LOD Methods for Solving Three-Dimensional Heat Equation

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Abstract

This research applied new splitting LOD (Locally One-Dimensional) method for solving three-dimensional time-dependent heat equation. In this work we will find an analytic solution of this equation and compare with numerical solutions.

Keywords: Mathematical Modeling

1. INTRODUCTION

Heat conduction is an everyday experience. Conduction of objects or the sun emit its thermal rays are examples of heat conduction. There are numerical methods for solving heat conduction problems. One numerical method used in solving heat conduction problems is Finite Difference Methods (FDM), which have been widely used for a few decades in teaching and modeling. Splitting FDM are popular methods for solving multidimension problems. In this research, we will extend the new splitting FDM in [2] to apply with three-dimensional time-dependent heat diffusion equation.

2. THEORY

In this research, we study the heat equation which is expressed in the form

$$u_t = \alpha(u_{xx} + u_{yy} + u_{zz}), 0 \le x \le A, 0 \le y \le B, 0 \le z \le C, 0 < t \le T \quad (1)$$

subject to initial condition

$$u(x,y,z,0) = F(x,y,z), 0 \le x \le A, 0 \le y \le B, 0 \le z \le C \quad (2)$$

and Dirichlet boundary conditions which are expressed in general form

$$u(0,y,z,t) = G_1(y,z), 0 < y < B, 0 < z < C, 0 < t \le T \quad (3)$$

$$u(A,y,z,t) = G_2(y,z), 0 < y < B, 0 < z < C, 0 < t \le T \quad (4)$$

$$u(x,0,z,t) = G_3(x,z), 0 < x < A, 0 < z < C, 0 < t \le T \quad (5)$$

$$u(x,B,z,t) = G_4(x,z), 0 < x < A, 0 < z < C, 0 < t \le T \quad (6)$$

$$u(x,y,0,t) = G_5(x,y), 0 < x < A, 0 < y < B, 0 < t \le T \quad (7)$$

$$u(x,y,C,t) = G_6(x,y), 0 < x < A, 0 < y < B, 0 < t \le T \quad (8)$$

Where F and G_1 to G_6 are known continuous functions of their arguments and all auxiliary equations (2)-(8) are linear. For heat conduction, the function $u(x,y,z,t)$ represents temperature distribution varying in time when α is thermal diffusivity.

3. THE LOD METHOD

In order to solve the equation (1) by using new LOD we split this equation into three one-dimensional heat equations,

$$\frac{1}{3}u_t = \alpha u_{xx}, \quad (9)$$

$$\frac{1}{3}u_t = \alpha u_{yy}, \quad (10)$$

$$\frac{1}{3}u_t = \alpha u_{zz}. \qquad (11)$$

Each of these equations is then solved one third of the time step used for the complete three-dimensional equation, for which the three-stage procedure is : which x-sweep in the first one third time step to solve (9) the formula used which i = 2,3,…,I-2 is

$$U^{n+1/3}_{i,j,k} = \frac{s_x}{12}(6s_x-1)\left(U^n_{i-2,j,k}+U^n_{i+2,j,k}\right) + \frac{2s_x}{3}(2-3s_x)\left(U^n_{i-1,j,k}+U^n_{i+1,j,k}\right) + \frac{1}{2}(2-5s_x+6s_x^2)U^n_{i,j,k} \qquad (12)$$

for each j = 0,1,…,J and k = 0,1,2,…,K.

When computing values of $U^{n+2/3}_{i,j,k}$ from the values of $U^{n+1/3}_{i,j,k}$ in the y-sweep used in the second stage, the formula used with
j = 2,3,…,J-2 for each i = 1,2,…,I and k = 1,2,3,…,K is

$$\begin{aligned} U^{n+2/3}_{i,j,k} &= \frac{s_y}{12}(6s_y-1)\left(U^{n+1/3}_{i,j-2,k}+U^{n+3}_{i,j+2,k}\right) \\ &+\frac{2s_y}{3}(2-3s_y)\left(U^{n+1/3}_{i,j-1,k}+U^{n+1/3}_{i,j+1,k}\right) \\ &+\frac{1}{2}(2-5s_y+6s_y^2)U^{n+1/3}_{i,j,k} \end{aligned} \qquad (13)$$

