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THE MODIFIED BASIS FUNCTION METHOD FOR TIME-VARYING FREQUENCY ESTIMATION

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ABSTRACT

A modified version of the basis function method, called the modified basis function method, is proposed for estimating time-varying frequency of nonstationary signals. The modification is accomplished by adjusting a TVAR process, used as a linear predictor, and applying a combination of both the forward and the backward predictors for calculating TVAR parameters. The time-varying frequency estimate was extracted from location of the closest pole to the unit circle in the complex z-plane. Two nonstationary signals, one is chirp and another is of sinusoidally time-varying frequency, are used as examples. From our results, the proposed approach yielded better accuracy in estimating the time-varying frequency in either noisy or noise-free situation than using the traditional basis function method.

KEYWORDS: Time-varying autoregressive, Modified Basic function method, Nonstationary signal, Frequency estimation

1. INTRODUCTION

Time-varying frequency estimation problems are addressed in many applications, including communication, system monitoring, diagnostic, and many more. Several approaches, such as, the short-time Fourier transform, the time-frequency distribution, and the parametric method, are cable of estimating the time-varying frequency and spectral. Among those, the parametric method yields the possible highest frequency resolution. The time-varying autoregressive (TVAR) process, i.e. $y[t] = -\sum_{i=1}^{p} a_i[t] y[t-i] + e[t]$, have been used for estimating time-varying frequency, where the estimation is done in two steps: first, estimation of the TVAR parameters $a_i[t]$ and then second, frequency extraction from the TVAR parameters $a_i[t]$. Two general approaches, adaptive algorithm and basis function method, have been discussed and shown that, while the adaptive algorithms such as LMS, RLS, and RLS with constraint, are cable of tracking the frequency jump, it fails to track the fast continuous variation. In contrast, the basis function method is in potent to estimate the fast varying frequency or spectrum [1].

In this work, we propose a method, called "the modified basis function method", by making some modification to the TVAR process used as a linear predictor and applying a combination of both the backward and the forward linear estimators for improving the frequency estimation accuracy. Superiority in the time-varying frequency estimation of the proposed method is shown and discussed.

2. THE BASIS FUNCTION METHOD

The TVAR model that has been used for the non-stationary signal [2] [3] [4] utilizes only forward linear estimation. The forward linear predictor defined as $\hat{y}[t] = -\sum_{i=1}^{p} a_i[t] y[t-i]$ is used in the basis function method, and the TVAR model parameters $a_i[t]$ are explicitly modeled as a summation of weighted time functions as

$$a_i[t] = \sum_{k=0}^{q} a_{ik} f_k(t) \tag{1}$$

where a_{ik}, $i = 1,2,\ldots,p$, $\mathrm{k} = 0,1,\ldots,q$ are constants, q is expansion dimension, and $f_k(t)$ are some predefined time-functions that are used as the basis function in the expansion.

With $a_i[t]$ defined in (1), the forward linear estimator can be rewritten as

$$\hat{y}[t] = -\sum_{i=1}^{p}\left(\sum_{k=0}^{q} a_{ik} f_k(t)\right) y[t-i] \tag{2}$$

and the estimation error is

$$e[t] = y[t] + \sum_{i=1}^{p}\left(\sum_{k=0}^{q} a_{ik} f_k(t)\right) y[t-i].$$

Parameters a_{ik} are calculated by solving a set of $p(q+1)$ linear equations, equation (3), which is the result from minimizing the mean squared error $\varepsilon = \frac{1}{2(T-p)}\sum_{t=p}^{T} | e[t] |^2$

$$\sum_{i=1}^{p}\sum_{k=0}^{q} a_{ik} c_{kl}(i,j) = -c_{0l}(0,j), \quad j = 1,2,\ldots p \text{ and } l = 0,1,\ldots q. \tag{3}$$

Here
$$c_{kl}(i,j) \doteq \frac{1}{T-p}\left(\sum_{t=p}^{T} f_k(t) f_l^*(t) y[t-i] y^*[t-j]\right). \tag{4}$$

The parameter estimates $\hat{a}_i[t]$ are obtained by $\hat{a}_i[t] = \sum_{k=0}^{q} a_{ik} f_k(t)$. Details of this method are in [3]. After obtaining the parameters $\hat{a}_i[t]$, the frequency estimate can be calculated.

