4393553094	1.135046×10^{-5}
5.9	5.4763275867	1.186103×10^{-5}
6.0	5.5108133285	1.229533×10^{-5}

Table 1

OP2	h = 0.01	
x	Calculated y	Error
5.1	5.0755037420	3.810506×10^{-6}
5.2	5.1430942445	6.500206×10^{-6}
5.3	5.2040852614	1.009503×10^{-5}
5.4	5.2594995841	1.161142×10^{-5}
5.5	5.3101421781	1.275022×10^{-5}
5.6	5.3566599151	1.502891×10^{-5}
5.7	5.3995843610	1.562768×10^{-5}
5.8	5.4393505982	1.606170×10^{-5}
5.9	5.4763216577	1.779009×10^{-5}
6.0	5.5108076708	1.795303×10^{-5}

Table 2

OP3	h = 0.01	
x	Calculated y	Error
5.1	5.0755075514	1.350494×10^{-9}
5.2	5.1431008412	2.437446×10^{-9}
5.3	5.2040953537	2.713922×10^{-9}
5.4	5.2595111920	3.507012×10^{-9}
5.5	5.3101549249	3.441528×10^{-9}
5.6	5.3566749400	4.023605×10^{-9}
5.7	5.3995999849	3.812602×10^{-9}
5.8	5.43936665555	4.307370×10^{-9}
5.9	5.4763394437	4.045432×10^{-9}
6.0	5.5108256194	4.474714×10^{-7}

Table 3

OP4	h = 0.01	
x	Calculated y	Error
5.1	5.0755075500	2.517481×10^{-9}
5.2	5.1431008396	4.118192×10^{-9}
5.3	5.2040953512	5.187758×10^{-9}
5.4	5.2595111896	5.908078×10^{-9}
5.5	5.3101549219	6.417395×10^{-9}
5.6	5.3566749372	6.781192×10^{-9}
5.7	5.3995999817	7.043127×10^{-9}
5.8	5.4393666526	7.246854×10^{-9}
5.9	5.4763394404	7.399649×10^{-9}
6.0	5.5108256163	7.516064×10^{-9}

Table 4

OP5	h = 0.01	
x	Calculated y	Error
5.1	5.0755075525	2.546585×10^{-10}
5.2	5.1431008437	4.874892×10^{-10}
5.3	5.2040953564	6.912160×10^{-10}
5.4	5.2595111957	8.876668×10^{-10}
5.5	5.3101549285	1.069566×10^{-9}
5.6	5.3566749442	1.193257×10^{-9}
5.7	5.3995999890	1.367880×10^{-9}
5.8	5.4393666603	1.520675×10^{-9}
5.9	5.4763394482	1.593435×10^{-9}
6.0	5.5108256244	1.760782×10^{-9}

Table 5

Example 2

Find the numerical solution of the equation

(11) $y' = \dfrac{xy}{y^2 - x^2}$, $x \in [0,1]$ with the initial condition

(12) $y(0) = 1$.

The analytical solution of the above equation is $y(x) = \dfrac{1}{\sqrt{3-2\sqrt{1+x^2}}}$.

The ten exact values are;

$$y(0.1) = 1.0050124371$$
$$y(0.2) = 1.0201959029$$
$$y(0.3) = 1.0459645461$$
$$y(0.4) = 1.0829215632$$
$$y(0.5) = 1.1317139243$$
$$y(0.6) = 1.1928228789$$
$$y(0.7) = 1.2663324102$$
$$y(0.8) = 1.3517639463$$
$$y(0.9) = 1.4480661205$$
$$y(1.0) = 1.5537739740.$$

The numerical results are in following table 6 to table 10.

OP1	1 h = 0.01	
x	Calculated y	Error
0.1	1.0050121908	2.462839×10^{-7}
0.2	1.0201949649	9.379783×10^{-7}
0.3	1.0459626233	1.922830×10^{-6}
0.4	1.0829186387	2.924542×10^{-6}
0.5	1.1317103672	3.557063×10^{-6}
0.6	1.1928194532	3.425730×10^{-6}
0.7	1.2663300924	2.317742×10^{-6}
0.8	1.3517635818	3.644145×10^{-7}
0.9	1.4480681110	1.990549×10^{-6}
1.0	1.5537781956	4.221509×10^{-6}

Table 6

OP2	h = 0.01	
x	Calculated y	Error
0.1	1.0050120714	3.657533×10^{-7}
0.2	1.0201945107	1.392214×10^{-6}
0.3	1.0459616609	2.885201×10^{-6}
0.4	1.0829171787	4.384572×10^{-6}
0.5	1.1317086023	5.321956×10^{-6}
0.6	1.1928177371	5.141781×10^{-6}
0.7	1.2663289348	3.475356×10^{-6}
0.8	1.3517634157	5.305956×10^{-7}
0.9	1.4480691036	2.983132×10^{-6}
1.0	1.5537803094	6.335387×10^{-6}

