An analytical description of asymmetric soliton states in a nonlinear optical fibre coupler

Michael A. Allen[1] *and George Rowlands*[2]

[1]Physics Department, Mahidol University, Rama 6 Road, Bangkok 10400
frmaa@mahidol.ac.th
[2]Physics Department, Warwick University, Coventry, CV4 7AL, UK
G.Rowlands@warwick.ac.uk

Abstract

Solitons in a nonlinear dual-core coupler can be in symmetric, antisymmetric, and also asymmetric states. There are two branches of asymmetric states. For one of the branches we apply perturbation theory to give an approximate expression for the soliton states near the ends of the branch, and combine the results to give a Padé approximant for the whole branch. Using a perturbative technique involving the regrouping of algebraically secular terms developed by us previously, we can analyse one end of the other branch. However, the remaining end does not appear to be amenable to this type of analysis.

1 Introduction

The envelope of a light wave propagating down a fibre obeys the cubic nonlinear Schrödinger (NLS) equation:

$$i\frac{\partial E}{\partial \xi} + \tfrac{1}{2}\frac{\partial^2 E}{\partial \tau^2} + |E|^2E = 0$$

where the normalized quantities E, ξ, and τ are proportional to the envelope function of the electric field, the distance along the fibre, and the time, respectively, and the coordinate system is moving with the group velocity [1]. The nonlinearity is a result of the Kerr effect which is the distortion of the electron orbitals due to an applied electric field. It produces a nonlinear refractive index, $n = n_0 + n_2|E|^2$, and this results in the wavenumber having a nonlinear dependence on the signal amplitude [2].

If two fibres are placed next to each other, optical signals propagating along them will be coupled via the evanescent fields. This form of nonlinear dual-core directional coupler can exhibit soliton switching and therefore has potential applications in optical processing [3]. The system may be described in terms of two linearly coupled NLS equations:

$$\begin{aligned} iU_\xi + \tfrac{1}{2}U_{\tau\tau} + |U|^2U + KV &= 0 \\ iV_\xi + \tfrac{1}{2}V_{\tau\tau} + |V|^2V + KU &= 0 \end{aligned} \quad (1)$$

where U, V are the electric field envelopes in the two fibres and K is the normalized coupling coefficient. Insisting on stationary pulse-like solutions by making

$$U(\xi,\tau) = u(\tau,q)e^{iq\xi}, \qquad V(\xi,\tau) = v(\tau,q)e^{iq\xi},$$

where q is a soliton parameter, one obtains the reduced equations for soliton states:

$$\begin{aligned} \tfrac{1}{2}\ddot{f} - f + f^3 + \kappa g &= 0 \\ \tfrac{1}{2}\ddot{g} - g + g^3 + \kappa f &= 0 \end{aligned} \quad (2)$$

where $f \equiv u/\sqrt{q}$, $g \equiv v/\sqrt{q}$, $\kappa \equiv K/q$, and the dot denotes the derivative with respect to the reduced variable $t \equiv \tau\sqrt{q}$ [4]. It is sometimes helpful to rewrite equations (2) as

$$\begin{aligned} \ddot{x} - \alpha^2x + x^3 + 3xy^2 &= 0 \\ \ddot{y} - \mu^2\alpha^2y + y^3 + 3yx^2 &= 0 \end{aligned} \quad (3)$$

where $x = (f+g)/\sqrt{2}$, $y = (f-g)/\sqrt{2}$ and

$$\alpha^2 = 2(1-\kappa), \quad \mu^2 = \frac{1+\kappa}{1-\kappa}. \quad (4)$$

From (2) it is apparent that both symmetric ($f = g$) and antisymmetric ($f = -g$) solutions exist. They take the forms

$$(x,y) = (\sqrt{2}\alpha\,\mathrm{sech}\,\alpha t, 0), \quad 0 < \kappa < 1,$$

and

$$(x,y) = (0, \sqrt{2}\alpha\mu\,\mathrm{sech}\,\alpha\mu t), \quad \kappa > 0,$$

respectively. It is shown in [4] that there are also two branches of asymmetric states. They appear as a result of bifurcations from the symmetric states at $\kappa = 3/5$ and from the antisymmetric states at $\kappa = 1$. The two branches are referred to as 'A' and 'B' respectively. All the asymmetric states can be conveniently parameterized by

$$\phi_0 \equiv \tan^{-1}\frac{y_0}{x_0},$$

where $x_0 \equiv x(0)$ and $y_0 \equiv y(0)$. The bifurcation diagram showing these states is plotted in Fig. 1.