3. PROPOSED METHOD: THE MODIFIED BASIS FUNCTION METHOD

The proposed method is a modified version of the basis function method. Here we define slightly different TVAR processes and use both forward and backward predictors.

Forward estimation:

$$\hat{y}^f[t+1] = -\sum_{i=1}^{p} a_i[t]y[t-i+1]\ , \quad t = p\text{-}1,\ p,\ \dots,\ T\text{-}1 \tag{5}$$

$$a_i[t] = \sum_{k=0}^{q} a_{ik} f_j(t)$$

Forward estimation error:

$$e^f[t+1] = y[t+1] - \hat{y}^f[t+1] = y[t+1] + \sum_{i=1}^{p}\left(\sum_{k=0}^{q} a_{ik} f_k(t)\right) y[t-i+1] \tag{6}$$

Backward estimation: $$\hat{y}^b[t-1] = -\sum_{i=1}^{p} b_i[t]y[t+i-1] \tag{7}$$

For stationary signals, the backward and the forward parameters are time-invariant, and they are related to each other. As a matter of fact, for stationary signals, the backward parameters b_i are simply complex conjugates of the forward parameters a_i. In the case of the non-stationary signal, the backward and the forward coefficients are not time-invariant, and they may or may not be equal to the complex conjugate of each other. However, to develop our approach, we will force them to be equal by using a constraint $b_i[t] \cong a_i^*[t]$. Therefore, we can rewrite the backward estimator as

$$\hat{y}^b[t-1] = -\sum_{i=1}^{p} a_i^*[t]y[t+i-1]\ , \quad t = 1,\ 2,\ \dots,\ T\text{-}p\text{+}1$$

and backward estimation error as

$$e^b[t-1] = y[t-1] - \hat{y}^b[t-1] = y[t-1] + \sum_{i=1}^{p}\left(\sum_{k=0}^{q} a_{ik} f_k(t)\right)^* y[t+i-1] \qquad (9)$$

In the basis function method, the estimation of a_{ik} is achieved by minimizing the mean square of only the forward prediction error, but here both the forward and the backward prediction errors are combined. We redefine the mean squared error as the combination of both the forward estimation error and the backward estimation error,

$$\varepsilon \doteq \frac{1}{2(T-p)}\left\{\sum_{t=p-1}^{T-1} | e^f[t+1]]^2 + \sum_{t=1}^{T-p+1} | e^b[t-1]|^2\right\} \qquad (10)$$

Since $| e^b[t]|^2$ and $| (e^b[t])^* |^2$ are equal, the mean squared error can be rewritten as

$$\varepsilon = \frac{1}{2(T-p)}\left\{\sum_{t=p-1}^{T-1} | e^f[t+1]]^2 + \sum_{t=1}^{T-p+1} | (e^b[t-1])^* |^2\right\}$$

$$= \frac{1}{2(T-p)}\left\{\begin{array}{l}\sum_{t=p-1}^{T-1}\left|y[t+1] + \sum_{i=1}^{p}\sum_{k=0}^{q} a_{ik} f_k(t) y[t-i+1]\right|^2 \\ \qquad + \sum_{t=1}^{T-p+1}\left|y^*[t-1] + \sum_{i=1}^{p}\left(\sum_{k=0}^{q} a_{ik} f_k(t)\right) y^*[t+i-1]\right|^2\end{array}\right\}$$

Taking derivative of ε with respected to a_{jl}, $j = 1,2,...p, l = 0,1,...q$, and equate to zero.

$$\frac{\partial \varepsilon}{\partial a_{jl}} = 0 = \sum_{t=p-1}^{T-1}\left(y[t+1] + \sum_{i=1}^{p}\sum_{k=0}^{q} a_{ik} f_k(t) y[t-i+1]\right) f_l^*(t) y^*[t-j+1]$$
$$+ \sum_{t=1}^{T-p+1}\left(y^*[t-1] + \sum_{i=1}^{p}\sum_{k=0}^{q} a_{ik} f_k(t) y^*[t+i-1]\right) f_l^*(t) y[t+j-1]$$

