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TVAR MODELLING AND TIME-VARYING FREQUENCY ESTIMATION OF NONSTATIONARY SIGNALS

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ABSTRACT

Time-varying autoregressive (TVAR) approach is used for modeling nonstationary signals, and then the frequency information of the signals is extracted from the TVAR parameters. Two methods may be used for estimating the TVAR parameters: the adaptive algorithm and the basis function approach. Adaptive algorithms, such as the least mean square (LMS) and the recursive least square (RLS), use a dynamic model for adapting the TVAR parameters and are capable of tracking time-varying frequency, provided that the variation is slow. The basis function method employs and explicit model for the TVAR parameter variation and the model parameters are estimated via a block calculation. It is observed in our study that, if the signals have a single time-varying frequency component, the RLS with a fixed pole on the unit circle yields the fastest convergence. But none of the adaptive algorithms can successfully track fast varying frequency. The basis function method, although is not so effective in tracking the frequency jump, but capable of tracking either the fast or the slow time-varying frequencies.

KEYWORDS: Time-varying autoregressive, Nonstationary signals, Basis function method, Adaptive algorithms, Frequency Estimation.

1. INTRODUCTION

An assumption, that a signal or system is stationary, is often made in the signal processing literatures and found useful in practical applications, including communication, system control, array processing, speech coding and encoding, and many more. This assumption is not always accurate for signals having time-varying frequency, because such the signals with the time-varying frequency are highly non-stationary. Frequency analysis of nonstationary signals is important since the time history of frequency is a powerful approach in signal characterization. The analysis of time-varying frequency is typically performed by using a short-time Fourier transform (STFT), a time-frequency distribution (TFD), or a parametric method. Among these methods, the STFT is the most direct and simplest approach, in which the signal is separated into small lengths fitting a sliding window. Both time and frequency resolution of the STFT is limited by the window length in an opposite manner (i.e. the short window length yields a good time resolution but poor frequency resolution and vice versa for the long window length). Hence, good frequency and time resolution cannot be obtained simultaneously by using the STFT. The TFD approaches, such as the Wigner-Ville distribution and Choi-Williams distribution, are mathematically complicated, but they yield possible higher time and frequency resolution. However, some of the TFD approaches have artifacts from cross-term interference when the signal have two or more frequency components, and in addition, the time and frequency resolutions are limited by the uncertainty principle, $\sigma_t \sigma_w \geq 1/2$ [1], where $\sigma_t^2 = \int (t - <t>)^2 \, |s(t)|^2 \, dt$ is the variance of energy in the time domain, which represents the sharpness in the time analysis, and $\sigma_w^2 = \int (w - <w>)^2 \, |S(w)|^2 \, dw$ is the variance of energy in the frequency domain, which represents the sharpness in the frequency analysis.

The best frequency resolution is possible by using the parametric method, in which a signal is modeled by using either the autoregressive, the moving average, or the autoregressive moving average models. The parametric method for the modern spectral estimation of stationary signals has been thoroughly studied and well documented [2] [3]. It was shown that, even though the available signal is very short, the parametric method yields very high frequency resolution in the spectral estimation.

In this study, we review a parametric method, especially the TVAR process, and compare two general methods for TVAR parameter estimation, the adaptive algorithm method (such as LMS, RLS, and RLS with a fixed pole on unit circle) and the basis function method. Some aspects about ability, in extracting the time-varying frequency of a non-stationary signal, between the two methods are shown and discussed.

