Mathematical Modeling of a Thin Two-link Flexible Robot Arm

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Abstract

More than three decades, flexible link robot arms have attracted attention from the researchers around the world due to advantages over conventional robot arms. The advantages include less overall mass, less energy consumption, smaller actuators and faster responses. This paper focuses on mathematical modeling for a thin two-link flexible robot arm operating in planar plane. The model was derived based on energy model and Lagrangian method. Computer simulation is also given to validate the effectiveness of the model. The results yield a satisfaction and imply the application of the area for applied science and engineering field in robotics and automation.

Keywords: mathematical modeling

1. INTRODUCTION

The first worldwide recognition as the first introduction of flexible structure in control areas has been presented by Gevarter [1]. Since then the flexible structures including the flexible manipulator have been attracted attention of researchers, scientists, and engineers around the world [2-6]. For flexible manipulators, there are advantages over the conventional ones. The advantages include less overall mass, less energy consumption requirement, and smaller actuators.

Since the flexible link robot arms are made of lightweight material, for example, the arms made of plastic, aluminum, or fiber; whereas, the conventional arms usually made of iron, this implies that vibration behavior in the link is an inherent property due to the flexible nature. In some engineering application such as assembling electronic parts and components which requires precision and accuracy during operation, the vibration in the flexible-link arms could reach unpleasant performance. Hence the vibration suppressions needs to be carried out during the operation. In doing so, control engineers have to design a control scheme that can track or regulate the work and suppress the vibration at the same time. In order to do that, most control schemes, both conventional and advanced control techniques, require a good mathematical model to be cooperated. This motivates the research of searching for an accuracy and reliable model for the needs.

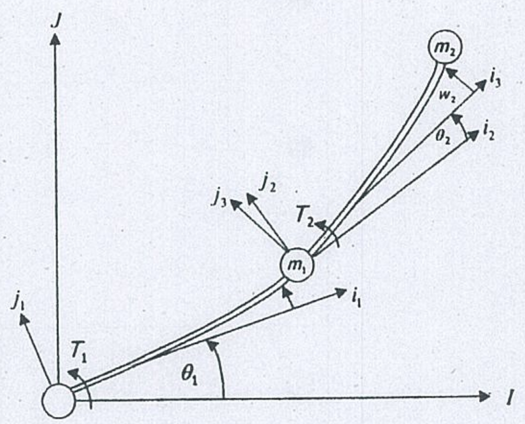

Figure 1 A model of a thin two-link flexible robot arm.

In this paper, a thin two-link flexible robot arm model is proposed. Structural damping is included in the model, which makes the model being an improved version of the model proposed by other researchers in the past [2-6].

2. MATHEMATICAL MODELING

A mathematical model of a two-link flexible robot arm in this research is depicted in Figure 1. Here θ_i, w_i, m_i; and L_i stand for an angle, the deflected distance away from the link i^{th}, mass at the end of the link i^{th}, and the length of the link i^{th} measured corresponding to Figure 1, respectively. It is worth mentioning that the model is assumed to work under no gravity effect, and the torsion and transverse forces in the links are negligible.

To begin with, let us now summarize coordinate transformation using to derive the model along with energy model corresponding to the problem in hand. The coordinate transformations using here are:

$$\begin{Bmatrix} i_1 \\ j_1 \end{Bmatrix} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{Bmatrix} I \\ J \end{Bmatrix} \qquad (2.1)$$

$$\begin{Bmatrix} i_2 \\ j_2 \end{Bmatrix} = \begin{bmatrix} \cos(w'_{1,L_1}) & \sin(w'_{1,L_1}) \\ -\sin(w'_{1,L_1}) & \cos(w'_{1,L_1}) \end{bmatrix} \begin{Bmatrix} i_1 \\ j_1 \end{Bmatrix} \qquad (2.2)$$

$$\begin{Bmatrix} i_3 \\ j_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{Bmatrix} i_2 \\ j_2 \end{Bmatrix} \qquad (2.3)$$