After rearrangement in the above equation, we will have

$$\sum_{i=1}^{p}\sum_{k=0}^{q} a_{ik} c_{kl}(i,j) = -c_{0l}(0,j), \quad j = 1,2,...p \text{ and } l = 0,1,...q \qquad (11)$$

where

$$c_{kl}(i,j) = \sum_{t=p+1}^{T-1} f_k(t) f_l^*(t) y[t-i+1] y^*[t-j+1] + \sum_{t=1}^{T-p-1} f_k(t) f_l^*(t) y^*[t+i-1] y[t+j-1] \quad (12)$$

Solving the equation (11) yields the constant parameters a_{ik}, and then the estimates of the time-variant parameters, $\hat{a}_i[t] = \sum_{k=0}^{q} a_{ik} f_i(t)$. As can be noticed, equation (11) is exactly the same as equation (3), but the $c_{ik}(i,j)$ defined in (12) are totally different from that in (4). Furthermore, the parameter estimate $\hat{a}_i[t]$ from the basis function and the modified basis function methods are also different in either their values or time-availability. In the basis function method, $\hat{a}_i[t]$ are available from time t = p until t = T, but in the proposed approach, $\hat{a}_i[t]$ can be calculated only from time t = 1 to t =T-1.

Equation (11) is a set of $p(q+1)$ linear equations. Solving them for a_{ik} might be tedious, especially if p and q are large. However, they can be changed into a matrix form, and then linear algebra techniques can be applied for achieving the solutions. One can write (11) in a matrix form as

$$\mathbf{Ca} = -\mathbf{d} \quad (13)$$

Where

$$\mathbf{C} = \begin{bmatrix} \Phi(1,1) & \Phi(2,1) & \cdots & \Phi(p,1) \\ \Phi(1,2) & \Phi(2,2) & \cdots & \Phi(p,2) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi(1,p) & \Phi(2,p) & \cdots & \Phi(p,p) \end{bmatrix}$$ is of $p(q+1)$ x $p(q+1)$ size.

$$\Phi(i,j) = \begin{bmatrix} c_{00}(i,j) & c_{10}(i,j) & \cdots & c_{q0}(i,j) \\ c_{01}(i,j) & c_{11}(i,j) & \cdots & c_{q1}(i,j) \\ \vdots & \vdots & \ddots & \vdots \\ c_{0q}(i,j) & c_{1q}(i,j) & \cdots & c_{qq}(i,j) \end{bmatrix}$$ is of $q+1$ x $q+1$ size.

$$\mathbf{a} = \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_p \end{bmatrix}, \bar{a}_i = \begin{bmatrix} a_{i0} \\ a_{i1} \\ \vdots \\ a_{iq} \end{bmatrix}, \mathbf{d} = \begin{bmatrix} \bar{\chi}_1 \\ \bar{\chi}_2 \\ \vdots \\ \bar{\chi}_p \end{bmatrix}, \text{ and } \bar{\chi}_j = \begin{bmatrix} c_{00}(0,j) \\ c_{01}(0,j) \\ \vdots \\ c_{0q}(0,j) \end{bmatrix}$$

Computational aspects of the matrix form as in the equation (13) were discussed in Hall et al. [1983], in which a symmetric property was utilized to reduce the steps of computation. For the proposed approach, the matrix **C** is still symmetric. Therefore, the computational reduction, mentioned in Hall et al. [1983], can still be used in our approach. However, since we employed both the forward and the backward estimations, the computation for forming the matrix **C** in (13) is unavoidably two times as much of that mentioned in Hall's paper.

4. FREQUENCY ESTIMATION; RESULTS AND DISCUSSION

The proposed approaches, the modified basis function method, were tested in noise-free and noisy situation, to estimate the time varying frequencies of two synthetic signals that have only a single frequency component. The synthetic signals were real and generated such that their frequencies were exactly known as shown in Figure 1. The first signal is a real (not complex) chirp signal whose normalized frequency increased linearly from $0.1Fs$ to $0.41Fs$ over 32 samples, where Fs = sampling frequency. The second signal is sinusoid whose frequency varies periodically from $0.1Fs$ to $0.3Fs$ with a sweep rate of $0.05Fs$. We estimated the time-varying frequency by first, solving for the vector **a** in equation (13), and then computing the TVAR parameter estimate by using $\hat{a}_i[t] = \sum_{k=0}^{q} a_{ik} f_j(t)$, i = 1, 2, ..., p. Once $\hat{a}_i[t]$ were available, the time-varying frequency estimate was obtained from $\hat{f}[t] = angle(z_o) \cdot Fs/(2\pi)$, where z_o is the closest pole to the unit circle in the complex z-plane, and is a root of the prediction error filter polynomial $z^p + a_1[t]z^{p-1} + a_2[t]z^{p-2} + \ldots + a_p[t] = 0$.