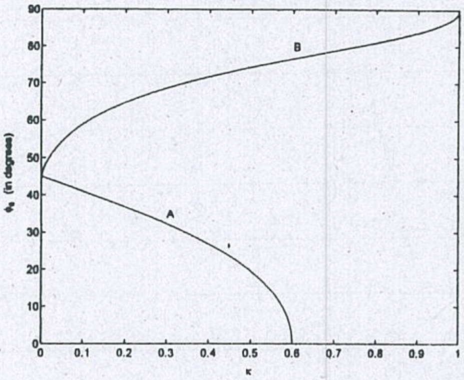

Figure 1: Bifurcation diagram showing asymmetric states

The asymmetric states are of particular interest because for some values of ϕ_0 their shape deviates significantly from the sech-form usually associated with non-topological solitons. This is important, as many calculations had been done with the assumption that the two solitons had sech-function shapes.

2 'A'-type asymmetric states

As can be seen from the figure, the two branches of asymmetric states meet at $\kappa = 0$ which corresponds to the decoupled states. One such state is a single soliton $(f, g) = (\sqrt{2}\,\mathrm{sech}\,\sqrt{2}t, 0)$. It turns out that if we perturb about this state, we obtain the 'A'-branch.

After introducing the new variables $T = \sqrt{2}t$, $F = f/\sqrt{2}$, and $G = g/\sqrt{2}$ from (2) we obtain:

$$\begin{aligned} F'' - F + 2F^3 &= -\kappa G \\ G'' - G + 2G^3 &= -\kappa F \end{aligned} \qquad (5)$$

where the prime denotes differentiation with respect to T. Using κ as our small parameter, we carry out a small-κ perturbation analysis and expand the envelope functions as

$$F = F_0 + \kappa F_1 + \dots, \quad G = \kappa G_1 + \kappa^2 G_2 + \dots .$$

The zeroth order solution is the state we are perturbing about, namely $F_0 = \mathrm{sech}\,T$. To first order we find

$$L_o(1)F_1 = 0 \qquad (6)$$

and

$$G_1'' - G_1 = -F_0, \qquad (7)$$

where we have denoted $d^2/dT^2 + 6\,\mathrm{sech}^2 T - p$ by $L_o(p)$. The solution to (6) is $F_1 = AF_0'$, but F must be even and therefore $A = 0$. The only solution to (7) which decays to zero at infinity is

$$G_1 = \cosh T \ln(2\cosh T) - T \sinh T. \qquad (8)$$

We now have enough information to determine the small-κ behaviour of ϕ_0 to first order for the 'A'-type asymmetric states. From our definition of ϕ_0 and our expression for G_1, we obtain

$$\tan\phi_0 \approx 1 - 2\kappa \ln 2 \qquad (9)$$

from which

$$\phi_0 \simeq \frac{\pi}{4} - \kappa \ln 2.$$

The gradient, $d\phi_0/d\kappa$, at $\kappa = 0$ agrees with that obtained numerically for the 'A'-type asymmetric states.

At the other end of the 'A'-branch is one of the symmetric soliton states, $y = 0$. To perturb about such a state, it is most convenient to use (3) rewritten using the reduced variables $(X, Y) = (x, y)/(\sqrt{2}\alpha)$ and $T = \alpha t$:

$$\begin{aligned} \frac{d^2X}{dT^2} - X + 2X^3 + 6XY^2 &= 0 \\ \frac{d^2Y}{dT^2} - \mu^2 Y + 2Y^3 + 6YX^2 &= 0. \end{aligned} \qquad (10)$$

Introducing an expansion parameter $\epsilon^2 = \mu_0^2 - \mu^2$ we write

$$X = X_0 + \epsilon X_1 + \dots, \quad Y = \epsilon Y_1 + \dots .$$

The zeroth order equation derived from (10) is

$$X_0'' - X_0 + 2X_0^3 = 0 \qquad (11)$$

and has the solution $X_0 = \mathrm{sech}\,T$. The first order equations are

$$X_1'' + 6X_0^2 X_1 - X_1 \equiv L_o(1)X_1 = 0 \qquad (12)$$

and

$$Y_1'' + 6X_0^2 Y_1 - \mu_0^2 Y_1 \equiv L_o(\mu_0^2)Y_1 = 0. \qquad (13)$$

The solution to (12) is $X_1 = A\,\mathrm{sech}\,T \tanh T$. As with the small-$\kappa$ analysis, since X must be an even function of T, we have to make $A = 0$.

$L_o(p)$ can be transformed into the associated Legendre operator if $z = \tanh T$:

$$(1-z^2)\frac{d}{dz}\left((1-z^2)\frac{d}{dz}\right)+N(N+1)(1-z^2)-\lambda$$

with $N = 2$ and eigenvalue, $\lambda = p$. From this it can be deduced that (13) has exactly two solutions which decay at infinity. The non-trivial solution is $Y_1 = B_1 \operatorname{sech}^2 T$ with $\mu_0^2 = 4$ (which corresponds to $\kappa = 3/5$). This is the result obtained in [4].