2. TVAR MODELLING OF NONSTATIONARY SIGNAL

According to Wold's decomposition theorem, any stationary random process $x[t]$ may be represented by a sum of two orthogonal components, predictable and regular processes [4]. The first component, denoted by $x_p[t]$ is deterministic and predictable, which can be calculated from an infinite number of its previous value without any estimation error (i.e. $x_p[t] = -\sum_{i=1}^{\infty} a_i x_p[t-i]$). The second component is nondeterministic, denoted by $x_r[t]$, and can be estimated as an output of a causal noise-shaping filter with infinite order (i.e. $x_r[t] = -\sum_{i=1}^{\infty} h_i w[t-i]$, where h_i is coefficient of a filter). In practice, using the infinite number of previous values and filter coefficients is impossible. A finite order has then been used, depending on a chosen structure of the filter; whether auto regressive (AR), moving average (MA), or autoregressive moving average (ARMA) filter is used. Since the AR process is simple and was shown that it is capable of approximating either the MA or the ARMA model [2], the AR process has been much of interest. In fact, it has been modified by allowing its coefficients to change with time for modeling non-stationary signals, and is called the time-varying autoregressive (TVAR).

The TVAR process, also known as the time-varying linear predictor, is to estimate the current or future value $y[t]$ by using a linear combination of previous values $y[t-i]$. The process is written as

$$\hat{y}[t] = -\sum_{i=1}^{p} a_i[t]y[t-i] \tag{1}$$

or

$$y[t] = -\sum_{i=1}^{p} a_i[t]y[t-i] + e[t]$$

Where $a_i[t], i = 1,2,\ldots,p$ are time-varying coefficients, p is model order, $\hat{y}[t]$ is the estimate of $y[t]$, and $e[t]$ is the estimation error. Although, there is no corresponding general theory for a nonstationary process, the TVAR has been successfully shown for modeling, analyzing, and synthesizing nonstationary signals, such as speech [5] [6].

3. TVAR PARAMETER ESTIMATION

In order to use the TVAR as a model for nonstationary signal processing, the model order p in the equation (1) must be chosen, and then time-variant parameters $a_i[t]$ must be determined, if not pre-known. We assume that the order p is known, and only the parameters $a_i[t]$ are needed to be estimated. Since $a_i[t]$ are time-variant, they cannot be estimated by using the same method as obtaining the solution of the Wiener-Hopf equations. Available approaches for estimating the time-variant coefficients fall into one of two categories, based on the model of the $a_i[t]$ variation. In one category, a dynamic model is used for the parameter variation, and the parameters are updated each time when new data are available. An adaptive algorithm is utilized in this category. In another category, the coefficient $a_i[t]$ is explicitly defined as a linear summation of weighted time-functions. The time functions are pre-defined. Only the weight must be determined, and this can be done via block processing. Some details of these two approaches for the TVAR parameter estimation are shown below.

3.1 Using the dynamic model and the adaptive algorithm

Variation of the time-varying parameters $a_i[t]$ are based on a dynamic model defined as

$$\bar{a}_i[t] = \bar{a}_i[t-1] + \Delta\bar{a}_i[t] \tag{2}$$

The parameters $a_i[t]$ are updated from their previous values $a_i[t-1]$. Terms $\Delta a_i[t]$ represent innovation terms, and they might be different depending on which rule or the adaptive algorithm being utilized (i.e., steepest descent, least mean square (LMS), recursive least square (RLS). algorithm, etc). If the steepest descent or the LMS algorithm is used, $\Delta\bar{a}_i[t]$ is equal to $\mu \cdot E\{e[t]\bar{\varphi}[t]\}$ or $\mu \cdot e[t]\bar{\varphi}[t]$, respectively, where $\bar{\varphi}[t] = [y[t] \quad y[t-1] \quad \ldots \quad y[t-p]]^T$ is a vector of the sampled nonstationary signal. If the RLS algorithm is applied, the $\Delta a_i[t]$ are a little bit more involved. Computation steps of the LMS and the RLS algorithms are listed below.