First, we test the proposed method for signal test 1 and 2 in noise-free environment. Figure 2 shows results from using basis function and modified basis function to estimate time-varying frequency of the signal tests 1 and 2. For signal test 1, we used the polynomial basis function $f_k(t) = \left(\frac{t-1}{N}\right)^k$ where N is the total sample number, and chose p = 2, q =4, while for the signal test 2, we used the cosine function defined as $f_k(t) = \cos\left(\frac{\pi kt}{N}\right)$, and selected p = 2, q

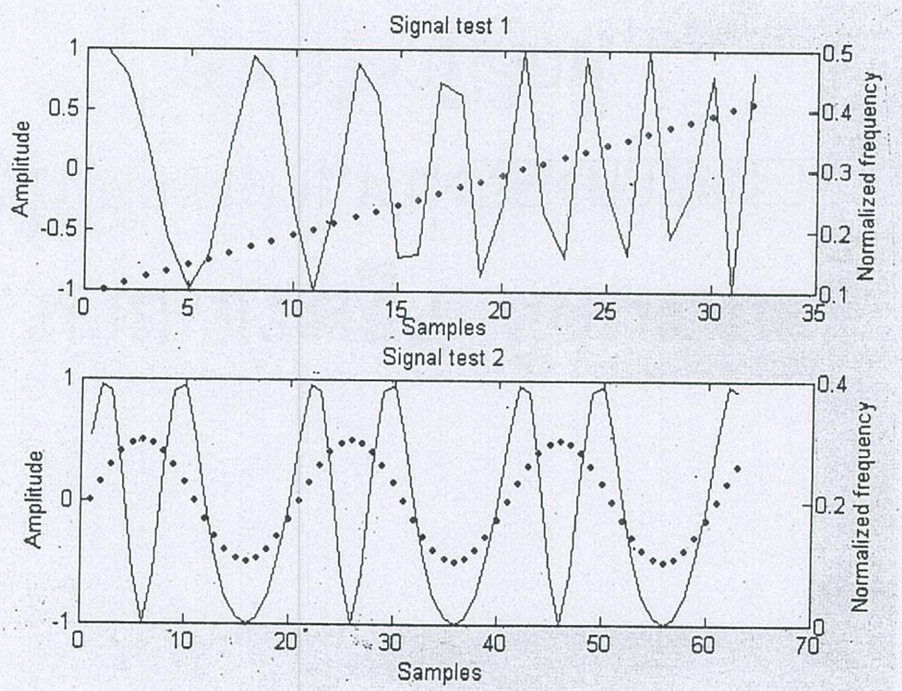
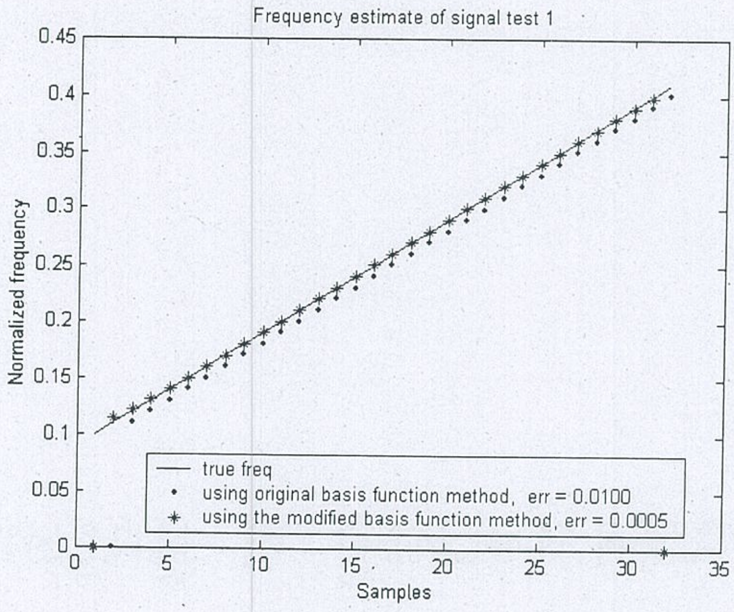


Figure 1: Two signals used for the test. Solid = real signals. Dot = Normalized frequency varying with time.