The total kinetic energy can be described as

$$\begin{aligned} T &= T_{L_1} + T_{L_2} \\ &= \frac{1}{2}\int_0^{L_1} (\rho A)_1 (\dot{\bar{r}}_{1,x} \cdot \dot{\bar{r}}_{1,x}) dx_1 + \frac{1}{2} m_1 (\dot{\bar{r}}_{1,x} \cdot \dot{\bar{r}}_{1,x}) \Big|_{x_1 = L_1} \\ &\quad + \frac{1}{2}\int_0^{L_2} (\rho A)_2 (\dot{\bar{r}}_{2,x} \cdot \dot{\bar{r}}_{2,x}) dx_2 + \frac{1}{2} m_2 (\dot{\bar{r}}_{2,x} \cdot \dot{\bar{r}}_{2,x}) \Big|_{x_2 = l_2} \end{aligned} \qquad (2.4)$$

where ρ_i is the volume density and A_i is a cross-section of link i^{th}, respectively, and

$$\dot{\bar{r}}_{1,x} \cdot \dot{\bar{r}}_{1,x} = L_1^2 \dot{\theta}_1^2 + 2L_1 \dot{\theta}_1 \dot{w}_{1,L_1} + \dot{w}_{1,L_1}^2 + w_{1,L_1}^2 \dot{\theta}_1^2 \qquad (2.5)$$

$$\begin{aligned} \dot{\bar{r}}_{2,x} \cdot \dot{\bar{r}}_{2,x} &= L_1^2 \dot{\theta}_1^2 + 2L_1 \dot{\theta}_1 \dot{w}_{1,L_1} + \dot{w}_{1,L_1}^2 + x_2^2 \omega^2 + 2x_2 \omega \dot{w}_2 + \dot{w}_2^2 \\ &\quad + 2L_1 \dot{\theta}_1 x_2 \omega \cos\theta_2 + 2L_1 \dot{\theta}_1 \dot{w}_2 \cos\theta_2 + 2\dot{w}_{1,L_1} x_2 \omega \cos\theta_2 \\ &\quad + 2\dot{w}_{1,L_1} \dot{w}_2 \cos\theta_2 + w_{1,L_1}^2 \dot{\theta}_1^2 + w_2^2 \omega^2 + 2w_{1,L_1} \dot{\theta}_1 w_2 \omega \cos\theta_2 \end{aligned} \qquad (2.6)$$

We assume that the displacements are small so that the higher order terms associated with the small deviation can be neglected. Now the potential energy can be expressed as

$$U = \frac{1}{2}\int_0^{L_1} (EI)_1 (w_1'')^2 dx_1 + \frac{1}{2}\int_0^{L_2} (EI)_2 (w_2'')^2 dx_2 \qquad (2.7)$$

where E_i is the Yong's modulus, and is the moment of inertia of the i^{th}.

We assume that there is existence of modes of vibration in the flexible arms so that we can utilize the assumed mode method. We now define modes in corresponding to the deflection as follows:

$$\begin{aligned} w_1(x,t) &= \sum_{i=0}^{\infty} \phi_i(x_1) a_i(t) = \phi^T a = a^T \phi \\ w_2(x,t) &= \sum_{j=0}^{\infty} \psi_j(x_2) c_j(t) = \psi^T c = c^T \psi \end{aligned} \qquad (2.8)$$

In nature, there are no cases such that the arms are still having vibrations in simple harmonic motion. Vibration is die out due to the law of entropy. With this picture in mind, we then define a dissipation function in the form of

$$D = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij} \dot{q}_i \dot{q}_j \qquad (2.9)$$

where d_{ij} is the damping coefficients and the q_i is a generalized coordinate i^{th} defined as

$$q = \begin{bmatrix} \theta_1 & \theta_2 & a^T & c^T \end{bmatrix}^T \qquad (2.10)$$

Using the Lagrangian equation, we have

$$\overline{M}\ddot{q} + \overline{D}\dot{q} + \overline{K}q = 0 \quad (2.11)$$

and after long time consuming calculation, we then arrive at equations of motions in state space formulation as:

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\phi}_L \\ \dot{\psi}_L \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\phi}_L \\ \ddot{\psi}_L \end{bmatrix} = \begin{bmatrix} 0 & I \\ -inv(M^{*}_{sys})K^{*}_{sys} & -inv(M^{*}_{sys})D^{*}_{sys} \end{bmatrix} \cdot \begin{bmatrix} \theta_1 \\ \theta_2 \\ \phi_L \\ \psi_L \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\phi}_L \\ \dot{\psi}_L \end{bmatrix} + \begin{bmatrix} 0 \\ inv(M^{*}_{sys}) \end{bmatrix} \cdot u \quad (2.12)$$

where u is an input vector and the rest of the corresponding terms will be shown in Appendix.

3. SIMULATION RESULTS

To validate the model and observe the effects of damping, following parameters are used in the below simulation: $L_1 = 0.5$, $L_2 = 0.5$, $m_1 = 0.4$, $m_2 = 0.1$, $d_1 = 0.05$, $d_2 = d_3 = 0$ and $d_4 = 0.07$, $\rho_i = 2710 kg/m^3$, $A_i = 1.613*10\text{^}-5 m^2$, $EI = 0.0684 N*m^2$. The reason to set d_2 and d_3 equal to zero is that to provide no direct interaction of damping effects via the middle joint between two links.