LMS algorithm:

Parameter: p = Filter order

μ = Step size

$$\bar{\varphi}[t] = [y[t] \quad y[t-1] \quad \ldots \quad y[t-p+1]]$$

$$\bar{a}[t] = [a_1[t] \quad a_2[t] \quad \ldots \quad a_p[t]]$$

Initialization: $\hat{\bar{a}}[0] = 0$

Computation: For t = 0, 1, 2, ...

$$\hat{y}[t] = -\sum_{i=1}^{p} a_i[t]y[t-i]$$

$$e[t] = y[t] - \hat{y}[t]$$
$$\hat{\bar{a}}[t] = \hat{\bar{a}}[t-1] + \mu \cdot e[t]\bar{\varphi}[t-1]$$

RLS algorithm:

Parameter: p = Filter order

λ = Exponential weighting or forgetting factor

δ = Value used to initialize P(0)

$$\bar{\varphi}[t] = \begin{bmatrix} y[t] & y[t-1] & \ldots & y[t-p+1] \end{bmatrix}$$

$$\bar{a}[t] = \begin{bmatrix} a_1[t] & a_2[t] & \ldots & a_p[t] \end{bmatrix}$$

Initialization: $\hat{\bar{a}}[0] = 0$

$P(0) = \delta^{-1} I$

Computation: For $t = 1, 2, \ldots$

$$\bar{z}[t] = P[t-1]\bar{\varphi}^*[t-1]$$
$$\bar{g}[t] = \frac{1}{\lambda + \bar{y}^T[t]\bar{z}[t]}\bar{z}[t]$$

$$e[t] = y[t] + \sum_{i=1}^{p} \bar{a}_i[t-1]y[t-i]$$
$$\hat{\bar{a}}[t] = \hat{\bar{a}}[t-1] + e[t]\bar{g}[t]$$
$$p[t] = \frac{1}{\lambda}\left[P[t-1] - \bar{g}[t]\bar{z}^H[t]\right]$$

Step size and forgetting factor affect the convergence rate of the LMS and the RLS algorithm, respectively. To ensure stability, the step size μ must be in the range $0 < \mu < 2/\lambda_{\max}$, where

$\lambda_{\max}$ is the largest eigenvalue of the correlation matrix[1] R. However, in general application, knowledge of the $\lambda_{\max}$ is unavailable. Therefore, a simple working rule, stepsize in the range $0 < \mu < 2/tap\_input\_power$ where $tap\_input\_power = \sum_{i=0}^{p-1} E[|y(t-i)|^2]$, can be used [7]. For the RLS algorithm, the exponential weighting or forgetting factor λ must be chosen in the range of $0 < \lambda \leq 1$. The factor λ is to ensure that the data in a distant past are forgotten. Details about these factors, the LMS, and the RLS algorithm are available [7].

The adaptive methods may be sufficient for the TVAR parameter estimation, if the nonstationay part of a signal changes slowly. However, due to its nature as an adaptive system, they are sensitive to noise. Increasing the step size or the forgetting factor can help in reducing sensitivity to the noise, but the convergence rate of the algorithm decreases as well, and this may result in a diminished ability in tracking the parameter change.

3.2 Using the basis function and the block processing: the basis function approach

Instead of using the dynamic model and the adaptive algorithm for estimation, $a_i[t]$ are explicitly modeled as a summation of weighted time functions as

$$a_i[t] = \sum_{k=0}^{q} a_{ik} f_k(t) \tag{3}$$

where a_{ik}, $i = 1,2,\ldots,p$, $\mathrm{k} = 0,1,\ldots,q$ are constants, q is expansion dimension, and $f_k(t)$ are some predefined time-functions that are used as the basis function in the expansion. With $a_i[t]$ defined in (3), equation (1) can be rewritten as

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[1] Correlation matrix is defined as $R = E\{\bar{y}[t-1]\bar{y}^H[t-1]\}$, where $\bar{y}[t-1] = [y[t-1] \quad y[t-2] \quad \ldots \quad y[t-p]]^T$

$$\hat{y}[t] = -\sum_{i=1}^{p}\left(\sum_{k=0}^{q} a_{ik} f_k(t)\right) y[t-i] \qquad (4)$$

and the estimation error is

$$e[t] = y[t] + \sum_{i=1}^{p}\left(\sum_{k=0}^{q} a_{ik} f_k(t)\right) y[t-i].$$