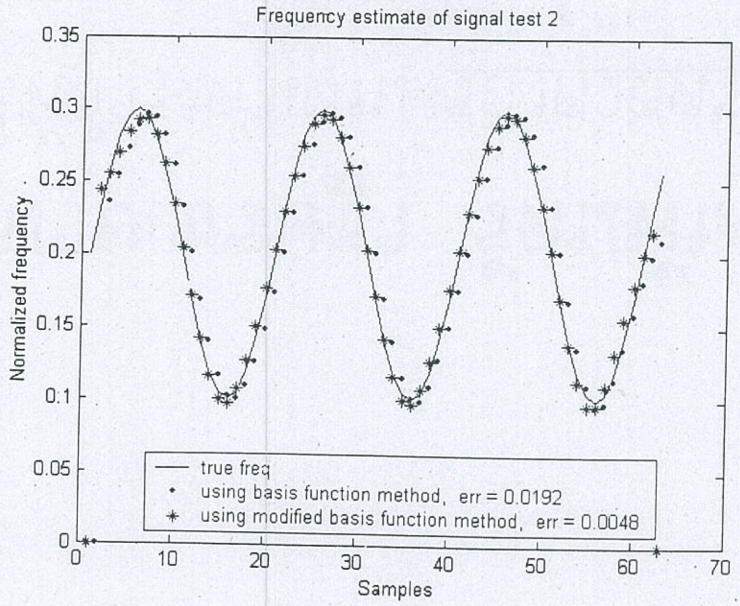
=12. The errors shown in the Figure are the averaged errors, calculated from the definitions $err = \frac{1}{N-p}\sum_{n=p+1}^{N}\left|f[n]-\hat{f}[n]\right|$, for the basis function method, and $err = \frac{1}{N-2}\sum_{n=2}^{N-1}\left|f[n]-\hat{f}[n]\right|$, for the proposed approach. As seen, the proposed approach using both the forward and the backward estimators yields less estimation error, compared to that from the traditional basis function method using only the forward estimator. As noticeable for the signal test 1, the average error from using basis function method is, in fact, about one-step difference between the adjacent true frequency, $\left|f[n]-f[n-1]\right| = 0.01$. This error verifies that the TVAR process used as the linear predictor in the traditional basis function method that have been used by several researchers, is at best a step-delay version of the true parameter. However, this error is negligible when the sampling frequency is high. Also noted was that the traditional basis function method could yield a frequency estimate only after the p^{th} sample, due to the p-delay required at the initial state, but the proposed approach 1 allowed the frequency estimate available for all sampled time except the first and the end samples.

Next, we used the proposed approach to estimate the frequency of the signal test 1 and 2 in the additive white noise that has signal-to-noise ratio (SNR) of 20 dB. We assumed that the true model order p is unknown, so we used $p > 2$, which is over-determined. We can see a benefit of the proposed approach that utilizes both the forward and the backward estimators, when the model order is over-determined.

Figure 3 displays results when the model order p = 4, q = 4 for signal test 1, and p = 4, q =12 for signal test 2. Both cases are of over-determined model orders. As seen from Figures 3(a) and 3(b), when the signal is noisy and the model order is over-determined, the traditional basis function method does not yield successfully the estimate of time-varying frequency. However, the proposed approach yields the frequency estimate that is reasonably accurate.