Figures 2 shows the simulation obtained from using the model and the given parameters. In this case, we simulate an impulse response acting on the arms. Subgraph (1,1) on the upper-left corner shows the effects of the impulse response applied at angle θ_1, which is corresponding to apply current U(1) to the motor located at the hub in Dirac delta function as a very narrow pulse signal, while other inputs are retained to be unchanged. Here label Y(4) means the observation point is at the tip of the second arm. It is no surprised that the smallest vibration amplitude can be observed. This is because the contribute of the impulse is away from the tip of the second arm.

Subgraph (1,2) on the upper-right corner shows the effects of an impulse response applied at angle θ_2, which is corresponding to apply current U(2) to the motor located at the joint between two arms in Dirac delta function as a very narrow pulse signal, while other inputs are retained to be unchanged. We can observe that higher order of magnitude of the vibration can be investigated. The closer the input point to the tip, the higher the vibration amplitude at the observation points.

Subgraph (2,1) on the lower-left corner shows the effects of an impulse response U(3) applied at link 1, which is corresponding to apply impulsive force hitting on the link 1, while other inputs are retained to be unchanged. We can see that the vibration from the first link is also effected on the tip of the arm.

Subgraph (2,2) on the lower-right corner shows the effects of an impulse response U(4) applied at link 2, which is corresponding to apply impulsive force hitting on the link 2, while other inputs are retained to be unchanged. As expected, compared to all subgraphs, the highest vibration amplitude can be observed. And for all above subgraphs, we have seen the damping effects that vibration is eventually die out.

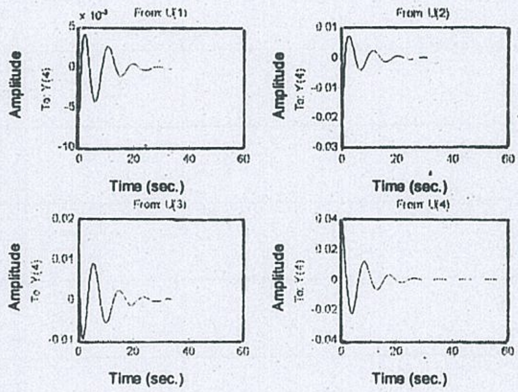

Figure 2 Simulation results corresponding to the given data.

Figure 3 illustrates the corresponding picture in frequency domain for each subgraph corresponds to each one in Figure

2, respectively. As we can see from poles and zeroes in below subgraphs, the system is not the minimum phase due to there are zeroes on the right half plan, and the poles lie on the real axis contribute to the damping behavior of the system; while pairs of poles indicate the vibration. This analysis is confirmed with the picture in time domain and the interpretation is exactly shown what would be expected and predicted in frequency domain analysis of classical stability theory.

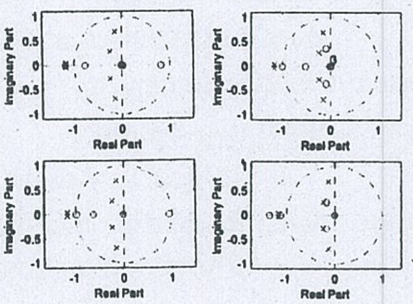

Figure 3 Simulation results in frequency domain correspond to Figure 2

4. CONCLUSION

We have presented a mathematical modeling for a thin two-link flexible robot arm. The model is derived by using assumed mode method along wi-th Lagrangial equation. The finding is arranged in state space form, which is a preferable form for engineers in most disciplines to work out for further application. The findings have been validated graphically by using computer simulation. As far as the analysis is concerned, the model provides important information in practical senses in which the damping effects can be significantly observed.

5. ACKNOWLEDGEMENTS

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APPENDIX

This Appendix describes important terms corresponding to the findings in state space form in Section 2. Those terms include:

$$M_{1,I}^{*} = \begin{bmatrix} J_{1,I}^{*} + J_{1,Ii}^{*} + a^{T} M_{1aa}^{*} a + c^{T} M_{\dot{c}\dot{c}}^{*} c \\ J_{2}^{*} + c^{T} M_{\dot{c}\dot{c}}^{*} c \\ M_{1\dot{a},I}^{*} + L_{2} c^{T} c M_{1cc\dot{a}}^{*} \\ M_{2\dot{c}}^{*} \end{bmatrix} \quad \text{(A1)}$$