To describe the asymmetric states, we need to proceed to higher order. The second order equations are

$$L_o(1)X_2 = -6X_0Y_1^2 \tag{14}$$

and

$$L_o(4)Y_2 = 0. \tag{15}$$

The solution of (14) is

$$X_2 = B_1^2(\operatorname{sech}^3 T - 2\operatorname{sech} T) \tag{16}$$

and from (15), Y_2=0. To third order we have

$$L_o(1)X_3 = 0$$

and

$$L_o(4)Y_3 = Y_1 - 2Y_1^3 - 12Y_1X_0X_2. \tag{17}$$

To determine B_j we can use the consistency condition

$$\int_0^\infty Y_1 L_o(4)Y_{j+2}\, dT = 0 \tag{18}$$

derived from the self-adjoint property of $L_o(4)$. Applying (18) to (17) yields $B_1 = \sqrt{5/48}$.

The expansion for X and Y so far is

$$X = \operatorname{sech} T + \epsilon^2 B_1^2(\operatorname{sech}^3 T - 2\operatorname{sech} T) + \ldots$$

$$Y = \epsilon B_1 \operatorname{sech}^2 T + \ldots .$$

To compare these results with the numerical results of Fig. 1 we must determine ϕ_0:

$$\tan\phi_0 = \frac{y(0)}{x(0)} = \frac{Y(0)}{X(0)} \simeq \frac{\epsilon B_1}{1-\epsilon^2 B_1^2}.$$

Expressing $\epsilon(=\sqrt{4-\mu^2})$ in terms of κ using (4) we then obtain

$$\phi_0 \approx \tan\phi_0 \approx \sqrt{\frac{25}{32}\left(1-\frac{5}{3}\kappa\right)} \tag{19}$$

for κ near 3/5.

We can combine the results (9) and (19) using a two-point Padé approximant. Given the form of the equations, it seemed most natural to obtain an approximant, P, for $\tan^2\phi_0$. We therefore write

$$P = \left(1-\frac{5}{3}\kappa\right)\left(\frac{1+a_1\kappa}{1+b_1\kappa}\right) \tag{20}$$

so that P automatically has a zero at $\kappa = 5/3$ and the correct value at $\kappa = 0$. Matching the expansion to first order of the Padé approximant with that of $\tan^2\phi_0$ for small κ gives

$$a_1 - b_1 - \frac{5}{3} = -4\ln 2\,,$$

and to ensure that P has the correct gradient around its zero we must have

$$\frac{1+3a_1/5}{1+3b_1/5} = \frac{25}{32}.$$

From these two relations we obtain the Padé coefficients $a_1 \simeq 2.28305$ and $b_1 \simeq 3.38898$. The error in ϕ_0 obtained using (20) is less than 2.5%.

3 'B'-type asymmetric states about $\kappa = 1$

In the case of the antisymmetric state we have $f = -g$ and so $x = 0$. We now rewrite (3) in terms of the reduced variables $X = x/(\sqrt{2}\alpha\mu)$, $Y = y/(\sqrt{2}\alpha\mu)$, $T = \alpha\mu t$:

$$\frac{d^2X}{dT^2} - \nu^2 X + 2X^3 + 6XY^2 = 0, \tag{21}$$

$$\frac{d^2Y}{dT^2} - Y + 2Y^3 + 6YX^2 = 0, \tag{22}$$

in which we have put $\nu = 1/\mu$. Notice that these equations are the same as (10) but with X and Y exchanged and μ replaced by ν. Demanding a solution with κ non-zero and X_1 vanishing at infinity, would similarly result in $\nu_0 = 4$ which would imply a negative κ. The numerical results suggest that we should insist that $\nu_0 = 0$ (i.e. $\kappa = 1$). Hence we use ν as our small parameter and write

$$X = \nu X_1 + \nu^2 X_2 + \ldots$$

$$Y = Y_0 + \nu Y_1 + \nu^2 Y_2 + \ldots .$$

To zeroth order in ν, we have $Y_0 = \operatorname{sech} T$. To first order we obtain

$$L_o(0)X_1 = 0$$

which has the general solution

$$X_1 = A_1\left(\frac{3}{2}s^2 - 1\right) + H_1\left(\frac{3}{2}Tt - T + \frac{3}{2}t\right)$$

in which $s \equiv \operatorname{sech} T$ and $t \equiv \tanh T$. Although this does not blow up for large T (with H_1 set to zero), it still does not vanish at infinity. It is therefore an algebraic secularity, and based on our previous work [5], we would hope that it can be regrouped with other such terms at higher order to give an expression which does tend to zero overall for large T.