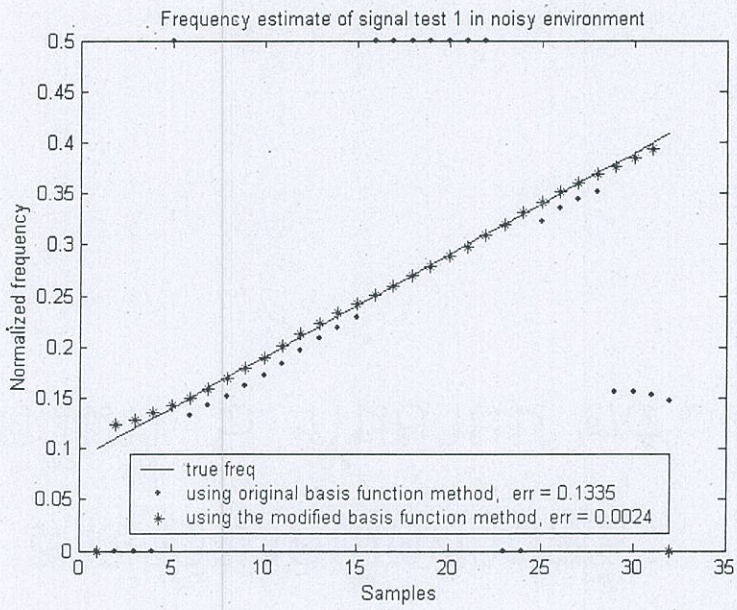


(a)

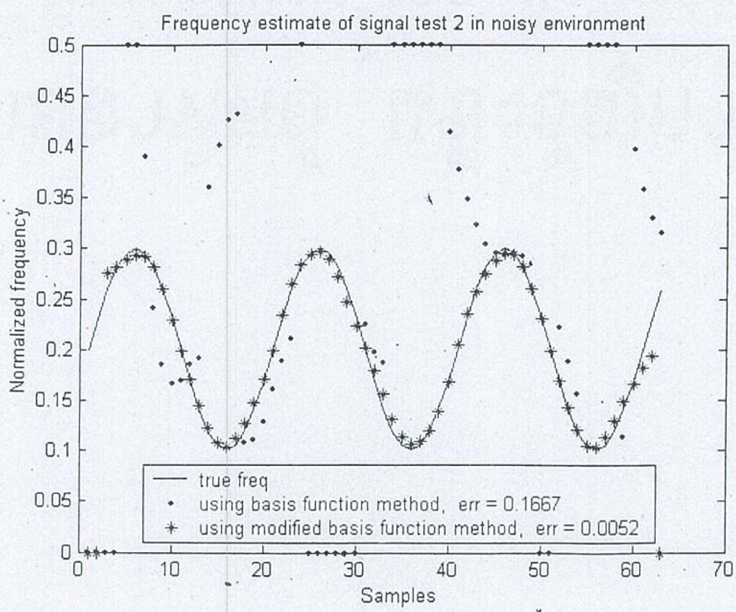


(b)

Figure 2: Results from using proposed approach to estimate the time-varying frequency of (a) signal test 1 and (b) signal test 2.



(a)



(b)

Figure 3: Results from using the proposed approach for frequency estimation of (a) signal test 1 and (b) signal test 2 in noisy environment and when model order is over-determined.

One might argue that the reason that the traditional basis function method that utilize only an forward predictor failed to estimate the time-varying frequency in Figure 3, was because of the total number of parameters that were needed to be estimated ($p.q$) was not small, compared to the number of available data, and that did not follow the rule of thumb or the implication of the parsimony principle, recommended by Niedzwiecki [2000][1]. This argument may be true because we used for the signal test 1, $p = 4$, $q = 4$ ($p.q = 16$), while the number of data = 32 samples, and for the signal test 2 $p = 4$, $q = 12$ ($p.q = 48$) while the number of available data = 96 samples. To see that the proposed approach, the modified basis function method did actually improve the frequency estimation when the model order was over-determined, we did a few more tests in noisy 20dB-SNR environments where the sample number of signal test 1 and signal test 2 were increased to be 128, and 224 samples, respectively. The increase in the sample number is to satisfy the parsimony principle. The results of these tests are summarized in table 1. We remarked that the true model order was two, since the signal test 1, or the signal test 2 only had a single time-frequency component. As seen, when the true model order p=2 was selected, the estimation errors from the forward and the proposed approach were not much different, but when the used model order was increased (over-determined), the estimation errors of both signal tests from the proposed approach were smaller than that from the traditional basis function method using the forward predictor alone. This confirms that the proposed method did improve the frequency estimation.

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[1] Niedzwiecki, M. [2000] suggested that, based on the principle of parsimony, the total number of estimated parameters should be much smaller than the number of available data points or satisfy the inequality $pq \leq 0.2N$, where p is the model order, q is dimension in time-function expansion, and N is total number of data. {Source: Maciej Niedzwiecki, *Identification of Time-varying processes*, Chapter 6, John Wiley & Sons, Ltd., 2000}

Table 1: Frequency estimation errors of the proposed approach (the modified basis function method) and the traditional basis function method as the model order was increased.