$$M_{2,I}^{*} = \begin{bmatrix} J_{2}^{*} + c^{T} M_{\dot{c}\dot{c}}^{*} c \\ J_{2}^{*} + c^{T} M_{\dot{c}\dot{c}}^{*} c \\ M_{2\dot{a},I}^{*} + L_{2} c^{T} c M_{1cc\dot{a}}^{*} \\ M_{2\dot{c}}^{*} \end{bmatrix} \quad \text{(A2)}$$

$$M^{\bullet}_{3,i} = \begin{bmatrix} M^{\bullet}_{1\dot{a},i} + L_2 c^T c M^{\bullet}_{1cc\dot{a}} \\ M^{\bullet}_{2\dot{a},i} + L_2 c^T c M^{\bullet}_{1cc\dot{a}} \\ M^{\bullet}_{1aa} + M^{\bullet}_{\dot{a}\dot{a},ii} + L_2^2 c^T c M^{\bullet}_{cc\dot{a}\dot{a}} \\ M^{\bullet}_{\dot{c}\dot{a},i} \end{bmatrix} \quad \text{(A3)}$$

$$M^{\bullet}_{4,i} = \begin{bmatrix} M^{\bullet}_{2\dot{c}} \\ M^{\bullet}_{2\dot{c}} \\ M^{\bullet T}_{\dot{c}\dot{a},i} \\ M_{\dot{c}\dot{c}} \end{bmatrix} \quad \text{(A4)}$$

$$M^{\bullet}_{1,ii} = \begin{bmatrix} J^{\bullet}_{1,ii} + 2c^T M^{\bullet}_{1ca} a \\ J^{\bullet}_{12,ii} + c^T M^{\bullet}_{1ca} a \\ M^{\bullet}_{1\dot{a},ii} + L_2 c^T M^{\bullet}_{1ca\dot{a}} a \\ M^{\bullet}_{1\dot{c},ii} \end{bmatrix} \quad \text{(A5)}$$

$$M^{\bullet}_{2,ii} = \begin{bmatrix} J^{\bullet}_{12,ii} + c^T M^{\bullet}_{1ca} a \\ 0 \\ M^{\bullet}_{2\dot{a},ii} \\ 0 \end{bmatrix} \quad \text{(A6)}$$

$$M^{\bullet}_{3,ii} = \begin{bmatrix} M^{\bullet}_{1\dot{a},ii} + L_2 c^T M^{\bullet}_{1ca\dot{a}} a \\ M^{\bullet}_{2\dot{a},ii} \\ M^{\bullet}_{\dot{a}\dot{a},ii} \\ M^{\bullet}_{\dot{c}\dot{a},ii} \end{bmatrix} \quad \text{(A7)}$$

$$M^{\bullet}_{4,ii} = \begin{bmatrix} M^{\bullet}_{1\dot{c},ii} \\ 0 \\ M^{\bullet T}_{\dot{c}\dot{a},ii} \\ 0 \end{bmatrix} \quad \text{(A8)}$$

$$M^{\bullet}_{sys,i} = \begin{bmatrix} M^{\bullet}_{1,i} & M^{\bullet}_{2,i} & M^{\bullet}_{3,i} & M^{\bullet}_{4,i} \end{bmatrix} \quad \text{(A9)}$$

$$M^{\bullet}_{sys,ii} = \begin{bmatrix} M^{\bullet}_{1,ii} & M^{\bullet}_{2,ii} & M^{\bullet}_{3,ii} & M^{\bullet}_{4,ii} \end{bmatrix} \quad \text{(A10)}$$

$$M^{\bullet}_{sys} = M^{\bullet}_{sys,i} + M^{\bullet}_{sys,ii} \cos\theta_2 \quad \text{(A11)}$$

$$K^{\bullet}_{sys} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 K^{\bullet}_{aa} & 0 \\ 0 & 0 & 0 & \eta_L^2 r_L^2 \mu_2 K^{\bullet}_{cc} \end{bmatrix} \quad \text{(A12)}$$

$$D^{\bullet}_{1,i} = \begin{bmatrix} I^{\bullet}_{1,i} + \frac{1}{2} c^T D^{\bullet}_{1cc} c \\ \frac{1}{2} I^{\bullet}_2 + \frac{1}{2} c^T D^{\bullet}_{12cc} c \\ D^{\bullet}_{1\dot{a},i} + \frac{1}{2} L_2 c^T c D^{\bullet}_{1cc\dot{a}} \\ \frac{1}{2} D^{\bullet}_{1\dot{c},i} \end{bmatrix} \quad \text{(A13)}$$