When computing values of $U^{n+1}_{i,j,k}$ from the values of $U^{n+1/3}_{i,j,k}$ in the z-sweep used in the third stage, the formula used with k = 2,3,… ,…,K-2 for each i = 1,2,…,I and j = 1,2,3,…,J is

$$\begin{aligned} U^{n+1}_{i,j,k} &= \frac{s_z}{12}(6s_z-1)\left(U^{n+2/3}_{i,j,k-2}+U^{n+2/3}_{i,j,k+2}\right) \\ &+\frac{2s_z}{3}(2-3s_z)\left(U^{n+3}_{i,j,k-1}+U^{n+21/3}_{i,j,k+1}\right) \\ &+\frac{1}{2}(2-5s_y+6s_y^2)U^{n+2/3}_{i,j,k} \end{aligned} \qquad (14)$$

For convenience, let Notation $U^n_{i,j,k}$ denotes an approximation of function $u(x,y,z,t)$ by using FDM approximate its at grid point $(i\Delta x, j\Delta y, k\Delta z, n\Delta t)$. First we let grid spacing $\Delta x, \Delta y, \Delta z$ and Δt, when $\Delta x = A/I$, $\Delta y = B/J$, $\Delta z = C/K$ and for $\Delta t = T/N$, where I, J, K and N are integers. Thus we can use the notation (x_i, y_j, z_k, t^n) to denote grid point $(i\Delta x, j\Delta y, k\Delta z, n\Delta t)$ for $i = 0,1,2,\dots,I$, $j = 0,1,2,\dots,J$, $k = 0,1,2,\dots,K$ and $n = 0,1,2,\dots,N$. This method has fourth order of accuracy and this new procedure is stable, in von Neumann sense, for

$$0 < s_x, s_y, s_z \le 2/3 \qquad (15)$$

when $s_x = \alpha\Delta t/(\Delta x)^2$, $s_y = \alpha\Delta t/(\Delta y)^2$ and $s_z = \alpha\Delta t/(\Delta z)^2$

4. RESULTS

In this section we will show the results by presenting a mathematical problem as an following example. Let us analyze our solution in the case in which the initial temperature is constant, $300\,K$. This corresponds to a physical problem that is easy to reproduce in the laboratory. Take a clay brick, measuring $0.2m \times 0.2m \times 0.3m$, and place it in a large tub of boiling liquid ($300\,K$). Let it sit there for a long time. After for a while (we expect) the brick will be at $300\,K$ throughout. Now put it in large well-stirred baths of liquid, $0\,K$. The mathematical problem is

$$u_t = \alpha(u_{xx}+u_{yy}+u_{zz}), 0 \le x \le 0.2, 0 \le y \le 0.2, 0 \le z \le 0.3 \qquad (16)$$

subject to initial condition

$$u(x,y,z,0) = 300, 0 \le x \le 0.2, 0 \le y \le 0.2, 0 \le z \le 0.3 \qquad (17)$$

and boundary conditions

$$\begin{aligned} u(0,y,z,t) &= u(0.2,y,z,t) = u(x,0,z,t) = 0 \\ &= u(x,0.2,z,t) = u(x,y,0,t) = u(x,y,0.3,t) \end{aligned} \qquad (18)$$

when thermal conductivity $k = 1.3 W/m\cdot K$, density $\rho = 1460\,kg/m^3$, specific heat $c_p = 880\,J/kg\cdot K$ and thermal diffusivity $\alpha = k/\rho c_p$.