Model Order (p)	Frequency estimation errors			
	Using basis function method		Using modified basis function method	
	Signal test 1	Signal test 2	Signal test 1	Signal test 2
p = 2	0.0029	0.0073	0.0015	0.0054
p = 4	0.0083	0.0084	0.0004	0.0020
p = 6	0.0086	0.0165	0.0003	0.0017

It should be noted that, as the model order p increases (still satisfying the principle of parsimony), the estimation errors from the proposed approach tends to decrease. This can be explained as well. Since the signal was in an additive noise, the TVAR model order-2 that could sufficiently represent the noise-free signal was no longer valid for the signal in the additive noise. In fact, the TVAR model with limited order is not completely valid to represent a process in an additive noise, since the signal in the additive noise algebraically becomes the TVARMA process. For the TVAR model to sufficiently represent the TVARMA processes, the model order of the TVAR process must be of infinite order or large. Proof can be given as following:

A. Show that a TVAR process with the additive white noise is equivalent to a TVARMA signal

Let the signal $x[t]$ be the impulse response of the TVAR(m) processes, and $y[t] = x[t] + w[t]$ where $w[t]$ is the additive white noise. The spectral density of $x[t]$ is

$$P_{xx}(t,\omega) = \frac{1}{\left|1+\sum_{k=1}^{m} a_k[t]e^{-jk\omega}\right|^2} = \frac{1}{D(t,\omega)}$$

The power spectral density of $y[t]$ is

$$P_{yy}(t,\omega) = P_{xx}(t,\omega) + \sigma_w^2 = \frac{1}{\left|1+\sum_{k=1}^{m} a_k[t]e^{-jk\omega}\right|^2} + \sigma_w^2$$

$$= \frac{1+\sigma_w^2 D(t,\omega)}{\left|1+\sum_{k=1}^{m} a_k[t]e^{-ik\omega}\right|^2},$$

which is actually the power spectral density of a TVARMA(m,m) process. Therefore, the TVAR(m) process in the additive white noise becomes a TVARMA(m,m) process, whose coefficients in the numerator of the TVARMA transfer function depend on the power spectral density of the signal and the noise variance. □

B. Show that the TVAR process with the infinite model order validly represents the TVARMA process with finite order.

To prove this, we will simply show that the TVARMA(1,1) can be changed into the TVAR form with the infinite order. The transfer function of ARMA(1) is

$$H_{ARMA}(z) = \frac{1+b_1[t]z^{-1}}{1+a_1[t]z^{-1}}$$

Let $\frac{1+b_1[t]z^{-1}}{1+a_1[t]z^{-1}} = \frac{1}{1+c_1[t]z^{-1}+c_2[t]z^{-2}+\ldots} = \frac{1}{C(z)} = H_{AR}(z)$

where the term in the right side is the transfer function of the TVAR(∞), and $C(z)$ is the z-transform of $c_k[t]$ with respect to *k*. That is

$$C(z) = Z\{z_k[t]\} = \frac{1+a_1[t]z^{-1}}{1+b_1[t]z^{-1}} = 1+\frac{(a_1[t]-b_1[t])z^{-1}}{1+b_1[t]z^{-1}}$$

Take inverse z-transform of $C(z)$,

$$c_k[t] = \begin{cases} 1 & , k = 0 \\ (a_1[t] - b_1[t])(-b_1[t])^{k-1} & , k = 1,2,3,.... \end{cases}$$

Hence the TVARMA(1,1) process is equivalent to the TVAR(∞), in which the TVAR parameters $c_k[t]$ must be defined as above.

5. SUMMARY

A TVAR model is used in our modified basis function method that utilizes the combination of both the forward and the backward prediction errors to estimate the TVAR parameters, and then the time-varying frequencies. Posing the constraint $\hat{b}_i[t] = \hat{a}_i^*[t]$ to the minimization of the summation of the forward and the backward error, was beneficial to the frequency estimate in that the estimates were approximately equal to the true frequency and could be obtained for all samples except the first and the last samples. In addition, for the noisy case where the model order was over-determined, our proposed approach yielded better results in estimating the time-varying frequency than using the traditional basis function method utilizing solely forward predictor, and the estimation error from our approach tended to decrease with the increase of modeled order, provided that the implication of the parsimony principle was still satisfied.

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