$$D^{\bullet}_{2,i} = \begin{bmatrix} \frac{1}{2} I^{\bullet}_2 + \frac{1}{2} c^T D^{\bullet}_{12cc} c \\ \frac{1}{2} I^{\bullet}_2 + \frac{1}{2} c^T D^{\bullet}_{2cc} c \\ \frac{1}{2} D^{\bullet}_{2\dot{a},i} + \frac{1}{2} L_2 c^T c D^{\bullet}_{2cc\dot{a}} \\ \frac{1}{2} D^{\bullet}_{2\dot{c}} \end{bmatrix} \quad \text{(A14)}$$

$$D^{\bullet}_{3,i} = \begin{bmatrix} D^{\bullet}_{1\dot{a},i} + \frac{1}{2} L_2 c^T c D^{\bullet}_{1cc\dot{a}} \\ \frac{1}{2} D^{\bullet}_{2\dot{a},i} + \frac{1}{2} L_2 c^T c D^{\bullet}_{2cca} \\ \frac{1}{2}\left(D^{\bullet}_{a\dot{a},i} + D^{\bullet T}_{\dot{a}\dot{a},i}\right) + \frac{1}{4} L_2^2 c^T c \left(D_{cc\dot{a}a} + D^T_{cc\dot{a}\dot{a}}\right) \\ \frac{1}{2} D^{\bullet}_{\dot{c}\dot{a},i} \end{bmatrix} \quad \text{(A15)}$$

$$D^{\bullet}_{4,i} = \begin{bmatrix} \frac{1}{2} D^{\bullet}_{1\dot{c},i} \\ \frac{1}{2} D^{\bullet}_{2\dot{c}} \\ \frac{1}{2} D^{\bullet T}_{\dot{c}\dot{a},i} \\ \frac{1}{4}\left(D^{\bullet}_{\dot{c}\dot{c}} + D^{\bullet T}_{\dot{c}\dot{c}}\right) \end{bmatrix} \quad \text{(A16)}$$

$$D^{\bullet}_{1,ii} = \begin{bmatrix} I^{\bullet}_{1,ii} + a^T D^{\bullet}_{1aa} a + c^T D^{\bullet}_{1ca} a \\ \frac{1}{2} I^{\bullet}_{12,ii} + \frac{1}{2} c^T D^{\bullet}_{12ca} a \\ D^{\bullet}_{1\dot{a},ii} + \frac{1}{2} L_2 c^T D^{\bullet}_{1ca\dot{a}} a \\ \frac{1}{2} D^{\bullet}_{1\dot{c},ii} \end{bmatrix} \quad \text{(A17)}$$

$$D^{\bullet}_{2,ii} = \begin{bmatrix} \frac{1}{2} I^{\bullet}_{12,ii} + \frac{1}{2} c^T D^{\bullet}_{12ca} a \\ 0 \\ \frac{1}{2} D^{\bullet}_{2\dot{a},ii} \\ 0 \end{bmatrix} \quad \text{(A18)}$$

$$D^{\bullet}_{3,ii} = \begin{bmatrix} D^{\bullet}_{1\dot{a},ii} + \frac{1}{2} L_2 c^T D^{\bullet}_{1ca\dot{a}} a \\ \frac{1}{2} D^{\bullet}_{2\dot{a},ii} \\ \frac{1}{2}\left(D^{\bullet}_{\dot{a}\dot{a},ii} + D^{\bullet T}_{\dot{a}\dot{a},ii}\right) \\ \frac{1}{2} D^{\bullet}_{\dot{c}\dot{a},ii} \end{bmatrix} \quad \text{(A19)}$$

$$D^{\bullet}_{4,ii} = \begin{bmatrix} \frac{1}{2} D^{\bullet}_{1\dot{c},ii} \\ 0 \\ \frac{1}{2} D^{\bullet T}_{\dot{c}\dot{a},ii} \\ 0 \end{bmatrix} \quad \text{(A20)}$$

$$D^{\bullet}_{sys,i} = \begin{bmatrix} D^{\bullet}_{1,i} & D^{\bullet}_{2,i} & D^{\bullet}_{3,i} & D^{\bullet}_{4,i} \end{bmatrix} \quad \text{(A21)}$$

$$D^{\bullet}_{sys,ii} = \begin{bmatrix} D^{\bullet}_{1,ii} & D^{\bullet}_{2,ii} & D^{\bullet}_{3,ii} & D^{\bullet}_{4,ii} \end{bmatrix} \quad \text{(A22)}$$

$$D^{\bullet}_{sys} = D^{\bullet}_{sys,i} + D^{\bullet}_{sys,ii} \cos\theta_2 \quad \text{(A23)}$$