By using eigenfunction expansion and separation of variables method. The analytical solution of this problem is

$$u(x,y,z,t) = 8/0.2\times 0.2\times 0.3$$
$$\times\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{r=1}^{\infty}\left\{\int_0^{0.2}\left\langle\int_0^{0.2}\left[\int_0^{0.3} I_r dz\right]\sin(q\pi y)dy\right\rangle\sin(p\pi x)dx\right\}\sin\left(\frac{p\pi x}{0.2}\right) \quad (19)$$
$$\times\sin\left(\frac{q\pi y}{0.2}\right)\sin\left(\frac{r\pi z}{0.3}\right)\exp\left[-\alpha\pi^2\left(\frac{p^2}{0.2^2}+\frac{q^2}{0.2^2}+\frac{r^2}{0.3^2}\right)t\right]$$

where $I_r = 300\sin\left(\frac{r\pi z}{0.3}\right)$ and the diffusivity $\alpha = 1.01183\times 10^{-6}$.

Next, we will consider approximation to the initial-boundary value problem (16)-(18). We have now obtained the solution to the problem for the heat diffusion with zero boundary conditions (18) and initial temperature distribution equaling $300^{\circ}K$. The solution is expressed in equation (19). The solution is quite complicated, involving an infinite series. First, we notice that $\lim_{t\to\infty} u(x,y,z,t) = 0$. The temperature distribution approaches a steady state, $u(x,y,z,t) = 0$. This is not surprising physically since all of edges are at $0^{\circ}K$; we expect all the initial heat energy contained in the brick to flow out those edges.

The equilibrium problem : $\alpha(u_{xx}+u_{yy}+u_{zz}) = 0$ with $u(0,y,z,t) = u(0.2,y,z,t) = u(x,0,z,t) = u(x,0.2,z,t) = 0$ and $u(x,y,0,t) = u(x,y,0.3,t) = 0$, has a unique solution, $u\equiv 0$, agreeing with the limit t tends to infinity of the time-dependent problem. We note that each term in (19) decays at a different rate (since decay of exponential function). We can then approximate the infinite series by summation of finite terms in (19). For studying of analytical solution when use only first term in (19), which easiest to compute (see Fig.1), it presents values of temperature distribution $u(x,0.05,0.05,\Delta t)$ when $\Delta x = 0.01\,m$, $\Delta t = 1235.38539\,s$. The peak amplitude occurring in the middle $x = 0.1\,m$ decays exponentially in time.

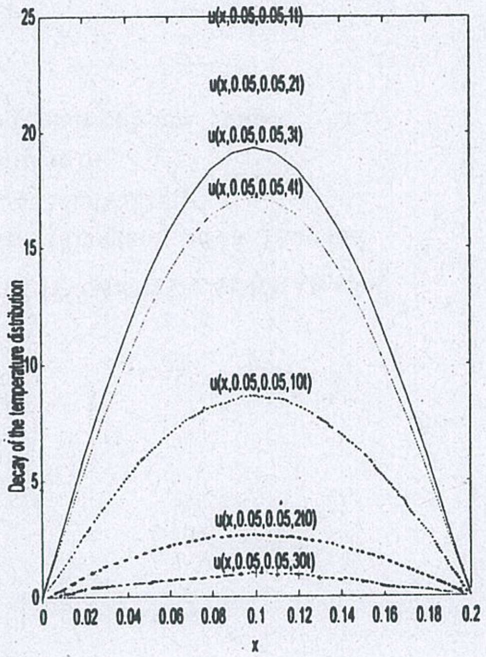

Figure 1 Decay of Temperature Distribution vary in x-direction

When $nt = n\times\Delta t; n = 1,2,3,\ldots$ notice that for higher level of time step we can then make a conclusion that $\lim_{t\to\infty} u(x,y,z,t) = 0$ as discussed above. For comparison of analytic solution with numerical solution. First, we let grid spacing $\Delta x = \Delta y = \Delta z = 1\,cm$ when $s = 2/3$. For this case $\Delta t = 49.41542\,s$, since Δt is small value, we then obtain the approximation of analytical solution by using summation of 37th first term in series (19) (the value of $u(x,0.01,0.01,\Delta t)$ when approximate the summation from $p,q,r = 1$ to 37 equal the summation from $p,q,r = 1$ to 39,40,41,...). Thus we can then use only 37[th] first term :