Parameters a_{ik} are calculated by solving a set of $p(q+1)$ linear equations, equation (5), which is

the result from minimizing the mean squared error $\varepsilon = \frac{1}{2(T-p)} \sum_{t=p}^{T} |e[t]|^2$

$$\sum_{i=1}^{p}\sum_{k=0}^{q} a_{ik} c_{kl}(i,j) = -c_{0l}(0,j), \quad j = 1,2,\ldots p \text{ and } l = 0,1,\ldots q \qquad (5)$$

Here
$$c_{kl}(i,j) \doteq \frac{1}{T-p}\left(\sum_{t=p}^{T} f_k(t) f_l^*(t) y[t-i] y^*[t-j]\right) \qquad (6)$$

The parameter estimates $\hat{a}_i[t]$ are obtained by $\hat{a}_i[t] = \sum_{k=0}^{q} a_{ik} f_k(t)$. Details of this method are in Hall et al. [1983]. It should be noted that, in this method not only the model order p, but also the basis time-function $f_k(t)$ and the expansion dimension q must be chosen. Some existing functions that can be used as a basis for parameter expansion are the time-polynomial, the Legendre, the Cosine, the Fourier function, and the discrete prolate spheroidal sequence.

4. TIME-VARYING FREQUENCY/SPECTRAL ESTIMATION EXTRACTION

The time-varying frequency can be extracted from the TVAR parameters $a_i[t]$. Since the nonstationary signal is modeled as the output of the TVAR process, with a zero-mean white noise input $w[t]$, the power spectral density $S_{yy}[t,f]$ of the stationary signal is given by

$$S_{yy}[t,f] = \frac{\sigma_w^2}{\left|1 + \sum_{i=1}^{p} a_i[t] e^{-j2\pi f \cdot i}\right|^2} \tag{7}$$

where σ_w^2 is the variance of the white noise $w[t]$. In practice, this variance is unknown, but can be approximated by

$$\sigma_w^2 \sim \sigma_e^2 = \frac{1}{T-p} \sum_{t=p+1}^{T} \left(y[t] + \sum_{i=1}^{p} a_i[t] y[t-i] \right)^2 \tag{8}$$

Time-varying frequencies can be extracted from the power spectral density by locating the peak of the $S_{yy}[t,f]$. If there are m frequency components in the signal, one can choose the frequency estimate as the locations of the m largest spectral peaks. Note that a threshold could be also set so that the peak below this threshold would be considered as that of an additive noise.

Another way to extract the time-varying frequency from the TVAR, is by calculating the roots z_i of the estimation error filter polynomial $z^p + a_1[t]z^{p-1} + a_2[t]z^{p-2} + \ldots + a_p[t] = 0$, and choosing the frequency estimate as the angles of the root z_i. For a real signal, the roots may be complex conjugate to each other. So only the roots in either upper or lower half of complex plane are selected. For example, if the number of the frequency components is m, the closest m poles to the unit circle in the upper half of the z-plane are chosen. The time-varying frequencies $f_i[t]$ are then calculated by $f_i[t] = \text{angle}\left(z_i[t]\right) \cdot Fs/2\pi$ According to Kay [1988], estimating the frequency in this way is slightly more accurate than locating the frequency as the largest spectral peak location.

5. ASPECT ABOUT ABILITY IN ESTIMATING TIME-VARYING FREQUENCY: ADAPTIVE ALGORITHM VS BASIS FUNCTION METHOD.

As explained previously, the frequency estimation via the TVAR model was accomplished in two steps, the TVAR parameter estimation and the frequency extraction. There exist several methods in the two general categories, and available for the estimation. One may wonder which approach should be chosen for estimating the time-varying frequency of a nonstationary signal. The adaptive algorithms are known for their ability to track time-variation in the statistics of the signal, provided that the variations are sufficiently slow, and its computation can be done on-line.