Table 7

OP3	h = 0.01	
x	Calculated y	Error
0.1	1.0050124373	1.709850×10^{-10}
0.2	1.0201959036	7.657945×10^{-10}
0.3	1.0459645480	1.846274×10^{-9}
0.4	1.0829215666	3.421519×10^{-9}
0.5	1.1317139294	5.078618×10^{-9}
0.6	1.1928228850	6.037226×10^{-9}
0.7	1.2663324157	5.509719×10^{-9}
0.8	1.3517639498	3.552486×10^{-9}
0.9	1.4480661216	1.109584×10^{-9}
1.0	1.5537739733	7.457857×10^{-10}

Table 8

OP4	h = 0.01	
x	Calculated y	Error
0.1	1.0050124374	3.019522×10^{-10}
0.2	1.0201959042	1.296939×10^{-9}
0.3	1.0459645493	3.143214×10^{-9}
0.4	1.0829215690	5.8043952×10^{-9}
0.5	1.1317139329	8.631105×10^{-9}
0.6	1.1928228892	1.025546×10^{-9}
0.7	1.2663324195	9.369614×10^{-9}
0.8	1.3517639523	6.040864×10^{-9}
0.9	1.4480661223	1.897206×10^{-9}
1.0	1.5537739728	1.255103×10^{-9}

Table 9

OP5	h = 0.01	
x	Calculated y	Error
0.1	1.0050124371	1.818989×10^{-12}
0.2	1.0201959029	1.818989×10^{-12}
0.3	1.0459645461	0.0
0.4	1.0829215632	5.456968×10^{-12}
0.5	1.1317139243	9.094947×10^{-12}
0.6	1.1928228789	1.637090×10^{-11}
0.7	1.2663324101	2.182787×10^{-11}
0.8	1.3517639462	1.637090×10^{-11}
0.9	1.4480661204	1.637090×10^{-11}
1.0	1.5537739740	2.728484×10^{-11}

Table 10

Example 3

Find the numerical solution of the equation

(13) $y' = y^3$, $x \in [0,1]$ with the initial condition

(14) $y(0) = 0.1$

The analytical solution of the above equation is $y(x) = \dfrac{1}{\sqrt{100-2x}}$.

The ten exact values are;

$y(0.1) = 0.1001001503$
$y(0.2) = 0.1002006020$
$y(0.3) = 0.1003013568$
$y(0.4) = 0.1004024161$
$y(0.5) = 0.1005-37815$
$y(0.6) = 0.1006054546$
$y(0.7) = 0.1007074368$
$y(0.8) = 0.1008097298$
$y(0.9) = 0.1009123352$
$y(1.0) = 0.1010152545$.

The numerical results are in following table 11 to table 15.

OP1	$h = 0.01$	
x	Calculated y	Error
0.1	0.1001001503	2.501110×10^{-12}
0.2	0.1002006020	5.229595×10^{-12}
0.3	0.1003013568	7.730705×10^{-12}
0.4	0.1004024161	1.023182×10^{-11}
0.5	0.1005037815	1.273293×10^{-11}
0.6	0.1006054546	1.546141×10^{-11}
0.7	0.1007074368	1.818989×10^{-11}
0.8	0.1008097298	2.103206×10^{-11}
0.9	0.1009123351	2.387424×10^{-11}
1.0	0.1010152544	2.637535×10^{-11}

Table 11

OP2	$h = 0.01$	
x	Calculated y	Error
0.1	0.1001001503	3.410605×10^{-12}
0.2	0.1002006020	6.707523×10^{-12}
0.3	0.1003013568	1.136868×10^{-11}
0.4	0.1004024161	1.500666×10^{-11}
0.5	0.1005037815	1.841727×10^{-11}
0.6	0.1006054546	2.330580×10^{-11}
0.7	0.1007074368	2.683009×10^{-11}
0.8	0.1008097298	3.046807×10^{-11}
0.9	0.1009123351	3.558398×10^{-11}
1.0	0.1010152544	3.910827×10^{-11}

Table 12

OP3	h = 0.01	
x	Calculated y	Error
0.1	0.1001001503	0.0
0.2	0.1002006020	1.136868×10^{-13}
0.3	0.1003013568	0.0
0.4	0.1004024161	0.0
0.5	0.1005037815	0.0
0.6	0.1006054546	1.136868×10^{-13}
0.7	0.1007074368	0.0
0.8	0.1008097298	0.0
0.9	0.1009123352	2.273737×10^{-13}
1.0	0.1010152545	0.0

Table 13

OP4	h = 0.01	
x	Calculated y	Error
0.1	0.1001001503	1.136868×10^{-13}
0.2	0.1002006020	0.0
0.3	0.1003013568	1.136868×10^{-13}
0.4	0.1004024161	1.136868×10^{-13}
0.5	0.1005037815	1.136868×10^{-13}
0.6	0.1006054546	1.136868×10^{-13}
0.7	0.1007074368	1.136868×10^{-13}
0.8	0.1008097298	0.0
0.9	0.1009123352	$1.1236868 \times 10^{-13}$
1.0	0.1010152545	$1.1236868 \times 10^{-13}$