$$u(x,y,z,t) \approx \sum_{p=1}^{37}\sum_{q=1}^{37}\sum_{r=1}^{37} 8\,/\,0.2\times 0.2\times 0.3$$
$$\times\left\{\int_0^{0.2}\left\langle\int_0^{0.2}\left[\int_0^{0.3} 300\sin(\pi z)dz\right]\sin(\pi y)dy\right\rangle\sin(\pi x)dx\right\}$$
$$\times\sin\left(\frac{\pi x}{0.2}\right)\sin\left(\frac{\pi y}{0.2}\right)\sin\left(\frac{\pi z}{0.3}\right)$$
$$\times\exp\left(-\alpha\pi^2 t\left\langle\frac{p^2}{0.2^2}+\frac{q^2}{0.2^2}+\frac{r^2}{0.3^2}\right\rangle\right)$$

$$\approx \sum_{p=1}^{37}\sum_{q=1}^{37}\sum_{r=1}^{37} \frac{77.4038}{pqr} \times (1-\cos \pi p)(1-\cos \pi q)(1-\cos \pi r) \times \sin\left(\frac{p\pi x}{0.2}\right)\sin\left(\frac{q\pi y}{0.2}\right)\sin\left(\frac{r\pi z}{0.3}\right) \times \exp\left(-\alpha\pi^2 t\left(\frac{p^2}{0.2^2}+\frac{q^2}{0.2^2}+\frac{r^2}{0.3^2}\right)\right) \quad (20)$$

The notation $u^n_{i,j,k}$ and $U^n_{i,j,k}$ denotes the analytic solution (20) and the numerical solution at grid point $(i\Delta x, j\Delta y, k\Delta z, n\Delta t)$ respectively. Representing the results in following table :

(i,j,k,n)	$u^n_{i,j,k}$	$U^n_{i,j,k}$
(1,1,0,1)	0	0
(1,1,1,1)	95.455	80.20932
(1,1,2,1)	133.456	139.80075
(1,1,3,1)	139.442	139.80075
(1,1,4,1)	139.812	139.80075
(1,1,5,1)	139.818	139.80075
(1,1,6,1)	139.82	139.80075
(1,1,7,1)	139.82	139.80075
(1,1,8,1)	139.819	139.80075
(1,1,9,1)	139.82	139.80075
(1,1,10,1)	139.82	139.80075
(1,1,11,1)	139.819	139.80075
(1,1,12,1)	139.82	139.80075
(1,1,13,1)	139.82	139.80075
(1,1,14,1)	139.819	139.80075
(1,1,15,1)	139.82	139.80075
(1,1,16,1)	139.819	139.80075
(1,1,17,1)	139.82	139.80075
(1,1,18,1)	139.82	139.80075
(1,1,19,1)	139.819	139.80075
(1,1,20,1)	139.82	139.80075
(1,1,21,1)	139.82	139.80075
(1,1,22,1)	139.819	139.80075
(1,1,23,1)	139.82	139.80075
(1,1,24,1)	139.82	139.80075
(1,1,25,1)	139.818	139.80075
(1,1,26,1)	139.812	139.80075
(1,1,27,1)	139.442	139.80075
(1,1,28,1)	133.456	139.80075
(1,1,29,1)	95.455	80.20932
(1,1,30,1)	0	0

Table 1 Compares analytic solutions with numerical solutions when $s = 2/3$

From table 1 we obtained the result that numerical solutions from numerical method give closely values to analytic solution. Showing the accuracy of the numerical method.

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REFERENCES

1. A.D. Snider, **Partial Differential Equations-Sources and Solutions**, Prentice-Hall, Inc., New Jersey, 1999.
2. B.J. Noye and K.J. Hayman, **New LOD and ADI Methods for the Two-Dimensional Diffusion Equation**, *J. Computer Mathematics*, Vol.51, pp. 215-228, 1994.
3. L. Lapidus and G.F. Pinder, **Numerical Solution of Partial Differential Equations in Science and Engineering**, Wiley International Science, New York, 1999.
4. R.F. Warming and B.J. Hyett, **The Modified Equation Approach to the Stability and Accuracy Analysis of Finite-Difference Methods**, *J. Computational Physics*, **Vol.14**, pp. 159-179, 1974.