We therefore apply a multiple-scale analysis [6] so that A_1 is now a function of the scaled variables T_1, etc., where $T_m \equiv \nu^m T$. The quantities X_i and Y_i are now functions of T_m as well as of T and so we must expand the derivatives in (21) and (22) using

$$\frac{d}{dT} \equiv \frac{\partial}{\partial T} + \nu\frac{\partial}{\partial T_1} + \nu^2\frac{\partial}{\partial T_2} + \dots .$$

Apart from A_1 and H_1 no longer being constants, the results obtained so far for the ordinary analysis are still valid. The solution to the other first order equation,

$$L(1)Y_1 = 0,$$

where we now have $L(p) \equiv \partial_T^2 + 6s^2 - p$, is

$$Y_1 = B_1 st + J_1\left(\tfrac{1}{2}\cosh T - \frac{3}{2}s + \frac{3}{2}Tst\right).$$

Evidently we must make $J_1 = 0$.

At second order we find

$$\begin{aligned} X_2 &= \frac{3}{2}A_{1,1}t + 3A_1B_1s^2t + 3B_1H_1\left(Ts^2t - t^2\right) \\ &\quad + H_{1,1}\left\{(1-T^2)\left(\frac{3}{2}s^2 - 1\right) - \frac{3}{2}Tt\right\}. \end{aligned}$$

Since the $H_{1,1}$ term cannot be removed by making it a function of A_1, B_1, or H_1, we put $H_1 = 0$. Our equation for Y_2 is then

$$\begin{aligned} Y_2 &= A_1^2\left(\frac{9}{4}s^3 - 3s + 3Tst\right) \\ &\quad + B_1^2 s - B_1^2 s^3 T - B_{1,1}Tst. \end{aligned}$$

Given the form of Y_1, it can be seen that the Tst terms in Y_2 are ghost secularities. Their removal gives a relation between B_1 and A_1:

$$B_{1,1} = 3A_1^2. \qquad (23)$$

At third order the most divergent term in X_3 is

$$(A_{1,11} - A_1 + 2A_1^3)\left\{\frac{T^2}{2}\left(1 - \frac{3}{2}s^2\right) - \frac{3}{2}Tt\right\}$$

which can be removed by choosing

$$A_{1,11} - A_1 + 2A_1^3 = 0 \qquad (24)$$

The only bounded solutions to (24) are

$$A_1 = \pm\,\mathrm{sech}\, T_1. \qquad (25)$$

From (23) and (25) we see that $B_1 = 3\tanh T_1$.

If we look at the ordinary perturbation analysis again and expand up to fifth order, ignoring the mounting algebraic secularities, we find that the leading terms in each order in the expansion of X sum to give

$$-\nu A\left[1 - \frac{2A^2-1}{2}\nu^2T^2 + (2A^2-1)\frac{6A^2-1}{24}\nu^4T^4\right]$$

These terms are consistent with the expansion of $-\nu A\,\mathrm{sech}\,\nu T$ if $A = \pm 1$. The positive solution in (25) is the one we have been taking throughout the previous analyses.

Using the results we have obtained so far, we find that

$$\tan\phi_0 \simeq \frac{1 - \frac{3}{4}\nu^2}{\frac{1}{2}\nu} \quad \text{with} \quad \nu = \sqrt{\frac{1-\kappa}{1+\kappa}}.$$

This agrees with the numerical results for κ close to unity.

4 Discussion

The analysis of the 'B'-type asymmetric states about the decoupled state has proved to be much more difficult. As κ approaches zero, the numerical calculations show that F tends to a sech T function while G has two sech-like humps (symmetrically placed about the origin) whose separation increases with decreasing κ. To carry out a small-κ analysis like that in §2 we would have to perturb about a state, G_0, that has two sech-like humps at infinity. With the G_m independent of κ, it is difficult to imagine what form they could take to relocate G_0's humps at infinity to some finite, κ-dependent distance from the origin.

References

[1] Hasegawa, A. & Tappert F. *Appl. Phys. Lett.* **23**, 142 (1973).

[2] Hasegawa, A. *Optical solitons in fibers* (Springer-Verlag, Berlin) (1989).

[3] Trillo, S., Wabnitz, S., Wright, E.M. & Stegeman, G.I. *Opt. Lett.* **13**, 672–4 (1988).

[4] Akhmediev, N. & Ankiewicz, A. *Phys. Rev. Lett.* **70**, 2395–8 (1993).

[5] Allen, M.A. & Rowlands, G. *J. Plasma Phys.* **50**, 413-24 (1993).

[6] Nayfeh, A.H. & Mook D.T. *Nonlinear Oscillations* (Wiley, New York, 1979).