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Poles and Straight Lines on Surfaces *

MINORU TANAKA

Abstract

There are open problems originated from classical differential geometry. In this note some interesting open problems will be introduced.

1. The existence of many poles

Let M be a 2-sheeted hyperboloid defined by

$$z = \sqrt{c}\sqrt{\frac{x^2}{a} + \frac{y^2}{b} + 1}, \qquad c > 0 < a \le b. \tag{1.1}$$

In 1881, von Mangoldt proved in [3] that the umbilic points, i.e., $(0, \pm\sqrt{a\frac{b-a}{a+c}}, \sqrt{c\frac{b+c}{a+c}})$, are poles. Furthermore if $a = c$, then any point sufficiently close to the unique umbilic point $(0, 0, \sqrt{c})$ is a pole. Here a point p on M is called a *pole* if any two geodesics emanating from p do not intersect again. He conjectured in the paper [3] that *for any point on M that is sufficiently close to an umbilic point would be a pole.* In [6] R. Sinclair checked by an experimental method that this conjecture is true.

Problem 1 Is any point on the 2-sheeted hyperboloid defined by (1.1) a pole, if the point is sufficiently close to an umbilic point?

2. The diameter of the set of poles

It is an interesting problem to measure the size of the set of poles on a 2-sheeted hyperboloid or any complete surface, since there are many poles on a 2-sheeted hyperboloid as I have introduced in Section 1. In 1989, M. Maeda found an upper bound of the diameter

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2000 *Mathematics Subject Classification.* Primary 53C22 of the set of poles on a nonnegatively curved Riemannian manifold M as follows. Let p be a point on M. For each $t > 0$ let $D(t)$ denote the diameter of the set

$$S_p(t) := \{q \in M; d(p, q) = t\},$$

where d denotes the Riemannian distance on M. Then the following theorem was proved by M. Maeda [2] and refined by K.Sugahara [7].

Theorem 2.1 *Let M be a complete connected Rieamnnian manifold with nonnegative sectional curvature. Then the number*

$$d_0(M) := \limsup \frac{D(t)^2}{t}$$

is independent of the choice of the point p and the diameter of the set of poles on M is not greater than $\frac{d_0(M)}{8}$.

Proposition 2.2 *Let M be a surface of revolution homeomorphic to R^2. If the Gaussian curvature of M is non-negative and $d_0(M)$ is finite, then M has a unique (trivial) pole.*

Definition 2.1 A complete Riemannian manifold homeomorphic to R^2 is called a *surface of revolution* if M has a point p such that for any points $q_1, q_2 \in M$ with $d(p, q_1) = d(p, q_2)$, there exists an isometry f on M satisfying $f(p) = p$, $f(q_1) = q_2$. Then the point p is called a *vertex* of M.

Definition 2.2 A surface of revolution M is called a *von Mangoldt surface of revolution* if $G \circ \mu$ is monotone non-increasing on $[0, \infty)$. Here G denotes the Gaussian curvature of M and $\mu : [0, \infty) \to M$ denotes a unit speed geodesic emanating from the vertex of M.

Example 2.1 Paraboloids and 2-sheeted hyperboloids are typical examples of a von Mangoldt surface of revolution.

Theorem 2.3([8]) *Let M be a surface of revolution homeomorphic to R^2. Then the vertex of M is a unique pole if and only if*

$$\int_1^\infty \frac{1}{L(t)^2} dt = \infty \quad or \quad \liminf_{t \to \infty} L(t) = 0.$$

Here $L(t)$ denotes the length of $S_p(t)$.

Theorem 2.4([8]) *Let M be a surface of revolution homeomorphic to R^2. Then for any point $q \in M$ with $d(p, q) \leq r(M)$, q is a pole. Here the point p denotes the vertex of M and*

$$r(M) := \sup\{d(p, q); q\, is\, a\, pole\, of\, M\}.$$

Remark The vertex is always a pole.

Theorem 2.5([8]) *Let M be a von Mangoldt surface of revolution. If* $c(L) := 4\int_0^{\infty} \frac{L(t) - tL'(t)}{L(t)^3} dt$ *is nonpositive, then* $r(M) = \infty$, *i.e., any point on M is a pole.*

Theorem 2.6 ([8]) *Let M be a von Mangoldt surface of revolution. If* $\int_1^{\infty} \frac{1}{L(t)^2} < \infty$ *and $c(L)$ is positive $r(M)$ is equal to the unique solution r of* $c(L) = \int_r^{\infty} \frac{1}{L(t)^2} dt$.

Problem 2 Find an upper bound of the set of poles on a surface of revolution.

3. The existence of straight lines

A geodesic $\gamma : R \to M$ on a Riemannian manifold M is called a *straight line* if any bounded closed (i.e. compact) interval $[a, b] \subset R$, $\gamma|_{[a,b]}$ is a minimal geodesic segment joining $\gamma(a)$ to $\gamma(b)$. Let M denote the complete Riemannian manifold $(R \times S^1, dt^2 + m(t)^2 d\theta^2)$ homeomorphic to a cylinder. Then $\theta = const.$ is called a *meridian* on M. It is clear that each meridian is a straight line on M.

Theorem 3.1(Inoue) *For each point $p \in M$, M has a straight line passing through p distinct from the meridian through p if and only if* $\inf_{t \in R} m(t) > 0$, *and* $\int_R \frac{1}{m(t)^2} dt < \infty$.

Problem 3 Find a necessary and sufficient condition for the manifold M to have many straight lines.

Problem 4 Find a sufficient condition for M to have a straight line γ and a constant C such that $t(\gamma(s)) \leq C$ for ant $t \in R$.

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DEPARTMENT OF MATHEMATICS
TOKAI UNIVERSITY
HIRATSUKA 259-1292
JAPAN

E-mail address: m-tanaka@sm.u-tokai.ac.jp