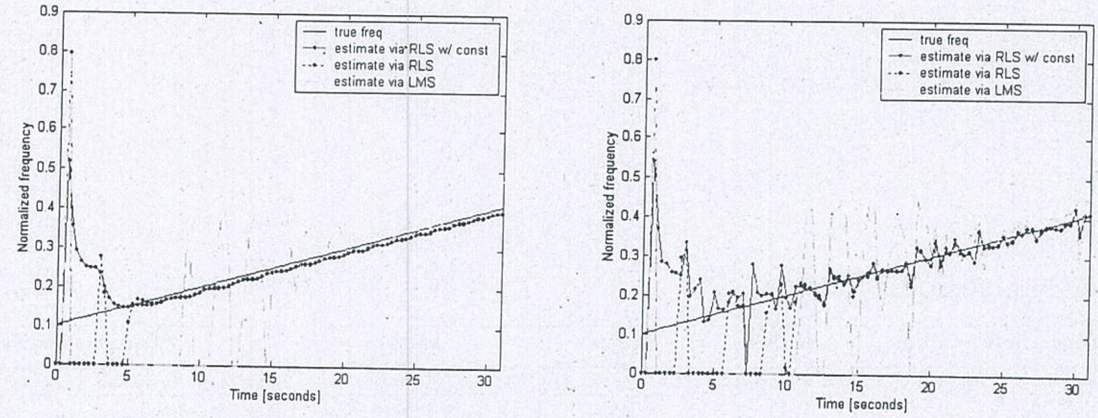
For block estimation, processing is done on a block-to-block basis. Traditionally, the length of the block is usually chosen short enough to maintain a pseudo-stationary assumption. However, for the basis function approach the TVAR parameters that contain the nonstationarity information of a signal are changed to the summation of a set of unknown constants multiplied by predefined time-functions. As a consequence, the block estimation in the basis function approach is to calculate these unknown constants, not the time-variant parameter, and the calculation is the same as if it is for a stationary signal. Hence the length of the block for the basis function approach can be of any length (i.e., the length of the block could be very long), provided that computational complexity and time are not restricted.

The following are some comparisons between the adaptive algorithm and the basis function approach about their ability in estimating the time-varying frequency of three nonstationary signals. The first signal was a chirp with unit amplitude and had a frequency increasing linearly. The next signal was a sinusoidal that its frequency varied periodically, and the last signal was a sinewave that has a frequency jump. All the signals were real and had one frequency component, then the TVAR model of the second order (p = 2) was used. Three adaptive algorithms, namely the LMS, the RLS, and the RLS with a pole constrained on the unit circle, were used to estimate the time-varying frequency of signals 1, 2, and 3, respectively. The step size for the LMS algorithm was set to 0.5 and the forgetting factor for the RLS method was set to 0.7. The results from using these algorithms are shown in Figures 1, 2 and 3.

Figure 1 shows the frequency estimate of the chirp signal where the frequency increase linearly, in (a) noise-free and (b) noisy (20dB-SNR) situations. As seen in figure 1a all algorithms can track the true frequency that changed slowly. However, the frequency estimate is in not equal to the true frequency, since there is always a delay when the adaptive algorithms were utilized. Among the algorithms, the LMS was slowest in the rate of convergence. The step size of the LMS algorithm could be increased so that it would converge faster, but the oscillation would also be higher. The RLS algorithm yielded fast convergence, especially when a pole of the TVAR(2) model was constrained to be on the unit circle. The reason that the RLS with the constrained pole converged very fast, was because after a pole was constrained, there was only another parameter needed to be estimated, instead of 2 parameters. Figure 1b shows the frequency estimate of the chirp in a noisy situation with the SNR ~20dB. Compared to the result in the noise-free situation, it is obvious that the adaptive algorithm is sensitive to the additive noise. This is due to the nature of its adaptive mechanism that automatically adjusted itself to the change.

The ability of the adaptive algorithm in tracking the time-varying frequency is more clearly seen from the result in figure 2, where the signal has a periodically changing frequency. When the frequency of the signal changes slowly (figure 2a), the adaptive algorithms yield the frequency estimate that is about the delayed version of the true time-varying frequency. However, when the frequency changes very fast (figure 2b), 4 times faster than that in figure 2a, the adaptive method fails to track the frequency change. This was obviously seen (in figure 2b) that the frequency estimate, neither from the LMS nor the RLS algorithm, reaches the peak and valley of the true frequency curve.