Table 14

OP5	h = 0.01	
X	Calculated y	Error
0.1	5.0755075525	0.0
0.2	5.1431008437	0.0
0.3	5.2040953564	2.273737×10^{-13}
0.4	5.2595111957	1.136868×10^{-13}
0.5	5.3101549285	1.136868×10^{-13}
0.6	5.3566749442	1.136868×10^{-13}
0.7	5.3995999890	1.136868×10^{-13}
0.8	5.4393666603	1.136868×10^{-13}
0.9	5.4763394482	1.136868×10^{-13}
1.0	5.5108256244	2.273737×10^{-13}

Table 15

Example 4

Find the numerical solution of the equation

(15) $y' = \dfrac{\sin x}{x^2} - \dfrac{2y}{x}$ $x \in [2,3]$ with the initial condition

(16) $y(2) = 1$

at the point $x = 3$ by using step size $h = 0.01$.

The analytical solution of the above equation is $y(x)=\dfrac{4+\cos 2-\cos x}{x^2}$ and $y(3)=0.5082050733$.

The results are in table 16 and
RK1 is Euler's Method
RK2 is 2-points Runge-Kutta method
RK3 is 3-points Runge-Kutta method
RK4 is 4-points Runge-Kutta method.

	Calculated y	Error
RK1	0.5067707775	1.434296×10^{-3}
RK2	0.5082072867	2.213332×10^{-6}
RK3	0.5082050692	4.243702×10^{-9}
RK4	0.5082050734	4.547474×10^{-11}
OP1	0.5082128146	7.741231×10^{-6}
OP2	0.5082160491	1.097568×10^{-5}
OP3	0.5082050756	2.187335×10^{-9}
OP4	0.5082050771	3.698915×10^{-9}
OP5	0.5082050733	7.003109×10^{-11}

Table 16

Example 5

Find the numerical solution of the equation

(17) $$y'=\frac{xy^3}{\sqrt{1+x^2}}$$ $x\in[0,1]$ with the initial condition

(18) $y(0)=1$

at the point $x=1$ by using step size $h=0.01$.

The analytical solution of the above equation is $y(x)=\dfrac{1}{\sqrt{3-2\sqrt{1+x^2}}}$ and

$y(1)=2.4142135625$

The results are in table 17.

		Error
RK1	2.2781717556	1.360418×10^{-1}
RK2	2.31608068522	9.813288×10^{-2}
RK3	2.4142056843	7.878192×10^{-6}
RK4	2.4142135222	4.035246×10^{-8}
OP1	2.4099523426	4.261220×10^{-3}
OP2	2.4078786870	6.334876×10^{-3}
OP3	2.4141286225	8.493999×10^{-5}
OP4	2.4140741639	1.393987×10^{-4}
OP5	2.4142071062	6.456310×10^{-6}

Table 17

Example 6

Find the numerical solution of the equation

(19) $$y'=\frac{1}{2\sqrt{x}}-\frac{1}{2x\sqrt{x}}$$ $x\in[1,100]$ with the initial condition

(20) $y(0)=1$

at the point $x = 10 -$ and $x = 100$ by using step size $h = 0.01$.

The analytical solution of the above equation is $y(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$ and $y(10) = 3.4785054262$ and $y(100) = 10.1$

The results are in table 18 and 19.

		Error
RK1	3.4777897010	7.157252×10^{-4}
RK2	3.4785012135	4.212602×10^{-6}
RK3	4.4785054261	4.001777×10^{-11}
RK4	3.4785054261	4.001777×10^{-11}
OP1	3.4785138507	8.424493×10^{-6}
OP2	3.4785180615	1.263532×10^{-5}
OP3	3.4785054305	4.361937×10^{-9}
OP4	3.4785054336	7.399649×10^{-9}
OP5	3.4785054262	0.0

Table 18

		Error
RK1	10.099748331	2.516690×10^{-4}
RK2	10.099995831	4.168949×10^{-6}
RK3	10.100000000	7.275958×10^{-11}
RK4	10.100000000	7.275958×10^{-11}
OP1	10.100008336	8.336094×10^{-6}
OP2	10.100012503	1.250285×10^{-5}
OP3	10.100000004	4.220554×10^{-9}
OP4	10.100000007	7.392373×10^{-9}
OP5	10.100000000	7.275958×10^{-11}

Table 19

Note that for the integration method we used the exact value for the initial points. The errors in on tables are the absolute true error.

4. CONCLUSION

All five new formulas are good numerical formulas for finding the numerical solutions of the initial value problem of the ordinary differential equations. These five new formulas will give us more freedom to find the way of finding the numerical solutions of the initial value problem of the ordinary differential equations. However we have to keep in mind that the above numerical solutions are just for only these six equations. We strongly recommend the formula OP1, OP3 and OP5.

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