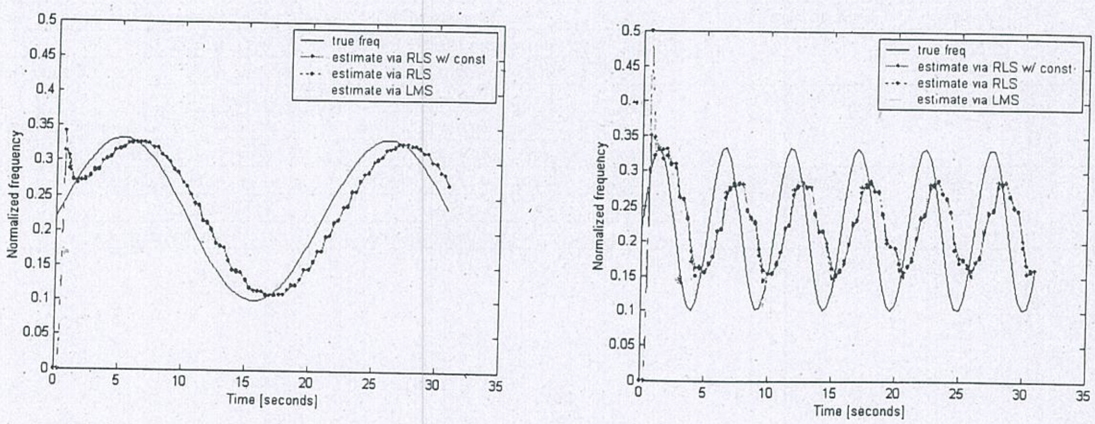
Figure 3 displays the frequency estimate of the signal that has a frequency discontinuity at about 20 seconds. The RLS and the RLS with a pole constraint can track the frequency jump, and so does the LMS algorithm. However, the LMS algorithm seems to have trouble in tracking the frequency that varies close to zero (as seen in figure 3 that the frequency estimate from the



(a) Noise-free case

(b) Noisy case

Figure 1: Frequency estimate of linear chirp (f = 0.01+0.1t Hz) in (a) noise-free and (b) noisy situation using 1. RLS algorithm with forgetting factor =0.75 with a pole constrained on unit circle, 2. RLS algorithm with forgetting factor =0.75 (no constraint), 3. LMS algorithm with step size = 0.5



(a) True frequency f = *0.35*sin(0.3*t)+.65*

(b) true frequency *f = 0.35*sin(1.2*t)+.65*

Figure 2: Frequency estimate of a real sinusoid where frequency changes periodically, in noise-free environment, using adaptive algorithm. a) Frequency change slowly. b) Frequency changes very fast, 4 times faster than in (a).

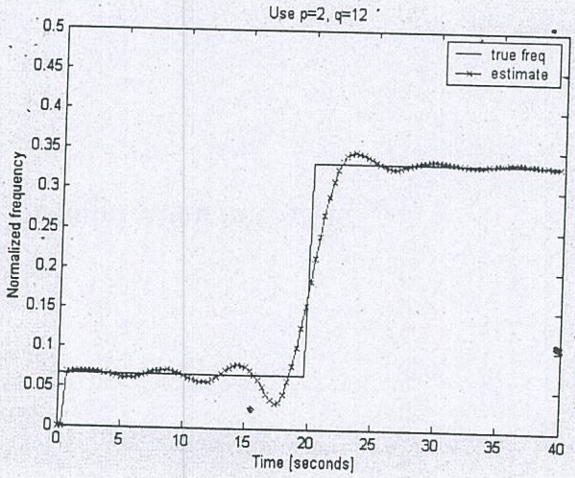
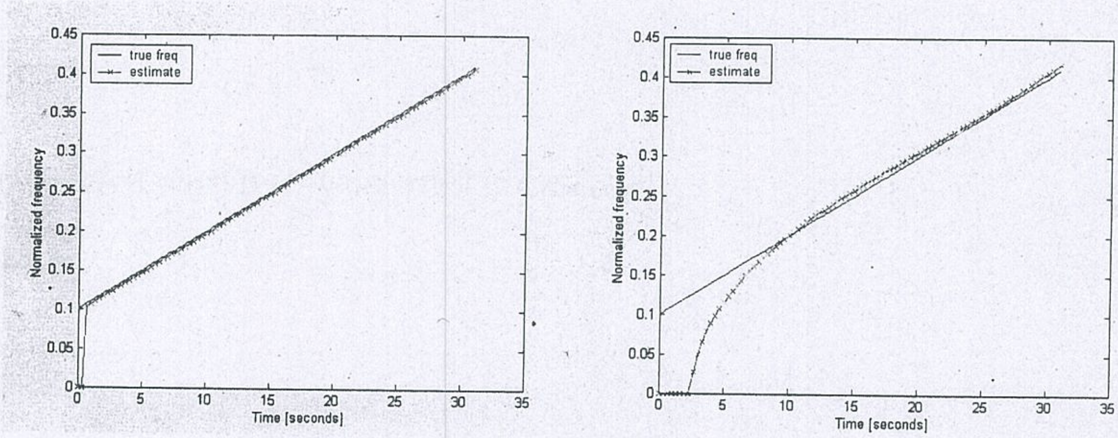


Figure 3: Frequency estimate of sinusoidal signal which has frequency jump, using adaptive algorithm

LMS algorithm does not converge to the true frequency when the true frequency is at 0.067Fs, where Fs is the 3 Hz sampling frequency used in this paper). This is because the eigenvalues of the autocorrelation matrix **R** for this signal spread wildly. The biggest and smallest eigenvalues are so different. The step size of the LMS algorithm must be set small, corresponding to the biggest eigenvalue, so that a convergence could be attained Hence, while the LMS does converse, it converges slowly at low frequency and is seen as if it cannot track the low frequency for a given signal range.

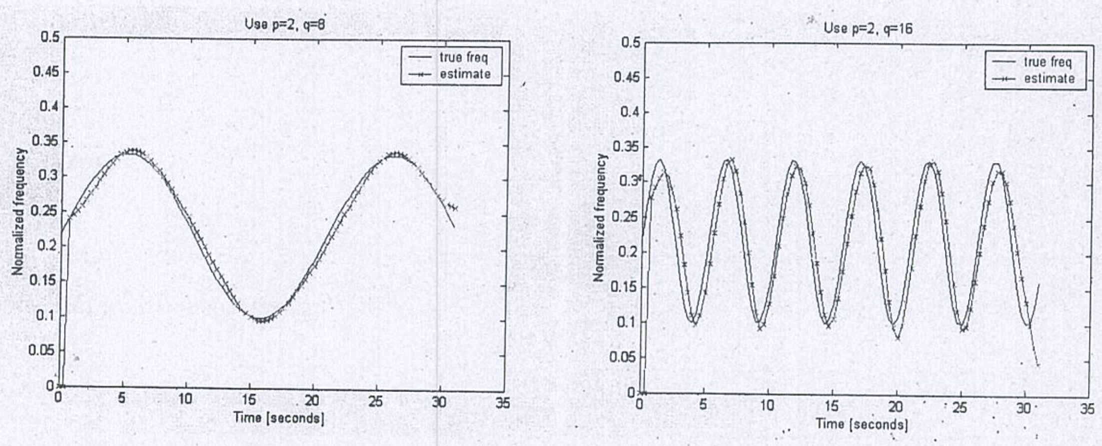
Results from using the basis function method are shown in Figures 4, 5, and 6 In using the basis function approach, the time function $f_k(t)$ and the expansion dimension q must be chosen. For the chirp signal, we used the time-polynomial function $f_k(t) = \left(\frac{t-1}{N}\right)^k$ and the dimension q = 2. For the sinusoid with the periodically time-varying frequency, the cosine function $f_k(t) = \cos\left(\pi k \frac{t}{N}\right)$ was used, and the dimension of q =8 and q =16 were utilized for



(a) No-noise

(b) Noisy case (20dB-SNR)

Figure 4: Frequency estimation of the chirp signal in (a) noise free and (b) noisy cases, using the TVAR parameter as a summation of the weighted time-function.



(a) True frequency $f = 0.35*sin(0.3*t)+.65$

(b) True frequency $f = 0.35*sin(1.2*t)+.65$

Figure 5: Frequency estimate of the sinusoid where frequency changes periodically, in noise-free environment, using the basis function method. a) Frequency change slowly. b) Frequency changes very fast, 4 times faster than in (a).

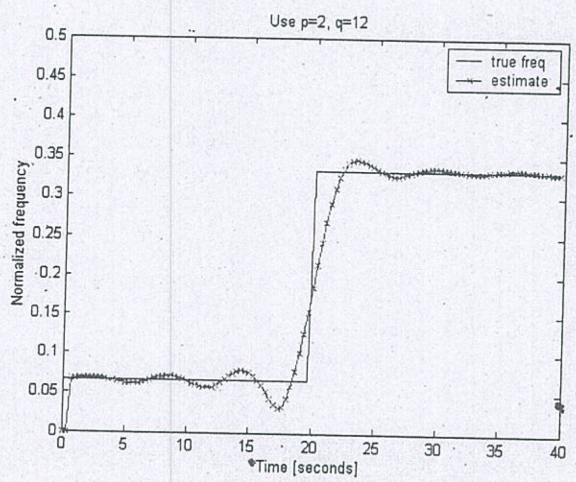


Figure 6: Frequency estimate of the signal that has a frequency jump, using the basis function method.

the slow and fast time-varying frequencies. The cosine function was also applied for the case of frequency jump.

As seen from figure 4a of the noise-free case, the frequency estimate is about the true frequency. The convergence rate was not a problem for the basis function method in the estimation, since the computation was done in a processing block. However, the estimate at the beginning (about p steps) was not available, since it required p initial values of the data before the calculation started, as it is for the AR(p) process. Figure 4b displays an ability of the basis function method to estimate the time-varying frequency in a noisy situation, 20dB-SNR. As also can be seen from the figure, the noise resulted in the shift of the frequency estimate from the true frequency, and many incorrect frequency estimates at the beginning. The ability of the basis function method in tracking the time-varying frequency that varies broadly can be more obviously seen from the results in figure 5a and 5b. The frequency estimates are close to the true frequencies for both slow and fast varying.

Figure 6 displays frequency estimate from using the basis function method in a situation where there is a frequency jump. The expansion dimension is $q = 12$. As shown, the frequency estimate bounces around the true frequency before and after the jump. The estimate could be closer to the true frequency, if a higher expansion dimension was used.

6. CONCLUDING REMARKS

We have presented TVAR processes as the models for nonstationary signals, of which the information, especially the time-varying frequency or spectrum, are contained in the time-variant coefficients of the TVAR model. Frequency information is extracted in two steps: first, the TVAR parameter estimation, and then frequency calculation. We have used adaptive algorithms such as the LMS, the RLS, and the RLS with constraint, and the basis function method for estimating the TVAR parameters and then calculate the time-varying frequency. It was demonstrated that while the adaptive algorithms had the ability to track the slowly time-varying frequency, it was sensitive to the noise. They also failed to track the time-varying frequency of the signal, if the frequency changed very fast or broadly. However, they were efficient in tracking the frequency jump. In contrast, the basis function method, in which the time-variant parameters are expanded as a summation of the weighted time-functions, are capable of tracking both the fast or the slow time-varying frequencies. It is ideally expected that when the expansion dimension is infinite, the result of the frequencies estimation from any basis function is the same, which will exactly equal to the true frequency. But this is impractical, since the computation may require infinite memory, and infinite computational time consumption. A result from the previous section showed that the effect of an additive noise could cause the shift of the frequency estimate from the true frequency. Also it was observed that the frequency estimate was not exactly equal to the true frequency, but it was a step-delayed version of the true frequency. An accuracy improvement of the basis function method for tracking time varying frequency is possible.

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